Z. Wahrscheinlichkeitstheorie verw. Gebiete
40, 147–155 (1977)

Zeitschrift für
Wahrscheinlichkeitstheorie
und verwandte Gebiete

Random Sheets

Ernst Eberlein

ETH Zürich, Mathematikdepartment, Ch-8092 Zürich

0. Introduction

The classical random walk played an important role in the study of sequences of random variables. Deep results could be proved using the simple geometry of its paths. On the following pages we introduce random sheets, a two-dimensional analogue. Section 1 gives a formal definition. Section 2 is devoted to its combinatorics. Results on cardinalities of certain sets are obtained. The proof of Theorem (2.1) using the generating function was suggested by the reviewer. Our original proof had been based on geometric arguments. In Section 3 probability theory enters the field. Random sheets define in a natural way an array of vector-valued random variables, the underlying distribution being the uniform one on rectangles. By means of these variables we study the asymptotic behavior of random sheets along a strip of arbitrary width. The result is a normal distribution which is centered at the origin. For width one we compute the covariances explicitly and get 2/3. Having in mind that a symmetric random walk – suitably normalized – is asymptotically normally distributed with variance 1 we can interpret this as follows. On such a strip a random sheet behaves qualitatively in the same way as a symmetric random walk but its variance is smaller. This is obviously the effect of the strong correlation between neighbor states.

The limit theorem got in this paper is "essentially" one-dimensional. Two-dimensional results are obtained in [2].

In a discussion on this paper with D. Abraham, G. Gallavotti, K. Jacobs and D. Ruelle it was realized that the interaction concept given by the random sheet construction is equivalent to the six vertex ice model studied in theoretical physics. In fact an isomorphism between the ice configurations on $n \times m$ lattice points and the random sheets on a rectangle of size $n \times m$ can easily be constructed. Thus it is not surprising that the total number of random sheets on a rectangle is computed by means of a matrix (Section 2). The transfer matrix method has already a certain tradition in theoretical physics. Main contributions to the six vertex ice model have been made by E.H. Lieb. We note that his transfer matrix is not the same as ours. The differences are briefly explained in Section 2. For a survey on the type of questions asked by physicists and the answers known so far see [5]. Further references are given there.

I thank the reviewer for his comments and A. Gubitz for writing a computer program for me in an early stage of this research.

1. Definitions

Let Z^d denote the d-dimensional lattice and Z^d_+ the subset of points having all coordinates non-negative. Given points (k, l) and (m, n) in Z^2 which satisfy $(k, l) \leqq (m, n)$, i.e. $k \leqq m$ and $l \leqq n$, we consider the following *rectangle* in Z^2

$$r((k, l), (m, n)) = \{(p, q) \in Z^2 \mid (k, l) \leqq (p, q) \leqq (m, n)\}.$$

Instead of $r((0, 0), (m, n))$ we shall usually write $r(m, n)$. Two points (k, l) and (m, n) in Z^2 are called *neighbors* iff they have Euclidean distance 1, i.e. either $k = m$ and $|l - n| = 1$ or $l = n$ and $|k - m| = 1$.

(1.1) *Definition.* A *random sheet (RS)* on $r(m, n)$ is a function $w: r(m, n) \to Z^1$ such that

(1) $w(0, 0) = 0$

(2) if $(p, q), (p', q') \in r(m, n)$ are neighbors then $|w(p, q) - w(p', q')| = 1$.

Let $\mathscr{S}(m, n)$ denote the family of RSs on $r(m, n)$. $|\mathscr{S}|$ means the cardinality of a finite set $\mathscr{S}$.

Recall that a *random walk (RW)* of length n is given by a $(n+1)$-tupel $x = (x(0), \ldots, x(n))$ where $x(k) \in Z^1$ and $|x(k+1) - x(k)| = 1$ for all $0 \leqq k \leqq n-1$. Set $\mathscr{X}(n)$ for the family of these tupels and $\mathscr{X}_0(n)$ for the subset of RWs satisfying $x(0) = 0$. Define τ_n: $\mathscr{X}(n) \to \{0, 1\}^n$ by the following rule

$$(\tau_n(x))_k = \begin{cases} 0 & \text{if } x(k) - x(k-1) = 1 \\ 1 & \text{if } x(k) - x(k-1) = -1. \end{cases}$$

Then

$$h_n(x) = \sum_{k=1}^{n} (\tau_n(x))_k \, 2^{k-1} + 1$$

defines a numbering $h_n: \mathscr{X}(n) \to \{1, \ldots, 2^n\}$. RWs of arbitrary starting value but consisting of the same sequence of ups and downs get the same number.

For every n we define ${}_nT = ({}_nT_{ij})_{1 \leqq i, j \leqq 2^n}$ to be the *transfer matrix* for RWs of length n where ${}_nT_{ij}$ by definition equals the number of RSs w on $r(n, 1)$ such that

$$w|_{r(n,0)} \in h_n^{-1}(i) \quad \text{and} \quad w|_{r((0,1),(n,1))} \in h_n^{-1}(j).$$

Here $w|_B$ means the restriction of w to a subset B of its domain. The following recursive procedure yields ${}_nT$ explicitly. To start we construct all RSs on $r(1, 1)$ and $r(2, 1)$ and get

$${}_1T = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \quad \text{and} \quad {}_2T = \begin{pmatrix} 2 & 1 & 1 & 0 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 0 & 1 & 1 & 2 \end{pmatrix}.$$

Now assume ${}_nT$ is known. We split it in four submatrices of equal size ${}_n^1T, \ldots, {}_n^4T$ in the following way

$$ {}_nT = \left(\begin{array}{c|c} {}_n^1T & {}_n^2T \\ \hline {}_n^3T & {}_n^4T \end{array}\right) $$

then

$$ {}_{n+1}T = \left(\begin{array}{cc|cc} \multicolumn{2}{c|}{\multirow{2}{*}{${}_nT$}} & {}_n^2T & 0 \\ & & {}_n^3T & {}_n^2T \\ \hline {}_n^3T & {}_n^2T & \multicolumn{2}{c}{\multirow{2}{*}{${}_nT$}} \\ 0 & {}_n^3T & & \end{array}\right). $$

We indicate how to prove this. Let $x \in \mathscr{X}(n)$ be given. Define

$$ x^{n+1}(k) = \begin{cases} x(k) & k = 0, \ldots, n \\ x(n)+1 & k = n+1, \end{cases} $$

$$ x_{n+1}(k) = \begin{cases} x(k) & k = 0, \ldots, n \\ x(n)-1 & k = n+1. \end{cases} $$

Then $h_{n+1}(x^{n+1}) = h_n(x)$ and $h_{n+1}(x_{n+1}) = h_n(x) + 2^n$. Thus the definition of the submatrix ${}_{n+1}T = ({}_{n+1}T_{ij})_{1 \leqq i, j \leqq 2^n}$ involves only $x \in \mathscr{X}(n+1)$ with the property $x(n+1) - x(n) = 1$. In other words for such a pair (i, j) the RWs $h_{n+1}^{-1}(i)$ and $h_{n+1}^{-1}(j)$ are given already by their values on $\{0, \ldots, n\}$. This implies ${}_{n+1}T = {}_nT$. Similar arguments apply to the remaining submatrices.

As the formula for ${}_{n+1}T$ shows the sequence of matrices $({}_nT)_{n \geqq 1}$ has the property that if $1 \leqq i, j \leqq 2^n$ then ${}_{n+k}T_{ij} = {}_nT_{ij}$ for all $k \geqq 0$. Therefore instead of considering the sequence $({}_nT)_{n \geqq 1}$ we can consider the matrix T having rows and columns of unbounded length and being defined by $T_{ij} = {}_nT_{ij}$ where n has to be sufficiently large, i.e. such that $i, j \leqq 2^n$. Writing only T it is clear from the domain of the running indices i, j which submatrix of T, i.e. which ${}_nT$ is actually meant.

2. Combinatorial Results

By the very definition of the transfer matrix the total number of RSs on $r(n, m)$ is given by the following formula

$$ |\mathscr{S}(n, m)| = \sum_{k, i = 1}^{2^n} T_{ki}^{(m)} $$

where $T^{(m)}$ stands for the m-th power of the $2^n \times 2^n$-matrix T. More precisely $T_{ki}^{(m)}$ gives the number of RSs w on $r(n, m)$ such that $w|_{r(n, 0)} \in h_n^{-1}(k)$ and $w|_{r((0, m), (n, m))} \in h_n^{-1}(i)$. Since this formula involves the m-th power of a possibly very large matrix we mention that for small m there are easier ways to get this number. For example

$$|\mathscr{S}(n,2)| = 4|\mathscr{S}(n-1,2)| + 2q_{n-1}$$

where q_n satisfies $q_n = |\mathscr{S}(n-1,2)| + q_{n-1}$ and the initial values are given by $|\mathscr{S}(1,2) = 18$ and $q_1 = 5$. By symmetry the number of RSs on $r(n,m)$ equals the number of RSs on $r(m,n)$. Thus the interpretation as cardinality of a set of RSs leads to the following combinatorial identity

$$\sum_{i,j=1}^{2^n} T_{ij}^{(m)} = \sum_{i,j=1}^{2^m} T_{ij}^{(n)}.$$

At this point let us quote Lieb's result (see [5]) which is essentially the computation of

$$\lim_{n,m\to\infty} |\mathscr{S}(n,m)|^{1/nm}.$$

Lieb assumes periodicity of the ice configurations in the vertical and horizontal direction. Horizontal periodicity simplifies the transfer matrix. Using a slightly different counting function than h_n Lieb's matrix for the $n \times m$-configurations can be written as a diagonal block matrix. There are $n+1$ blocks of size $\binom{n}{k} \times \binom{n}{k}$ for $k = 0, \ldots, n$ arranged in this order and each block has 2's on the diagonal and 1's elsewhere. Since there are no physical reasons to justify the periodicity assumption it would be of interest to compute the limit using our T.

Now we shall determine the number of RSs on $r(n,1)$ given a fixed value at the endpoint. Define

$$\mathscr{S}(n,1;k) = \{w \in \mathscr{S}(n,1) \mid w(n,1) = k\}.$$

Of course $\mathscr{S}(n,1;k) \neq \emptyset$ only if $k = n+1-2l$ $(0 \leqq l \leqq n+1)$. Remember that if k_0, k_1, k_2 are nonnegative integers such that $k_0 + k_1 + k_2 = n$ then $\binom{n}{k_0, k_1, k_2}$ denotes the multinomial coefficient. For every $n \geqq 1$ and $0 \leqq i \leqq 2n$ we introduce numbers $D_{n,i}$ setting

$$D_{n,i} = \sum_{k_1 + 2k_2 = i} \binom{n}{k_0, k_1, k_2}.$$

More precisely the sum ranges over all $(k_0, k_1, k_2) \in Z_+^3$ satisfying $k_0 + k_1 + k_2 = n$ and $k_1 + 2k_2 = i$. For completeness put $D_{n,i} = 0$ if i is not as above.

(2.1) **Theorem.** *For* $n \geqq 1$ *and* $0 \leqq i \leqq n+1$

$$|\mathscr{S}(n,1; n+1-2i)| = D_{n,2i} + 2D_{n,2i-1} + D_{n,2i-2}.$$

Proof. We shall use during this proof one of the alternative ways to describe RSs. Since only RSs on $r(n,1)$ are involved here we shall deal only with this special case. It will be clear how the characterization runs for RSs on arbitrary $r(n,m)$. Given $w \in \mathscr{S}(n,1)$ we put

$$x_k(w) = w(k,0) - w(k-1,0) \qquad (1 \leqq k \leqq n),$$
$$y_k(w) = w(k,1) - w(k,0) \qquad (0 \leqq k \leqq n).$$

Then the sequence $(y_0(w), x_1(w), y_1(w), \ldots, x_n(w), y_n(w))$ consisting of $+1$'s and -1's is an identification or a code for w. If we order the possible states (x_k, y_k) in the following way $(1, 1)$, $(-1, 1)$, $(1, -1)$, $(-1, -1)$ then the corresponding transfer matrix is

$$\tilde{T} = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}.$$

More precisely $\tilde{T}((x_k, y_k); (x_{k+1}, y_{k+1})) = 1$ if there is a RS w such that

$$(x_k(w), y_k(w)) = (x_k, y_k) \quad \text{and} \quad (x_{k+1}(w), y_{k+1}(w)) = (x_{k+1}, y_{k+1}).$$

Otherwise $\tilde{T}((x_k, y_k); (x_{k+1}, y_{k+1})) = 0$. The generating function for the number of RSs with $w(n, 1) = x_1 + \cdots + x_n + y_n$ given can then be written as follows

$$g(z) = \sum_{\substack{y_0, \ldots, y_n \\ x_1, \ldots, x_n}} \tilde{T}((1, y_0); (x_1, y_1)) \ldots \tilde{T}((x_{n-1}, y_{n-1}); (x_n, y_n)) \cdot z^{x_1 + \cdots + x_n + y_n}.$$

So if we put $\tilde{T}_z((x_{k-1}, y_{k-1}); (x_k, y_k)) = \tilde{T}((x_{k-1}, y_{k-1}); (x_k, y_k)) \cdot z^{x_k}$, i.e.

$$\tilde{T}_z = \begin{pmatrix} z & z^{-1} & z & 0 \\ z & z^{-1} & z & 0 \\ 0 & z^{-1} & z & z^{-1} \\ 0 & z^{-1} & z & z^{-1} \end{pmatrix}$$

and $a = (1, 0, 1, 0)$, $b = (z, z, z^{-1}, z^{-1})$ then $g(z) = a\, \tilde{T}_z^{(n)} b^T$. The eigenvalues of $\tilde{T}_z$ are $0, 0, z + 1 + z^{-1}$ and $z - 1 + z^{-1}$. As corresponding eigenvectors arranged as columns of a matrix we find

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ -z^2 - 1 & -2z^2 & 1 & 1 \\ z^{-2} & 1 & z^{-1} & -z^{-1} \\ z^2 & z^2 & z^{-1} & -z^{-1} \end{pmatrix}.$$

Now $g(z)$ can be got explicitly via the spectral representation of $\tilde{T}_z^{(n)}$.

$$g(z) = 2^{-1}(z + 1 + z^{-1})^n (z + 2 + z^{-1}) + 2^{-1}(z - 1 + z^{-1})^n (z - 2 + z^{-1}).$$

Expanding the n-th power by the multinomial theorem our result follows. ⌟

Basicly the method applies to solve the same problem for $\mathcal{S}(n, m; k)$ where $m > 1$. But to get $g(z)$ explicitly becomes difficult for larger matrices.

3. Probabilistic Results

In this section we interpret random sheets probabilistically. Our main goal is to describe the asymptotic behavior of a random sheet on $r(k, m)$ when $k \to \infty$.

Let m_0 be an arbitrary positive integer which will be fixed throughout this section. Put $\mathcal{Y}(m_0) = \{y = (y(0), y(1), \ldots, y(m_0)) \mid y(i) \in \{1, -1\}\}$. We consider the product space $\Omega^k = \mathcal{X}_0(m_0) \times \prod_{j=1}^{k} \mathcal{Y}_j(m_0)$ where $\mathcal{Y}_j(m_0) = \mathcal{Y}(m_0)$ for each j. A subset

Ω_a^k of Ω^k will be used. $\omega = (x, y_1, \ldots, y_k) \in \Omega_a^k$ if there is a RS $w \in \mathcal{S}(k, m_0)$ such that

$$(w(0,0), \ldots, w(0, m_0)) = x$$

and

$$(w(j,0) - w(j-1,0), \ldots, w(j, m_0) - w(j-1, m_0)) = y_j \quad \text{for } 1 \leqq j \leqq k.$$

The family of random sheets $(\mathcal{S}(k, m_0))_{k \geqq 1}$ defines now in a natural way a triangular array of vector-valued random variables $(X_{k,j})_{1 \leqq k, 0 \leqq j \leqq k}$. For every k $X_{k,0}$ takes values in $\mathcal{X}_0(m_0)$ whereas $X_{k,j}$ for $j \geqq 1$ takes values in $\mathcal{Y}(m_0)$. The distribution we are interested in is the uniform distribution on each $\mathcal{S}(k, m_0)$ which reads explicitly as follows

$$P[X_{k,0} = x, X_{k,1} = y_1, \ldots, X_{k,k} = y_k] = \begin{cases} |\mathcal{S}(k, m_0)|^{-1} & \text{if } (x, y_1, \ldots, y_k) \in \Omega_a^k \\ 0 & \text{otherwise.} \end{cases}$$

We denote by $\mathfrak{B}_{i,j}^k$ the σ-algebra generated by $X_{k,l}$ $(0 \leqq i \leqq l \leqq j \leqq k)$. Remember that an array is called *φ-mixing* if for events $E_1 \in \mathfrak{B}_{0,l}^k$ satisfying $P(E_1) > 0$ and $E_2 \in \mathfrak{B}_{l+n,k}^k$ we have

$$|P(E_2 | E_1) - P(E_2)| \leqq \varphi(n)$$

where φ is a real-valued function satisfying $\lim_{n \to \infty} \varphi(n) = 0$.

(3.1) **Proposition.** *The triangular array of vector-valued random variables $(X_{k,j})_{1 \leqq k, 0 \leqq j \leqq k}$ is φ-mixing.*

Proof. We start with sets $E_1 = [X_{k,0} = x, \ldots, X_{k,l} = y_l]$ and $E_2 = [X_{k,l+n} = y_{l+n}, \ldots, X_{k,k} = y_k]$. Of course we can assume both $P(E_i) > 0$ since otherwise there is nothing to prove. In order to compute the values $P(E_2 | E_1)$ and $P(E_2)$ explicitly we have to count certain sets of RSs. The transfer matrix involved here is ${}_{m_0}T$ which we can write T. Furthermore we shall use the abbreviations

$$T_i^p = \sum_j T_{ij}^{(p)} = \sum_j T_{ji}^{(p)}$$

and

$$T_0^p = \sum_{i,j} T_{ij}^{(p)}.$$

Put $m = k - (l+n)$ then obviously

$$P(E_1) = (T_0^k)^{-1} T_i^{n+m}$$

where the index i is determined by $(x, y_1, \ldots, y_l)$. For any positive integer p with dual representation $p = \sum_{i \geqq 0} z_i 2^i$ we shall write $d(p) = (z_0, z_1, \ldots)$. Now consider the vector $y_k = (y_k(0), y_k(1), \ldots, y_k(m_0))$ occurring above. Define $s(y_k) = \{0 \leqq t \leqq m_0 - 1 \mid y_k(t+1) \neq y_k(t)\}$ which we shall write $s(y_k) = \{t_1, \ldots, t_q\}$ such that

$t_1 < t_2 < \dots < t_q$. Then if $s(y_k)$ is empty we put $g_k = g(y_k) = \{1, \dots, 2^{m_0}\}$. If $s(y_k)$ is nonempty and $y_k(t_1) = 1$ say we put

$$g_k = g(y_k) = \{1 \leqq p+1 \leqq 2^{m_0} \mid d(p)_{t_i} = 0 \text{ for } i \text{ odd, } d(p)_{t_i} = 1 \text{ for } i \text{ even } (1 \leqq i \leqq q)\}.$$

In case $y_k(t_1) = -1$ the words odd and even are exchanged. Now it is easy to see that

$$P[X_{k,k} = y_k] = (T_0^k)^{-1} \Big(\sum_{j \in g_k} T_j^{k-1} \Big).$$

More generally we get

$$P(E_2) = (T_0^k)^{-1} \Big(\sum_{j \in g} T_j^{l+n-1} \Big)$$

with the changement that $s(y_{l+n}, \dots, y_k) = \{s_1, \dots, s_r\}$ is the union of the $s(y_i)$ $(l+n \leqq i \leqq k)$. Note that the 0's and 1's at the coordinates s_i in the definition of $g = g(y_{l+n}, \dots, y_k)$ are no longer necessarily alternating as they are in the definition of g_k. The same reasoning gives

$$P(E_1 \cap E_2) = (T_0^k)^{-1} \Big(\sum_{j \in g} T_{ij}^{(n-1)} \Big).$$

We conclude

$$|P(E_2 | E_1) - P(E_2)| = \Big| (T_i^{n+m})^{-1} \Big(\sum_{j \in g} T_{ij}^{(n-1)} \Big) - (T_0^k)^{-1} \Big(\sum_{j \in g} T_j^{l+n-1} \Big) \Big|.$$

$T^{(p)}$ is diagonalizable with spectral representation

$$T^{(p)} = \lambda_1^p B_1 + \dots + \lambda_{2m_0}^p B_{2m_0}$$

where we can assume $\lambda_1 \geqq |\lambda_2| \geqq \dots \geqq |\lambda_{2m_0}|$. Since T is aperiodic we know $\lambda_1 > |\lambda_2|$. This implies

$$\lim_{p \to \infty} T_{ij}^{(p)} \lambda_1^{-p} = (B_1)_{ij}$$

where the rate of convergence is exponential. We determine constants L, L_1 such that uniformly in the sets $g = g(y_{l+n}, \dots, y_k)$ defined above

$$\Big| \sum_{j \in g} T_{ij}^{(p)} - \sum_{j \in g} \lambda_1^p (B_1)_{ij} \Big| \leqq |\lambda_2|^p L$$

and

$$\Big| \sum_{j \in g} T_j^p - \sum_{j \in g} \lambda_1^p B_1(j) \Big| \leqq |\lambda_2|^p L_1$$

with $B_1(j) = \sum_i (B_1)_{ij}$ and – used below – $B_1 = \sum_{i,j} (B_1)_{ij}$. Now write

$$\begin{aligned}|P(E_2|E_1)-P(E_2)| &\leqq |(T_i^{n+m})^{-1}(\sum_{j\in g} T_{ij}^{(n-1)})-(\lambda_1^{n+m}B_1(i))^{-1}(\sum_{j\in g}\lambda_1^{n-1}(B_1)_{ij})| \\ &\quad +|(T_0^k)^{-1}(\sum_{j\in g} T_j^{l+n-1})-(\lambda_1^k B_1)^{-1}(\sum_{j\in g}\lambda_1^{l+n-1}B_1(j))| \\ &\quad +|(\lambda_1^{m+1}B_1(i))^{-1}(\sum_{j\in g}(B_1)_{ij})-(\lambda_1^{m+1}B_1)^{-1}(\sum_{j\in g}B_1(j))| \\ &= e_1+e_2+e_3.\end{aligned}$$

With the approximations made above an upper bound for e_1 and e_2 can be given using the following inequality which holds for real numbers $a>0$, $x>0$ and small $c>0$, $d>0$

$$|(a\pm c)(x\pm d)^{-1}-ax^{-1}|\leqq 2(a+c)\,dx^{-2}+cx^{-1}.$$

The result for e_1 is

$$\begin{aligned}e_1 &\leqq \lambda_1^{-m-1}[2(1+(|\lambda_2|/\lambda_1)^{n-1}LB_1(i)^{-1})(|\lambda_2|/\lambda_1)^{n+m}L_1B_1(i)^{-1} \\ &\quad +(|\lambda_2|/\lambda_1)^{n-1}LB_1(i)^{-1}] \\ &= O(|\lambda_2|^{n-1}\lambda_1^{-n-m}).\end{aligned}$$

Analogously $e_2=O(\lambda_1^{-k}|\lambda_2|^{l+n-1})$. The transformation of T into a diagonal matrix can be done by an orthogonal transformation which implies $B_1(i)\,B_1(j)=(B_1)_{ij}B_1$. Consequently $e_3=0$.

To finish the proof we have to consider sets $F_1=\sum_{i=1}^{s}F_1^i$ and $F_2=\sum_{j=1}^{t}F_2^j$ where each F_1^i is assumed to be of the same type as E_1 above and F_2^j being as E_2. A straightforward computation shows that if the bounds hold for each F_1^i then the same bounds hold for F_1. We have only to use the fact that given q pairs $a_i, b_i>0$ and $c>0$ such that $|a_ib_i^{-1}-c|<\varepsilon$ then $|(\sum_i r_ia_i)(\sum_i r_ib_i)^{-1}-c|<\varepsilon$ for any positive integers r_i $(1\leqq i\leqq q)$. Concerning F_2 it is clear that the bounds got for F_2^j add up t times. But the number of sets F_2^j such that $P(F_2^j)>0$, i.e. the upper bound for t, is of the order $O(\lambda_1^{m+1})$. Thus

$$|P(F_2|F_1)-P(F_2)|\leqq \varphi(n)=O((|\lambda_2|/\lambda_1)^{n-1}). \quad \lrcorner$$

We consider now

$$S_k=X_{k,0}+X_{k,1}+\cdots+X_{k,k}.$$

The variables $X_{k,j}$ $(1\leqq j\leqq k)$ stand for successive increments of random sheets $w\in\mathscr{S}(k,m_0)$ and S_k describes the values at $(k,0),\ldots,(k,m_0)$. The asymptotic behavior of these values is therefore contained in the following theorem.

(3.2) **Theorem.** *Let $(X_{k,j})_{1\leqq k,0\leqq j\leqq k}$ be the triangular array of vector-valued random variables defined before then $S_k/\sqrt{k}$ has asymptotically a normal distribution centered at the origin.*

Proof. Each $X_{k,j}$ is centered and the array is φ-mixing with $\sum_{n=1}^{\infty}\varphi(n)^{1/2}<\infty$.

Therefore the multivariate version of the central limit theorem for a φ-mixing array applies ([1, 6]). $\lrcorner$

Finally let us compute explicitly the covariances in the case $m_0=1$. We have $|\mathscr{S}(k,1)|=2\cdot 3^k$ and one shows easily that – omitting the k in the index – $(X_j)_{j\geqq 1}$ is a stationary process. The structure underlying this process is actually Markovian which would appear explicitly if we had used the description introduced in Section 2. Writing $E[X]$ for expectation we have to compute for $0\leqq i,j\leqq 1$

$$\sigma_{ij}=E[X_1(i)\,X_1(j)]+\sum_{k=2}^{\infty} E[X_1(i)\,X_k(j)]+\sum_{k=2}^{\infty} E[X_k(i)\,X_1(j)].$$

Of course $E[X_1^2(0)]=E[X_1^2(1)]=1$. Furthermore we find

$$E[X_1(0)\,X_k(0)]=E[X_1(1)\,X_k(1)]=-3^{-k} \qquad (k\geqq 2)$$

and

$$E[X_1(0)\,X_k(1)]=E[X_1(1)\,X_k(0)]=3^{-k} \qquad (k\geqq 2).$$

Summing up the result is $\sigma_{ij}=2/3$ $(0\leqq i,j\leqq 1)$.

There is a more elementary way, i.e. without using the central limit theorem, to get this asymptotic variance. The squares of $\tilde{S}_n(1)=\sum_{k=0}^{n-1} X_k(1)$ satisfy the following difference equation

$$E[\tilde{S}_{n+1}^2(1)]-E[\tilde{S}_n^2(1)]=E[\tilde{S}_n^2(1)]-E[\tilde{S}_{n-1}^2(1)]+2\cdot 3^{-(n+1)}.$$

From this one derives

$$E[\tilde{S}_n^2(1)]=2n/3+o(n)$$

which implies

$$\lim_{n\to\infty} E[\tilde{S}_n^2(1)/n]=2/3.$$

References

1. Billingsley, P.: Convergence of probability measures. New York-London-Sidney: John Wiley & Sons 1968
2. Eberlein, E.: An invariance principle for lattices of dependent random variables. [To appear]
3. Feller, W.: An introduction to probability theory and its applications. Vol. I, 3rd edition. Vol. II, 2nd edition. New York-London-Sydney: John Wiley & Sons 1968 1971
4. Ibragimov, I.A., Linnik, U.B.: Independent and stationary sequences of random variables. Moskau: Nauka 1965 (in Russian)
5. Lieb, E.H., Wu, F.Y.: 'Two-dimensional ferroelectric models' in 'Phase transitions and critical phenomena'. Vol. 1 (edited by C. Domb and M.S. Green). London-New York: Academic Press 1972
6. Philipp, W.: The central limit problem for mixing sequences of random variables. Z. Wahrscheinlichkeitstheorie verw. Gebiete **12**, 155–171 (1969)

Received November 22, 1976; in revised form May 6, 1977