Z. Wahrscheinlichkeitstheorie verw. Gebiete 31, 351 – 356 (1975)

Random Walks and the Strong Law

M. F. Driscoll and N. A. Weiss*

§ 1. Introduction and Statement of Results

A need for the following generalization of the strong law of large numbers for strictly stationary processes arises in the study of infinite particle systems.

Let $X_1, X_2 \ldots$ be a sequence of i.i.d. integer-valued random variables and let P_n be the n-th-step transition function of the random walk $S_n = X_1 + \cdots + X_n$. The required result is that, for any finite nonempty set B,

$$\lim_{n\to\infty} \sum_x A(x)\, P_n(x, B) = \Lambda\, |B| \qquad \text{a.s.} \tag{1.1}$$

for any G-process $A(\cdot)$ with parameter variable Λ. (A process $A(\cdot)$ is called a G-process with parameter variable Λ if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{x=1}^{n} A(x) = \lim_{n\to\infty} \frac{1}{n} \sum_{x=0}^{-n+1} A(x) = \Lambda$$

with probability one.)

In this paper we establish this result for a large class of random walks $\{S_n\}$. The interpretation to infinite particle systems will be given in § 3.

Theorem 1. *If* $EX_1 = 0$ *and* $EX_1^4 < \infty$, *then* (1.1) *holds for any G-process with parameter variable* Λ *and for any finite nonempty set B.*

The key lemma in the proof of Theorem 1 is obtained by applying refined versions of the central and local central limit theorems.

Theorem 2. *Let* $X_1, X_2, \ldots$ *be as in Theorem* 1. *Then we have*

$$\limsup_{n\to\infty} \sum_x |x|\, |P_n(0, x) - P_n(0, x+1)| < \infty. \tag{1.2}$$

Remark. In a study of infinite particle systems, Stone [5], discussed a condition similar to (1.2); namely

$$\lim_{n\to\infty} \sum_x |P_n(0, x) - P_n(0, x+1)| = 0. \tag{1.3}$$

He proved that (1.3) holds for any strongly aperiodic random walk. However, it is not true that (1.2) holds for arbitrary strongly aperiodic random walk. To see

* This research was supported by Arizona State University Research Grants 7805-817 and 7805-890.
AMS Subject Classifications: 60F15 and 60J15.

this take $X_1=0$ or 1 with probability 1/2. Then computations show that

$$\sum_x |x|\,|P_{2n}(0,x)-P_{2n}(0,x+1)|$$
$$=4^{-n}\left\{\sum_{k=1}^{n-1} k\left[\binom{2n}{k+1}-\binom{2n}{k}\right]+\sum_{k=n}^{2n-1} k\left[\binom{2n}{k}-\binom{2n}{k+1}\right]+2n+1\right\}$$
$$=4^{-n}\left[(2n-1)\cdot\binom{2n}{n}+2\right]\sim 2\sqrt{n/\pi}.$$

Consequently, (1.2) does not hold for this random walk.

§ 2. Proofs

Throughout the discussion the following assumptions will be made. $X_1, X_2, \ldots$ is a sequence of integer-valued i.i.d. random variables with mean zero, variance $\sigma^2 \neq 0$, and finite fourth moment.

Let α_k and β_k denote, respectively, the k-th and absolute k-th moments of X_1. Also set $P_n(0,x)=P(X_1+\cdots+X_n=x)$, $f_n(x)=P_n(0,\sigma\sqrt{n}x)$, and $F_n(x)=\sum_{t\leq x} f_n(t)$. ϕ denotes the standard normal density function and Φ the distribution function of ϕ.

Finally, for convenience we let $b_n(x)=\phi(x/\sqrt{n}\sigma)$ and $B_n(x)=\Phi(x/\sqrt{n}\sigma)$.

The proof of Theorem 2 is accomplished in several steps. The first step is to decompose (1.2) into components about which some information can be obtained.

Computations show that

$$\sum_x |x|\,|b_n(x)-b_n(x-1)|$$
$$\leqq \sum_x |x|\,|b_n(x)-b_n(x+1)|+2\sum_x b_n(x).$$

and that

$$\sum_{|x|\geqq m_n} |x|\,|P_n(0,x)-b_n(x)|$$
$$\leqq \sum_{|x|\geqq m_n} |x|\{|P_n(0,x)-[B_n(x)-B_n(x-1)]|+|b_n(x)-b_n(x-1)|\}.$$

From these facts the following result is obtained:

Lemma 1. *Let $\{m_n\}$ be any sequence of positive real numbers. Then*

$$\begin{aligned}\sum_x |x|\,|P_n(0,x)-P_n(0,x+1)| &= O\Big(1+\sum_{|x|\leqq m_n} |x|\,|P_n(0,x)-b_n(x)| \\ &\quad+\sum_{|x|\geqq m_n} |x|\,|P_n(0,x)-[B_n(x)-B_n(x-1)]| \\ &\quad+\sum_x |x|\,|b_n(x)-b_n(x+1)|+\sum_x b_n(x)\Big).\end{aligned} \tag{2.1}$$

The next problem is to control each of the terms on the right hand side of (2.1). The following fact will be required. Suppose $\{a(x): x\in Z\}$ is a sequence of nonnegative real numbers such that (i) $\sum |x|\, a(x) < \infty$; (ii) $a(-x) = a(x)$, $x \in Z$; and (iii) $a(x+1) \leqq a(x)$, for $x \geqq 0$. Then

$$\sum_x |x|\, |a(x) - a(x+1)| = \sum_x a(x) < \infty .$$

With this fact in mind an estimate is obtained for the fourth term on the right of (2.1).

Lemma 2. *With the notation as before*

$$\sum_x |x|\, |\phi(x/\sigma\sqrt{n}) - \phi((x-1)/\sigma\sqrt{n})| = O(n^{1/2}). \tag{2.2}$$

Proof. Clearly $b_n(x)$ satisfies (ii) and (iii) of the above result, for each n. To verify (i) just note that

$$\sum_x |x|\, b_n(x) = 2(\sigma\sqrt{n})^{-1} \sum_{x=1}^{\infty} x\,\phi(x/\sigma\sqrt{n}) \leqq 1 + \beta_1^* \sigma\sqrt{n}$$

where $\beta_1^* = \int_{-\infty}^{\infty} |y|\, \phi(y)\, dy$. Consequently,

$$\sum_x |x|\, |b_n(x) - b_n(x+1)| = \sum_x b_n(x) \leqq 1 + (2\pi n \sigma^2)^{-1/2}$$

and (2.2) follows easily from this.

We now turn to the problem of bounding the second term on the right hand side of (2.1). It follows from a result of Esseen [1] that if $\beta_k < \infty$ for k an integer $\geqq 3$ then

$$\begin{aligned}\sum_{|x| \leqq m_n} |x|\, |P_n(0,x) - b_n(x)| &\leqq \sum_{v=1}^{k-2} \sum_{j=1}^{v} \frac{|C_{vj}|\, n^{-(v+1)/2}}{\sigma} \sum_{|x| \leqq m_n} |x\,\phi^{(v+2j)}(x/\sigma\sqrt{n})| \\ &\quad + o(n^{(1-k)/2}) \sum_{|x| \leqq m_n} |x|\end{aligned} \tag{2.3}$$

where the coefficients C_{vj} are constants determined by the distribution of X_1. Consequently, if $m_n \to \infty$ and $m_n = O(n^{(k-1)/4})$ then it follows that

$$\begin{aligned}\sum_{|x| \leqq m_n} |x|\, |P_n(0,x) - b_n(x)| \leqq \sum_{v=1}^{k-2} \sum_{j=1}^{v} \frac{|C_{vj}|\, n^{-(v+1)/2}}{\sigma} \sum_{|x| \leqq m_n} |x\,\phi^{(v+2j)}(x/\sigma\sqrt{n})| + o(1).\end{aligned} \tag{2.4}$$

Thus to bound the seond term on the right hand side of (2.1) it is only necessary to demonstrate that the first term on the right hand side of (2.4) is bounded.

To accomplish this first note that because of the exponential nature of ϕ, for each $m = 0, 1, 2, \ldots$ we have $\phi^{(m)}(y) = p_m(y)\,\phi(y)$ where p_m is a polynomial of degree m. With this in mind the following lemma is obtained.

Lemma 3. *For* $\sigma > 0$ *and* m *fixed*

$$\sum_x |x\,\phi^{(m)}(x/\sigma\sqrt{n})| = O(n).$$

Proof. Choose constants $b_0, b_1, \ldots, b_m$ such that $p_m(y)=\sum_{k=0}^m b_k y^k$. Setting $d_m=\max\{|b_k|: k=0, 1, \ldots, m\}$ we then obtain

$$|\phi^{(m)}(y)| \leqq \phi(y) \max\left\{(m+1)\, d_m, d_m \sum_{k=0}^{m} |y|^k\right\}$$
$$\leqq \phi(y)\, C_m(1+|y|^m)$$

where $C_m=(m+1)d_m$. It follows that

$$\frac{1}{n}\sum_x |x\,\phi^{(m)}(x/\sigma\sqrt{n})| \leqq \frac{C_m}{n}\sum_x |x|\,\phi(x/\sigma\sqrt{n}) + \frac{C_m}{\sigma^m n^{1+m/2}}\sum_x |x|^{m+1}\phi(x/\sigma\sqrt{n}).$$

Now, for $k=0, 1, 2, \ldots$, set $a_k=\max\left\{\binom{k}{j}: 0\leqq j\leqq k\right\}$. Then,

$$\sum_x |x|^k \phi(x/\sigma\sqrt{n}) \leqq \int_0^\infty (y+1)^k \phi(y/\sigma\sqrt{n})\, dy$$
$$\leqq 2(k+1)\, a_k \int_0^\infty (1+y^k)\, \phi(y/\sigma\sqrt{n})\, dy$$
$$=(k+1)\, a_k \sigma\sqrt{n}\,[1+\sigma^k n^{k/2} \beta_k^*]$$

where β_k^* is the k-th absolute moment of the standard normal distribution.

By using this last fact the following result is now obtained.

$$\frac{1}{n}\sum_x |x\,\phi^{(m)}(x/\sigma\sqrt{n})|$$
$$\leqq 2\sigma\, C_m[n^{-1/2}+\sigma\beta_1^*] + \frac{(m+2)\, a_{m+1}\, C_m}{\sigma^{m-1}}[n^{-(m+1)/2}+\sigma^{m+1}\beta_{m+1}^*].$$

For each m, the quantity on the right remains bounded as $n\to\infty$ and hence the lemma is proved.

Combining Lemma 3 and (2.4) it is easy to see that the following lemma is true.

Lemma 4. *Let k be an integer with $k\geqq 3$ and suppose $\{m_n\}$ is a sequence of positive real numbers such that $m_n\to\infty$ and $m_n=O(n^{(k-1)/4})$. If $\beta_k<\infty$, then*

$$\limsup_{n\to\infty} \sum_{|x|\leqq m_n} |x|\, |P_n(0, x)-b_n(x)| < \infty.$$

It remains to bound the third term on the right hand side of (2.1). Again a theorem due to Esseen [1] will be applied: If $\beta_k<\infty$ for k an integer $\geqq 3$ then

$$|F_n(y)-\Phi(y)| \leqq \frac{C(\delta, \beta)}{1+|y|^k}\, n^{-(k-2)/2} \tag{2.5}$$

uniformly for $|y|\geqq\sqrt{(1+\delta)(k-2)\log n}$. Here $C(\delta, \beta)$ is a finite positive constant depending only on δ ($0<\delta<1$, fixed) and the moments $\beta_2, \ldots, \beta_k$.

In view of this result the sequence $\{m_n\}$ can now be specified. Set m_n equal to the smallest integer greater than

$$\sigma\sqrt{(1+\delta)(k-2)\, n\log n} \tag{2.6}$$

where δ is any fixed number in $(0, 1)$. Notice that $m_n \to \infty$ and $m_n = o(n^{(k-1)/4})$ if $k \geqq 4$.

Because of (2.5) we now obtain for large n and $|x| > m_n$

$$|P_n(0, x) - [B_n(x) - B_n(x-1)]|$$
$$\leqq C(\delta, \beta)\, n^{-(k-2)/2} \left[\frac{1}{\left|\frac{x}{\sigma\sqrt{n}}\right|^k} + \frac{1}{\left|\frac{x-1}{\sigma\sqrt{n}}\right|^k}\right] \leqq 3\, C(\delta, \beta)\, \sigma^k\, n |x|^{-k}.$$

In view of this last inequality it is clear that for sufficiently large n

$$\sum_{|x| > m_n} |x|\, |P_n(0, x) - [B_n(x) - B_n(x-1)]| \leqq 3\, C(\delta, \beta)\, \sigma^k n \sum_{|x| > m_n} |x|^{-k+1}$$
$$= O\left(n^{(4-k)/2} (\log n)^{(2-k)/2}\right).$$

From this result the following lemma is deduced.

Lemma 5. *If $\beta_4 < \infty$ and m_n denotes the smallest integer greater than $\sqrt{3\sigma^2 n \log n}$ then*

$$\sum_{|x| > m_n} |x|\, |P_n(0, x) - [B_n(x) - B_n(x-1)] = O\left(\frac{1}{\log n}\right)$$

and so in particular the term on the left tends to zero as $n \to \infty$.

Combining Lemmas 1, 3, 5, and 6 the validity of Theorem 2 is now apparent.

Now that Theorem 2 has been proved we proceed to verify Theorem 1. To prove (1.1) it suffices to show that

$$\lim_{n \to \infty} \sum_x A(x)\, P_n(0, x) = A \tag{2.7}$$

with probability one.

To begin, let $\{\beta(x)\}$ be Césaro summable to λ and set

$$\sigma(x) = \begin{cases} -\dfrac{1}{x} \displaystyle\sum_{y=x+1}^{0} \beta(y); & x < 0 \\ 1; & x = 0 \\ \dfrac{1}{x} \displaystyle\sum_{y=1}^{x} \beta(y); & x > 0. \end{cases}$$

Then, $x\sigma(x) - (x-1)\,\sigma(x-1) = \beta(x)$ for all x and consequently,

$$\begin{aligned} \sum \beta(x)\, P_n(0, x) = \sum x\sigma(x)\, [P_n(0, x) - P_n(0, x+1)] \\ + \sum [\sigma(x-1) - \sigma(x)]\, P_n(0, x). \end{aligned} \tag{2.8}$$

Since $|\sigma(x-1) - \sigma(x)| \to 0$ as $|x| \to \infty$ and $P_n(0, x) \to 0$ as $n \to \infty$ the second sum on the right hand side of (2.8) tends to zero as $n \to \infty$. It remains to prove that the first sum on the right hand side of (2.8) tends to λ as $n \to \infty$.

For convenience let $\gamma_n(x) = x[P_n(0, x) - P_n(0, x+1)]$. Because $E|X_1| < \infty$ it follows that $\sum_x \gamma_n(x) = 1$ for all n. Since, for all x, y, $P_n(x, y) \to 0$ as $n \to \infty$ for random walks it is obvious that $\gamma_n(x) \to 0$ as $n \to \infty$ for each x. Finally, because of Theorem 2, the sequence $\{\sum |\gamma_n(x)|\}$ is bounded. Consequently, from a well-known summability theorem (see, e.g. Hardy [3]) we conclude that $\sum \sigma(x)\gamma_n(x) \to \lambda$ an $n \to \infty$. The proof of Theorem 1 now follows.

§ 3. Interpretation to Infinite Particle Systems

Suppose that initially particles are distributed throughout Z according to the point process $A_0(\cdot)$. Subsequently the particles are translated independently according to stochastic processes isomorphic to some fixed stochastic process $\{Y_n\}$. Let v_n be the distribution of Y_n. If $A_n(\cdot)$ represents the distribution of particles at time n then

$$E(A_n(B)|A_0) = A_0 * v_n(B). \tag{3.1}$$

This quantity is important for results concerning convergence in distribution of A_n to a Poisson process. See, for example, Stone [5] and Goldman [2]. The asymptotic behavior of $A_0 * v_n$ is also needed for obtaining important generalizations of results in Weiss [6] and [7].

Now, assume A_0 is any G-process with parameter variable Λ and that $\{Y_n\}$ is a random walk with mean zero and finite fourth moment. Then according to Theorem 1,

$$\lim_{n\to\infty} E(A_n(B)|A_0) = \Lambda |B| \tag{3.2}$$

with probability one. Since (see Spitzer [4]) $\lim_{n\to\infty} \sup_x P_n(x, B) = 0$, it follows by a result of Goldman [2] that the process A_n is asymptotically mixed Poisson with parameter variable Λ.

References

1. Esseen, C. G.: Fourier analysis of distribution functions. A mathematical study of the Laplace-Gaussian law. Acta Math. **77**, 1–25 (1945)
2. Goldman, J.: Stochastic point processes: limit theorems. Ann. Math. Statist. **38**, 771–779 (1967)
3. Hardy, G. H.: Divergent Series. London: Oxford University Press 1948
4. Spitzer, F.: Principles of Random Walk. Princeton: Van Nostrand 1964
5. Stone, C. J.: On a theorem by Dobrushin. Ann. Math. Statist. **39**, 1391–1401 (1968)
6. Weiss, N. A.: Limit theorems for infinite particle systems. Z. Wahrscheinlichkeitstheorie verw. Gebiete **20**, 87–101 (1971)
7. Weiss, N. A.: The occupation time of a set by countably many recurrent random walks. Ann. Math. Statist. **43**, 293–302 (1972)

M. F. Driscoll and N. A. Weiss
Department of Mathematics
Arizona State University
Tempe, Arizona 85281, USA

(Received April 18, 1973)