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On a.s. and r-Mean Convergence of Random Processes with an Application to First Passage Times

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1. Introduction

Let $X_1, X_2, \dots$ be i.i.d. random variables, centered at expectation whenever this is finite and let S_n, $n \geqq 1$, denote their partial sums. Set $S_0 = 0$.

Suppose that $E|X|^r < \infty$, $0 < r < 2$. According to a result of Kolmogorov ($r = 1$) and Marcinkiewicz ($r \neq 1$) this is equivalent to $n^{-1/r} \cdot S_n \xrightarrow{\text{a.s.}} 0$ as $n \to \infty$, ([11], 243) and in [12] it is proved that this is also equivalent to $n^{-1/r} \cdot S_n \xrightarrow{r} 0$ as $n \to \infty$. In [2] and [3] generalizations to certain dependent sequences are obtained and [8] contains an extension to randomly indexed partial sums of i.i.d. random variables.

In Section 2 it is first proved that the first mentioned results also are equivalent to $n^{-1/r} \cdot \max\limits_{1 \leqq k \leqq n} S_k \xrightarrow{r} 0$ as $n \to \infty$.

Now, let $\{X(t);\ t \geqq 0\}$ be a separable random process with independent, stationary increments and with mean zero whenever the mean is finite. It is no restriction to assume stochastic continuity ([11], 542) and that the sample functions are continuous from the right ([5], 168). The aim of Section 2 is to establish a theorem for this class of random processes, which corresponds to the above mentioned results for i.i.d. random variables.

Let $T = T(c) = \inf\{t;\ X(t) > c \cdot a(t)\}$, where now $EX(t) = t\theta$, $0 < \theta < \infty$, $c \geqq 0$ and $a(y)$ is positive, bounded on compact subsets of $[0, \infty)$ and such that $a(y) = o(y)$ as $y \to \infty$. In Section 3 the results of Section 2 and [7] are applied in order to investigate the finiteness of the moments of T and $X(T)$ and, under further restrictions on $a(y)$, asymptotic properties when $c \to \infty$, thereby extending results obtained in [7] for processes without positive jumps. See also [1, 14] and [16], where the case $a(y) \equiv 1$ has been treated.

Throughout $Y^+ = \max\{0, Y\}$, $Y^- = -\min\{0, Y\}$ and $\|Y\|_r = (E|Y|^r)^{1/r}$, where Y is a random variable.

2.

Let $X_1, X_2, \dots$ be i.i.d. random variables centered at expectation whenever this is finite and let S_n denote their partial sums, $(S_0 = 0)$.

Lemma 2.1. *Let $0 < r < 2$. The following statements are equivalent:*

$$E|X_1|^r < \infty, \qquad (EX_1 = 0 \ \text{if}\ r \geqq 1), \tag{2.1}$$

$$n^{-1/r} \cdot S_n \xrightarrow{\text{a.s.}} 0 \qquad \text{as } n \to \infty, \tag{2.2}$$

$$E|S_n|^r = o(n), \qquad \text{i.e. } n^{-1/r} \cdot S_n \xrightarrow{r} 0 \ \text{as } n \to \infty, \tag{2.3}$$

$$E \max_{1 \leqq k \leqq n} |S_k|^r = o(n), \qquad \text{i.e. } n^{-1/r} \cdot \max_{1 \leqq k \leqq n} S_k \xrightarrow{r} 0 \ \text{as } n \to \infty. \tag{2.4}$$

If $\{\tau(c), c \geqq 0\}$ *is a non-decreasing family of integer valued stopping times such that* $E\tau(c) < \infty$ *and* $E\tau(c) \nearrow \infty$ *as* $c \to \infty$, *then* (2.1) *implies that*

$$E|S_{\tau(c)}|^r = o(E\tau(c)) \quad \text{as } c \to \infty. \tag{2.5}$$

If, moreover, $c^{-1} \cdot E\tau(c) \to \mu$ *as* $c \to \infty$, *where* μ *is a positive constant, then*

$$E|S_{\tau(c)}|^r = o(c) \quad \text{as } c \to \infty. \tag{2.6}$$

The equivalence of (2.1) and (2.2) is proved in [11], the equivalence of (2.1) and (2.3) is proved in [12], (2.4) obviously implies (2.3) and the converse follows from the first part of the following lemma. (2.5) and (2.6) are proved in [8].

Lemma 2.2. *If* $E|X|^r < \infty$, *then*

$$E \max_{1 \leqq k \leqq n} |S_k|^r \leqq 8 \cdot \max_{1 \leqq k \leqq n} E|S_k|^r, \quad r > 0, \tag{2.7}$$

$$E \max_{1 \leqq k \leqq n} |S_k|^r \leqq 8 \cdot E|S_n|^r, \quad r \geqq 1. \tag{2.7'}$$

If $E(X^+)^r < \infty$, *then*

$$E \max_{1 \leqq k \leqq n} (S_k^+)^r \leqq 8 \cdot \max_{1 \leqq k \leqq n} E(S_k^+)^r, \quad r > 0, \tag{2.8}$$

$$E \max_{1 \leqq k \leqq n} (S_k^+)^r \leqq 8 \cdot E(S_n^+)^r, \quad r \geqq 1. \tag{2.8'}$$

Note. (2.7′) follows from [4], 337. Since $\{S_n\}_{n=1}^\infty$ is a martingale when $r \geqq 1$, $\{|S_n|^r\}_{n=1}^\infty$ and $\{(S_n^+)^r\}_{n=1}^\infty$ are non-negative submartingales. Therefore,

$$\max_{1 \leqq k \leqq n} E|S_k|^r = E|S_n|^r \quad \text{and} \quad \max_{1 \leqq k \leqq n} E(S_k^+)^r = E(S_n^+)^r$$

and hence (2.7) and (2.7′) are the same if $r \geqq 1$ and so are (2.8) and (2.8′).

Proof. Let $\mu(X)$ denote a median of X. By integrating the Lévy-inequalities ([11], 247), it follows that

$$E \max_{1 \leqq k \leqq n} |S_k - \mu(S_k - S_n)|^r \leqq 2E|S_n|^r, \tag{2.9}$$

$$E \max_{1 \leqq k \leqq n} ((S_k - \mu(S_k - S_n))^+)^r \leqq 2E(S_n^+)^r. \tag{2.10}$$

By Markov's inequality, $P(|X| \geqq (2 \cdot E|X|^r)^{1/r}) \leqq \frac{1}{2}$. Thus, $|\mu(X)|^r \leqq 2E|X|^r$ and hence

$$|\mu(S_n)|^r \leqq 2E|S_n|^r \quad \text{and} \quad \mu(S_n^+)^r \leqq 2E(S_n^+)^r. \tag{2.11}$$

Thus, (2.9), (2.11) and the c_r-inequalities yield

$$\begin{aligned} E \max_{1 \leqq k \leqq n} |S_k|^r &\leqq c_r \cdot E \max_{1 \leqq k \leqq n} |S_k - \mu(S_k - S_n)|^r + c_r \cdot \max_{1 \leqq k \leqq n} |\mu(S_k - S_n)|^r \\ &\leqq 2c_r \cdot E|S_n|^r + 2c_r \cdot \max_{1 \leqq k \leqq n} E|S_k|^r \leqq 4c_r \cdot \max_{1 \leqq k \leqq n} E|S_k|^r. \end{aligned}$$

Since $4c_r \leqq 8$ if $0 < r \leqq 2$, (2.7) is completely proved.

Similarly, since $(a+b)^+ \leqq a^+ + b^+$, where a and be are real numbers,

$$\begin{aligned} E \max_{1 \leqq k \leqq n} (S_k^+)^r &\leqq c_r \cdot E \max_{1 \leqq k \leqq n} ((S_k - \mu(S_k - S_n))^+)^r + c_r \cdot \max_{1 \leqq k \leqq n} ((\mu(S_k - S_n))^+)^r \\ &\leqq 2c_r \cdot E(S_n^+)^r + c_r \cdot \max_{1 \leqq k \leqq n} (\mu((S_k - S_n)^+))^r \\ &\leqq 4c_r \cdot \max_{1 \leqq k \leqq n} E(S_k^+)^r \leqq 8 \cdot \max_{1 \leqq k \leqq n} E(S_k^+)^r \quad \text{if } 0 < r \leqq 2 \end{aligned}$$

and thus (2.8) is proved for $0<r\leqq 2$ and (2.8′) for $1\leqq r\leqq 2$. If $r>1$, $E\max_{1\leqq k\leqq n}(S_k^+)^r\leqq \left(\frac{r}{r-1}\right)^r\cdot E(S_n^+)^r$ by Doob's inequality, ([4], 317). Since $\left(\frac{r}{r-1}\right)^r\leqq 8$ if $r>1.3$, (2.8) and (2.8′) also hold when $r>2$ and so the proof is complete.

Now, let $\{X(t), t\geqq 0\}$ be a separable random process with independent, stationary increments. As pointed out in the introduction, it is no restriction to assume that the process is stochastically continuous and that the sample functions are continuous from the right. Suppose that the mean is zero whenever it exists.

Lemma 2.3. *If* $E|X(1)|^r<\infty$, *then*

$$E\sup_{0\leqq s\leqq t}|X(s)|^r\leqq 8\cdot\sup_{0\leqq s\leqq t}E|X(s)|^r<\infty,\qquad r>0,\tag{2.12}$$

$$E\sup_{0\leqq s\leqq t}|X(s)|^r\leqq 8\cdot E|X(t)|^r<\infty,\qquad r\geqq 1.\tag{2.12′}$$

If $E(X(1)^+)^r<\infty$, *then*

$$E\sup_{0\leqq s\leqq t}(X(s)^+)^r\leqq 8\cdot\sup_{0\leqq s\leqq t}E(X(s)^+)^r<\infty,\qquad r>0,\tag{2.13}$$

$$E\sup_{0\leqq s\leqq t}(X(s)^+)^r\leqq 8\cdot E(X(t)^+)^r<\infty,\qquad r\geqq 1.\tag{2.13′}$$

Proof. For a separable, stochastically continuous random process every countable dense subset of the parameter set is a separating set (see [11], 510). Let $I_n^t=\left\{0,\frac{t}{2^n},\frac{t}{2^{n-1}},\ldots,t\right\}$ and set $I^t=\bigcup_{n=0}^{\infty}I_n^t=\lim_{n\to\infty}I_n^t$. By (2.7),

$$E\max_{s\in I_n^t}|X(s)|^r\leqq 8\cdot\max_{s\in I_n^t}E|X(s)|^r\leqq 8\cdot\sup_{0\leqq s\leqq t}E|X(s)|^r.$$

Since the sets I_n^t are monotonically increasing, it follows that

$$E\sup_{s\in I^t}|X(s)|^r\leqq 8\cdot\sup_{0\leqq s\leqq t}E|X(s)|^r$$

and since I^t is a separating set for $\{X(s), 0\leqq s\leqq t\}$ it follows that

$$\sup_{s\in I^t}|X(s)|^r=\sup_{0\leqq s\leqq t}|X(s)|^r.$$

Thus, if $r\geqq 1$, then

$$E\sup_{0\leqq s\leqq t}|X(s)|^r\leqq 8\cdot\sup_{0\leqq s\leqq t}E|X(s)|^r=8\cdot E\,|X(t)|^r<\infty,$$

where the last equality follows from martingale theory.

Now, let $0<r<1$. In this case the finiteness is no longer obvious, since the supremum is taken over all s belonging to an interval. Set $g(u)=E|X(u)|^r$, $u\geqq 0$. Then, $g(0)=0$, and, by the c_r-inequalities, $g(u)$ is a measurable, non-negative subadditive function. It now follows from [10], Chapter VII, that $g(u)$ is bounded on finite intervals and so

$$E\sup_{0\leqq s\leqq t}|X(s)|^r\leqq 8\cdot\sup_{0\leqq s\leqq t}E|X(s)|^r<\infty,$$

which completes the proof of (2.12).

The proofs of (2.13) and (2.13′) are similar and therefore omitted.

Theorem 2.1. *Let $0<r<2$. The following statements are equivalent:*

$$E|X(1)|^r<\infty, \qquad (EX(t)=0 \text{ if } r\geqq 1), \tag{2.14}$$

$$t^{-1/r}\cdot X(t)\xrightarrow{\text{a.s.}}0 \qquad \text{as } t\to\infty, \tag{2.15}$$

$$E|X(t)|^r=o(t), \qquad \text{i.e. } t^{-1/r}\cdot X(t)\xrightarrow{r}0 \text{ as } t\to\infty, \tag{2.16}$$

$$E\sup_{0\leqq s\leqq t}|X(s)|^r=o(t), \qquad \text{i.e. } t^{-1/r}\cdot \sup_{0\leqq s\leqq t}X(s)\xrightarrow{r}0 \text{ as } t\to\infty. \tag{2.17}$$

If $\{\tau(c),\ c\geqq 0\}$ is a non-decreasing family of stopping times such that $E\tau(c)<\infty$ and $E\tau(c)\nearrow\infty$ as $c\to\infty$, then (2.14) *implies that*

$$E|X(\tau(c))|^r=o(E\tau(c)) \qquad \text{as } c\to\infty. \tag{2.18}$$

If, moreover, $c^{-1}\cdot E\tau(c)\to\mu$ as $c\to\infty$, where μ is a positive constant, then

$$E|X(\tau(c))|^r=o(c) \qquad \text{as } c\to\infty. \tag{2.19}$$

Proof. Set $Y_0=0$, $Y_k=X(k)-X(k-1)$, $k=1,2,\ldots$. Obviously, $X(n)=\sum_{k=1}^n Y_k$, $n\geqq 1$. By Lemma 2.1, (2.15)-(2.17) each imply (2.14).

Now, suppose that (2.14) holds and define

$$V_n=\sup_{n-1\leqq s,t\leqq n}|X(t)-X(s)| \quad \text{and} \quad U_n=\sup_{n-1\leqq s\leqq n}|X(s)-X(n-1)|, \qquad n\geqq 1. \tag{2.20}$$

$V_n\leqq 2U_n$ and by the stationarity and Lemma 2.3,

$$EV_n^r\leqq 2EU_n^r=2EU_1^r=2E\sup_{0\leqq s\leqq 1}|X(s)|^r<\infty. \tag{2.21}$$

Thus, $\{V_n\}_{n=1}^\infty$ and $\{U_n\}_{n=1}^\infty$ are sequences of i.i.d. random variables having finite moments of order r.

From (2.2) we obtain that $n^{-1/r}\cdot X(n)\xrightarrow{\text{a.s.}}0$ as $n\to\infty$. Furthermore, by the strong law of large numbers, $n^{-1}\cdot\sum_{k=1}^n U_k^r\xrightarrow{\text{a.s.}}EU_1^r$ as $n\to\infty$, from which it follows that

$$n^{-1/r}\cdot U_n\xrightarrow{\text{a.s.}}0 \qquad \text{as } n\to\infty. \tag{2.22}$$

(2.15) now follows by observing that

$$t^{-1/r}\cdot X(t)=t^{-1/r}\cdot X([t])+t^{-1/r}\cdot(X(t)-X([t]))$$

and that

$$t^{-1/r}\cdot|X(t)-X([t])|\leqq 2\cdot t^{-1/r}\cdot U_{[t]+1}.$$

Next (2.16)-(2.17) are considered.

$$\begin{aligned}E|X(t)|^r&\leqq c_r\cdot E|X([t])|^r+c_r\cdot E|X(t)-X([t])|^r\leqq c_r\cdot E|X([t])|^r+c_r\cdot E\sup_{0\leqq s\leqq 1}|X(s)|^r\\&=o([t])+c_r\cdot EU_1^r=o(t) \qquad \text{as } t\to\infty,\end{aligned}$$

by (2.3) and (2.21). Thus, (2.14) implies (2.16). (2.17) now follows from (2.16) and Lemma 2.3 and the first part of the theorem is proved.

Let $\tau'=\tau'(c)=\tau$ if τ is a positive integer and let $\tau'=[\tau]+1$ otherwise. This makes τ' an integer valued stopping time because

$$\{\tau'=1\}=\{0\leqq\tau\leqq 1\}\in\sigma\{X(t);\ t\leqq 1\}$$

and

$$\{\tau'=n\}=\{n-1<\tau\leqq n\}\in\sigma\{X(t);\ t\leqq n\}, \qquad n\geqq 2.$$

Let $1\leqq r<2$.

$$\|X(\tau)\|_r\leqq \|X(\tau')\|_r+\|X(\tau)-X(\tau')\|_r\leqq \left\|\sum_{k=1}^{\tau'} Y_k\right\|_r+\|V_{\tau'}\|_r$$
$$=\big(o(E\tau')\big)^{1/r}+2\cdot\|U_{\tau'}\|_r=\big(o(E\tau)\big)^{1/r}+2\cdot\|U_{\tau'}\|_r,$$

by (2.5). Let $0<r<1$. Similarly, $E|X(\tau)|^r\leqq E|X(\tau')|^r+2\cdot EU_{\tau'}^r=o(E\tau)+2\cdot EU_{\tau'}^r$.

To prove (2.18) it therefore remains to show that

$$EU_{\tau'}^r=o(E\tau')=o(E\tau) \quad \text{as } c\to\infty. \tag{2.23}$$

But this follows by exactly the method used in the proofs of Lemmas 2.4 and 3.2 in [7], (which were inspired by [6]). Thus, (2.18) follows, from which (2.19) is immediate.

Remark. If $r=1$, (2.15) is known as the strong law of large numbers. See [4], 364.

3.

Let $\{X(t);\ t\geqq 0\}$ be a separable random process with independent, stationary increments, such that $EX(t)=t\theta$, $0<\theta<\infty$, and suppose, without restriction, that the process is stochastically continuous and that the sample functions are continuous from the right.

Define the first passage time

$$T=T(c)=\inf\{t;\ X(t)>c\cdot a(t)\}, \tag{3.1}$$

where $c\geqq 0$ and $a(y)$ is positive, bounded on compact subsets of $[0,\infty)$ and such that $a(y)=o(y)$ as $y\to\infty$.

The overshoot is defined as

$$Z(T)=X(T)-c\cdot a(T). \tag{3.2}$$

By the strong law of large numbers, T is a proper random variable, i.e. $P(T<\infty)=1$.

The purpose of this section is to investigate the behaviour of the first passage time, and to derive asymptotic properties as $c\to\infty$. Most results have been obtained earlier for processes without positive jumps, (see [7]), but with this assumption the overshoot vanishes, $\big(P(Z(T)=0)=1\big)$. By the results of Section 2 this extra assumption can be removed. The proofs of the theorems below are essentially as follows: The results of Section 2 are used to take care of the overshoot and the rest then follows as in [7]. We therefore confine ourselves to show how the results of Section 2 are applied and simply refer to [7] for the rest of the proofs.

Theorem 3.1. *Let* $r\geqq 1$.

$$E\big(X(1)^-\big)^r<\infty \Rightarrow ET^r<\infty, \tag{3.3}$$

$$E|X(1)|^r<\infty \Rightarrow E\big(X(T)\big)^r<\infty. \tag{3.4}$$

If $E\exp\{s\cdot X(1)^-\}<\infty$, $|s|<s_0$, $s_0>0$, *then there exists an* $s_1>0$ *such that* $E\exp\{sT\}<\infty$, $|s|<s_1$.

Proof. The proofs given in [7], Section 4, for processes without positive jumps, of (3.3) and the last assertion are also applicable here. Thus, only (3.4) has to be considered.

Throughout this section, let V_n and U_n be defined by (2.20), set $\tau=T$ and define τ' as in the proof of (2.18). Obviously,

$$X(T)\leqq c\cdot a(T-)+X(T)-X(T-), \tag{3.5}$$

$$|X(T)-X(T-)|\leqq V_{\tau'}\leqq 2U_{\tau'} \tag{3.6}$$

and by Lemma 2.1 of [7], (2.21) and (3.3) it follows that

$$EU_{\tau'}^r\leqq(E\tau'\cdot EU_1^r)^{1/r}<\infty.$$

If $a(y)\equiv 1$, then

$$\|X(T)\|_r\leqq c+\|X(T)-X(T-)\|_r<\infty$$

and the general case now follows as in [7], Theorem 3.1.

Assume furthermore, during the rest of this section that $a(y)$ is non-decreasing and, for sufficiently large values of y, concave, differentiable and regularly varying at infinity with exponent p, $0\leqq p<1$, i.e. $a(y)=y^p\cdot L(y)$, where $L(y)$ is slowly varying at infinity. Let $\lambda=\lambda(c)$ be the solution of the equation $c\cdot a(y)=\theta y$, see [15] and [7]. This solution is unique if c is sufficiently large. In the following, asymptotic results for the first passage time and the overshoot are established.

Let us start by showing how the results of Section 2 can be applied to investigate the behaviour of the overshoot.

Define

$$\hat{V}_n=\sup_{n-1\leqq s,t\leqq n}(X(t)-X(s))^+ \quad\text{and}\quad \hat{U}_n=\sup_{n-1\leqq s\leqq n}(X(s)-X(n-1))^+,\qquad n\geqq 1. \tag{3.7}$$

By stationarity and (2.13′)

$$E\hat{V}_n^r\leqq 2E\hat{U}_1^r\leqq 16\cdot E(X(1)^+)^r<\infty. \tag{3.8}$$

Thus, $\{\hat{V}_n\}_{n=1}^\infty$ and $\{\hat{U}_n\}_{n=1}^\infty$ are sequences of i.i.d. random variables with finite moments of order r.

Since $a(y)$ is non-decreasing, (3.6) may be replaced by

$$0\leqq Z(T)\leqq X(T)-X(T-)\leqq \hat{V}_{\tau'}\leqq 2\hat{U}_{\tau'}\leqq 2U_{\tau'}. \tag{3.9}$$

Lemma 3.1. *Let* $r\geqq 1$. *If* $E|X(1)|^r<\infty$, *then*

$$T^{-1/r}\cdot U_{\tau'}\xrightarrow{\text{a.s.}}0 \quad\text{and}\quad T^{-1/r}\cdot Z(T)\xrightarrow{\text{a.s.}}0 \qquad\text{as } c\to\infty. \tag{3.10}$$

If $E(X(1)^+)^r<\infty$, *then*

$$E\hat{U}_{\tau'}^r=o(ET) \quad\text{and}\quad E(Z(T))^r=o(ET) \qquad\text{as } c\to\infty. \tag{3.11}$$

Proof. Since $\tau'\xrightarrow{\text{a.s.}}+\infty$ as $c\to\infty$ it follows from (2.22) that $(\tau')^{-1/r}\cdot U_{\tau'}\xrightarrow{\text{a.s.}}0$ as $c\to\infty$, and so $T^{-1/r}\cdot U_{\tau'}\xrightarrow{\text{a.s.}}0$ as $c\to\infty$ (because $0\leqq\tau'-T\leqq 1$).

By exactly the same method as in the proof of [7], Lemmas 2.4 and 3.2, (cp. (2.23) above), it follows that $E\hat{U}^r_{\tau'} = o(E\tau') = o(ET)$ as $c \to \infty$. The proof is completed by applying (3.9).

Theorem 3.2. $\lambda^{-1} \cdot T \xrightarrow{\text{a.s.}} 1$ *as* $c \to \infty$.

Proof. Since $T \xrightarrow{\text{a.s.}} +\infty$ as $c \to \infty$, the strong law of large numbers yields

$$T^{-1} \cdot X(T) \xrightarrow{\text{a.s.}} \theta \quad \text{as } c \to \infty,$$

which, combined with (3.10) and the fact that

$$c \cdot a(T) < X(T) = c \cdot a(T) + Z(T), \tag{3.12}$$

implies that $\dfrac{c \cdot a(T)}{T} \xrightarrow{\text{a.s.}} \theta$ as $c \to \infty$ and, since $c \cdot a(\lambda) = \lambda\theta$,

$$\frac{\lambda \cdot a(T)}{T \cdot a(\lambda)} \xrightarrow{\text{a.s.}} 1 \quad \text{as } c \to \infty. \tag{3.13}$$

The result now follows as in [15], Lemma 4 and [7], Theorem 3.3.

Theorem 3.3. *Let* $r \geqq 1$.

$$E(X(1)^-)^r < \infty \Rightarrow \lambda^{-r} \cdot ET^r \to 1 \qquad as\ c \to \infty, \tag{3.14}$$

$$E(X(1)^+)^r < \infty \Rightarrow \lambda^{-1} \cdot E(Z(T))^r \to 0 \qquad as\ c \to \infty. \tag{3.15}$$

Proof. In [7], Section 4, (3.14) is proved for processes without positive jumps, but the same proof applies here. (3.15) follows from (3.11) and (3.14).

Theorem 3.4. *Suppose that* $\operatorname{Var} X(1) = \sigma^2 < \infty$. *Then*

$$P(T - \lambda \leqq \theta^{-1} \cdot (1-p)^{-1} \cdot \sigma\sqrt{\lambda} \cdot x) \to \Phi(x) \qquad as\ c \to \infty, \tag{3.16}$$

where

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-y^2/2}\, dy.$$

Proof. The proof follows the ideas of [13] and [7].

$$\begin{aligned}\frac{X(T) - T\theta}{\sigma\sqrt{\tau'}} &= \frac{X(\tau') - \tau'\theta}{\sigma\sqrt{\tau'}} + \frac{(\tau' - T)\theta}{\sigma\sqrt{\tau'}} + \frac{X(T) - X(\tau')}{\sigma\sqrt{\tau'}} \\ &= A_1(T) + A_2(T) + A_3(T).\end{aligned}$$

$P(A_1(T) \leqq x) \to \Phi(x)$ as $c \to \infty$ by [13], Theorem 1. $A_2(T) \xrightarrow{\text{a.s.}} 0$ as $c \to \infty$, because $0 \leqq \tau' - T \leqq 1$ and $\tau' \xrightarrow{\text{a.s.}} +\infty$ as $c \to \infty$. $|X(T) - X(\tau')| \leqq V_{\tau'} \leqq 2U_{\tau'}$ and thus, by (3.10), it follows that $A_3(T) \xrightarrow{\text{a.s.}} 0$ as $c \to \infty$. By Cramér's theorem,

$$P(X(T) - T\theta \leqq \sigma\sqrt{\tau'} \cdot x) \to \Phi(x) \quad \text{as } c \to \infty$$

and by Theorem 3.2 and Cramér's theorem,

$$P(X(T) - T\theta \leqq \sigma\sqrt{\lambda} \cdot x) \to \Phi(x) \quad \text{as } c \to \infty, \tag{3.17}$$

which combined with (3.10), (3.12) and Cramér's theorem yields

$$P(c \cdot a(T) - T\theta \leqq \sigma\sqrt{\lambda} \cdot x) \to \Phi(x) \quad \text{as } c \to \infty. \tag{3.18}$$

The conclusion now follows as in [7].

Theorem 3.5. *If* $E|X(1)|^r<\infty$, $1\leqq r<2$, *then*

$$\lambda^{-1/r}\cdot(T-\lambda)\xrightarrow{\text{a.s.}}0 \quad as\ c\to\infty, \tag{3.19}$$

$$\lambda^{-1/r}\cdot Z(T)\xrightarrow{\text{a.s.}}0 \quad as\ c\to\infty. \tag{3.20}$$

Proof. (3.20) follows from (3.10) and Theorem 3.2.

Since $T\xrightarrow{\text{a.s.}}+\infty$, Theorem 2.1 yields

$$T^{-1/r}\cdot\big(X(T)-T\theta\big)\xrightarrow{\text{a.s.}}0 \quad \text{as } c\to\infty, \tag{3.21}$$

which, together with Theorem 3.2, (3.12) and (3.20), implies that

$$\lambda^{-1/r}\cdot\big(c\cdot a(T)-T\theta\big)\xrightarrow{\text{a.s.}}0 \quad \text{as } c\to\infty. \tag{3.22}$$

The rest of the proof follows as in [7].

Finally, a result under the assumption that $a(y)\equiv 1$, $y\geqq 0$, i.e. $T=\inf\{t;\,X(t)>c\}$, $c\geqq 0$, in which case $\lambda=\lambda(c)=c/\theta$.

Theorem 3.6. *If* $E\big(X(1)^+\big)^2<\infty$, *then*

$$c/\theta\leqq ET\leqq c/\theta+O(1) \quad as\ c\to\infty. \tag{3.23}$$

If $\operatorname{Var}X(1)=\sigma^2<\infty$, *then*

$$\operatorname{Var}T=\frac{\sigma^2 c}{\theta^3}+o(c) \quad as\ c\to\infty. \tag{3.24}$$

If $E|X(1)|^r<\infty$, $1\leqq r<2$, *then*

$$c^{-1/r}\cdot\left(T-\frac{c}{\theta}\right)\xrightarrow{r}0 \quad as\ c\to\infty. \tag{3.25}$$

Proof. Define $T_N=\inf\{t;\,X(t)>c,\ t\geqq 1 \text{ is an integer}\}$ as in [7], Section 4. Obviously, $0\leqq T\leqq T_N$ and by [7], Theorems 2.6 and 2.7, $ET_N=\frac{c}{\theta}+O(1)$ as $c\to\infty$ if $E\big(X(1)^+\big)^2<\infty$. By Wald's lemma, $EX(T)=\theta\cdot ET$. Thus, $c<EX(T)=\theta\cdot ET\leqq \theta\cdot ET_N=c+O(1)$ as $c\to\infty$ and thus (3.23) is proved.

Now suppose that $\operatorname{Var}X(1)=\sigma^2<\infty$. $\operatorname{Var}T=E\left(T-\frac{c}{\theta}\right)^2-\left(E\left(T-\frac{c}{\theta}\right)\right)^2$ and by (3.23), $\left(E\left(T-\frac{c}{\theta}\right)\right)^2=O(1)$ as $c\to\infty$. It therefore suffices to prove

$$E\left(T-\frac{c}{\theta}\right)^2=\frac{\sigma^2 c}{\theta^3}+o(c) \quad \text{as } c\to\infty. \tag{3.26}$$

By Theorem 3.4 and Fatou's lemma we obtain

$$\varliminf_{c\to\infty} E\left\{\left(\frac{\sigma^2 c}{\theta^3}\right)^{-1}\cdot\left(T-\frac{c}{\theta}\right)^2\right\}\geqq 1$$

and it therefore remains to prove the opposite inequality.

$$\begin{aligned}\|\theta T-c\|_2 &\leqq \|X(T)-T\theta\|_2+\|X(T)-c\|_2=\|X(T)-T\theta\|_2+\|Z(T)\|_2\\ &=(\sigma^2\cdot ET)^{1/2}+o(c^{1/2})\leqq\left(\frac{\sigma^2 c}{\theta}+O(1)\right)^{1/2}+o(c^{1/2}) \quad \text{as } c\to\infty,\end{aligned}$$

by [9], Theorem 3, (3.15) and (3.23). This proves (3.26).

If $E|X(1)|^r<\infty$, $1\leqq r<2$, the above reasoning yields

$$\|\theta T-c\|_r\leqq\|X(T)-T\theta\|_r+\|Z(T)\|_r=o(c^{1/r}) \quad \text{as } c\to\infty,$$

by (2.19) and (3.15), and the opposite inequality follows from (3.19) and Fatou's lemma.

The proof is terminated.

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