Z. Wahrscheinlichkeitstheorie verw. Gebiete 31, 271–285 (1975)

Limit Theorems on the Self-Normalized Range for Weakly and Strongly Dependent Processes*

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$X(t)$ being a random function of time, with $X^*(t)=\int_0^t X(s)\,ds$, let

$$R(d)=(\sup-\inf)_{0\leqq u\leqq d}\{X^*(u)-(u/d)\,X^*(d)\},$$
$$S^2(d)=d^{-1}X^{2*}(d)-d^{-2}X^{*2}(d),$$
$$Q(d)=R(d)/S(d).$$

R is a range and Q a self-normalized range. For certain r.f. $X(t)$, one can select the weight function $A(d;Q)$, so that (in some sense) the $d\to\infty$ limit of either $Q(d)/A(d;Q)$ or $Q(e^{\Phi}d)/A(e^{\Phi}d;Q)$ is nondegenerate; if so, $A(d;Q)$ is the key factor in a new statistical technique, called R/S analysis. The theorems in this paper describe some aspects that have already been founded fully upon theorems (easy to prove but unexpected) concerning weak convergence of certain r.f., while the conjectures relate to other aspects of R/S analysis that still rely, at this stage, upon properties suggested by heuristics and by computer simulation. For iid processes satisfying $EX^2<\infty$, or attracted to a stable process of exponent α, it is shown that $A=\sqrt{d}$ independently of α. For processes that are weakly dependent (e.g., Markov or autoregressive) one still has $A=\sqrt{d}$. Conversely, whenever $A=\sqrt{d}$, the r.f. $X(t)$ will be said to have a finite R/S memory. On the other hand, if $X(t)$ are the finite increments of a proper fractional Brownian motion – defined as the fractional integral of order $H-0.5\neq 0$ of ordinary Brownian motion – one has $A=d^H$. This $X(t)$ is strongly dependent – rather than strongly mixing – and it can be said to have an infinite memory. Conversely, whenever $A\neq\sqrt{d}$, the r.f. $X(t)$ will be said to have an infinite R/S memory. When $A=d^H L(d)$, with $H\neq 0.5$ and

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* AMS (MOS) subject classification (1970). Primary 60F99, 60G15, 62M99; Secondary 86A05, 86A10, 90A20.

The substance of this paper was summarized in Part II of [22] and in [23]. For help in its elaboration, thanks are due, in chronological order, to Harold A. Thomas of Harvard for showing me, in 1963, the papers by Hurst [12, 13], to James R. Wallis of IBM for stimulating discussions during our joint study of R/S by computer simulation; to Murad Taqqu of Columbia University for assistance in further simulations supported by the National Bureau of Economic Research, for permission to quote unpublished theorems from the thesis he wrote under my supervision [35], and for stimulating discussions; and to Hirsch Lewitan of IBM for programming assistance.

Concerning the form of this paper, particular thanks are due to Ward Whitt of Yale University, who read the text as submitted in May 1974 and who has been extremely generous with time and advice. He tightened up Theorem 5 and Conjecture 5, suggested in passing the first variant in Section 13, disproved some wrong conjectures, and generally helped identify spots where further detail and/or more mathematical pedantry were required.

$L(d)$ a slowly varying function, H will be called the R/S exponent of long range dependence. The definitions are of practical usefulness because, for many natural records, Hurst has shown that H is clearly above 0.5.

1. Introduction to *R/S* Analysis

Let $X(t)$ be an integrable function of real time t, functions of discrete time being interpolated as right continuous step functions of real t. Define, for every real t, $X^*(t) = \int_0^t X(u)\,du$, and, for every real $d \geqq 0$, to be called the lag, define

$$R(d) = (\sup - \inf)_{0 \leqq u \leqq d} \{X^*(u) - (u/d)\, X^*(d)\},$$
$$S^2(d) = d^{-1} X^{2*}(d) - d^{-2} X^{*2}(d), \qquad Q(d) = R(d)/S(d).$$ [1]

These functions are, respectively, a form of the range, a variance, and a form of self-normalized range, where self-normalization involves the sample variance and also the sample mean. In particular, Q is the same for the function $X(t)$ and for all functions of the form $\sigma[X(t)+\mu]$; thus, σ and μ need not be known to calculate Q. Also, the first d values of Q can be evaluated directly from the first d values of $X(t)$. When $X(t)$ is a r.f. of time, $Q(d)$ is a r.f. of d.

To the best of my knowledge, the statistic Q was first used in the works of Harold Edwin Hurst [12–14]. He was an English physicist working in Cairo, who advanced the idea of the Aswan High Dam and, as possibly the greatest Nilologist of all time, was nicknamed Abu Nil, the Father of the Nile [2]. Steiger [34] also defined R/S – independently of Hurst – but did not pursue the matter. In 1965, I became interested in Hurst's work; since 1968 [22–24, 26–29, 35], the $d \to \infty$ behavior of Q has been the object of intensive study, and has opened up the new field of R/S analysis. This is an extremely effective method of statistical estimation and testing directed towards distinguishing between r.f. that can be either weakly or strongly dependent. In practical statistics, this last possibility had not been faced, so that methods addressed expressly to the distinction in question are few if not nonexistent.

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[1] A few elementary properties are as follows. Q is well determined except when $S=0$, which is the case iff all the $X(u)$ $(0<u\leqq d)$ are identical, so that $R(d)=0$ and Q takes the indeterminate form $Q=0/0$. In particular, this indeterminate form is always encountered when time is discrete and $d=1$. When $X(t)$ is random and is not a.s. constant, the probability of indeterminacy, defined as the probability of finding a string of d identical values of $X(u)$, tends to 0 with $1/d$. When time is continuous and Q is determined, $0 \leqq Q \leqq d/2$. When time is discrete and Q is determined, its lower attainable limit is 1 when d is even, and $d/(d-1)$ when d is odd. Its upper limit is $d/2$ when d is even, and $\sqrt{d^2-1}/2$ when d is odd. The two limits coincide for $d=2$, where $Q \equiv 1$. Roughly, $1 \leqq Q \leqq d/2$, meaning that $\log Q$, as a function of $\log d$, lies in an eighth of a plane with apex at $d=2$, $Q=1$.

[2] The numerator R was suggested to Hurst by an old method of preliminary design of water reservoirs, due to Rippl, and the denominator S was added as a natural normalizing factor, with no indication that its special virtues – to be described – had been noticed. There is a coincidental similarity between this self-normalization and one used in [18]. The latter work – which is entirely independent of the present one – only attacks questions relative to independent $X(t)$, and, by hard analysis, it achieves strong results; their counterpart lies beyond my present technical capability; luckily – as we shall – it is not indispensable to the practical application.

In the practice of R/S analysis, in order to be able to utilize as fully as possible the information available in a sample,[3] one defines $R(t, d)$ and $S^2(t, d)$ for each "starting point" t, applying the above formulas to the translates $X(t+s)$ of the original $X(t)$, viewed as r.f. of s. When $X(t)$ is a stationary r.f., then, for each fixed value of the lag, $Q(t, d)$ is another stationary r.f. of t (dependent on d in distribution). In his first application of R/S (working with the Nile, then with other rivers and other empirical records) Hurst estimated $EQ(d)$ by averaging the values of $R(t, d)/S(t, d)$ for several t's. He found that, roughly speaking, EQ "fluctuates around" d^H, with H "typically" about 0.74. The property of Nature that is embodied in this loose statement has come to be known as the "Hurst phenomenon" (or, for reasons that will transpire momentarily, "paradox" or "puzzle"). The following is a sharper statement of it, one more readily open to precise analysis: there often exists an $H \neq 0.5$ such that sample values of $Q(t, d)/d^H$ lie close to each other over the whole range where they can be estimated. One is therefore tempted to conclude that, if one could extend this sample indefinitely, Q/d^H would neither tend to 0 nor to ∞; rather, to a nondegenerate limit r.v. This is our reason for studying the r.v. $Q(d)/d^H$ for $d \to \infty$, or – more generally – for studying the r.v. $Q(d)/A(d; Q)$, in search of r.f. $X(t)$ such that $Q(d)/A(d; Q)$ has a nondegenerate limit when $A \sim d^H$. A further sharpening reexamines the empirical evidence, using a statistical procedure I recommend whenever it is feasible, which consists in tracing the sample $\log Q(t, d)$ as a function of $\log d$ for several values of t. One finds that it is possible to choose H so that, for sufficiently large d, $\log Q - H \log d$ looks increasingly like a stationary r.f. of $\log d$. This finding suggests that $Q(\mu d)/(\mu d)^H$, viewed as a function of the multiplier μ, may perhaps have some nondegenerate $d \to \infty$ limit that is a stationary r.f. of $\log \mu$. Finally, rewriting the multiplier μ as e^ϕ, we are led to conjecture that the r.f. $Q(e^\phi d)/(e^\phi d)^H$ has a non-degenerate limit that is a stationary r.f. of ϕ. This is our reason for studying the r.f. $Q(e^\phi d)/A(e^\phi d; Q)$.

Hurst immediately perceived that the phenomenon he discovered is in contradiction with what might have been expected; it may therefore express some deep characteristic of the underlying records, and it may yield the long-sought conceptual device that would allow cyclic but non-periodic records to be handled properly. Indeed, Hurst (whose treatment was very rough) and soon afterwards Feller [8] (who was more rigorous, but not quite completely, since he took a weak convergence lemma for granted) showed that for the simplest r.f., namely the iid process with $EX^2 < \infty$, the theory predicts that H should be equal to 0.5 – contrary to evidence! Several would-be explanations were presented, but none was conclusive. Moran [30, 31] claims to have proven that $H \neq 0.5$ if the r.f. $X(t)$ is far enough from being normal; however, many of the records in question are near-normal; in addition, Moran had misread the evidence, believing it to be relative to the $d \to \infty$ behavior of R itself rather than of R/S; this point, as we shall see, is a major one. Thus, the phenomenon long remained a puzzle (and a spur to hydrological model making) until a r.f. for which $H \neq 0.5$ was first exhibited in

[3] Note that several obvious alternatives are entirely acceptable, such as the statistics inspired by the Kolmogorov-Smirnov tests: $\sup_{0 \leq u \leq d} |X^*(u) - (u/d) X^*(d)|$, $\sup_{0 \leq u \leq d} \{X^*(u) - (u/d) X^*(d)\}$ or $\inf_{0 \leq u \leq d} \{X^*(u) - (u/d) X^*(d)\}$. All our theorems continue to hold, with only the obvious changes, if $R(d)$ is replaced by any of these expressions. On the other hand, such variants as $\sup_{0 \leq u \leq d} |X^*(s)|$ or $\sup_{0 \leq u \leq d} X^*(s)$ would not be acceptable.

my first paper on this topic [19]. The Hurst phenomenon was thus shown to be due to very long run dependence in the r.f. $X(t)$. Later, in [29], computer simulation was used to demonstrate that, thanks to the normalizing denominator S, Hurst's ratio has independent interest in statistics and probability in the sense that the behavior of $A(d; Q)$ for large d constitutes a powerful method for distinguishing between short-term (weak) and long-term (strong) dependence in r.f.; it makes no assumption about EX^2, and in fact is essentially insensitive to the distribution of X. Thus, the main virtue of this method lies in its robustness. Its statistical background is further developed in [23].

The purpose of the present paper is to describe the current mathematical foundation of this technique, and to solicit proof (or disproof) of various conjectures which it uses. We shall first state a stronger form of one part of the result of [8].

Notation. w-lim refers to weak convergence in function space; d-lim refers to weak convergence of a real valued r.v.; f-lim refers to convergence of all finite dimensional distributions for almost all values of the argument; as-lim is almost sure limit; l-lim is the limit of order l. The number given to a Theorem indicates the Section in which it occurs (thus, there is no Theorem 3).

Prototype Theorem 1. *If* $X(t)$ *is* iid *in discrete time, with* $EX^2<\infty$, *then* $w\text{-lim}_{d\to\infty} Q(e^{\phi} d)/\sqrt{e^{\phi} d}$ *in* $C]-\infty, \infty[$ *is a non-degenerate stationary* r.f. $\Psi(\phi)$. *In particular,* $d\text{-lim}_{d\to\infty} Q(d)/\sqrt{d}$ *is a non-degenerate* r.v. $\Psi(0)$.

Theorem 1, which is near obvious, will be proven in Section 3; then, having examined the continuity of certain transformations, we shall derive counterparts concerning the behavior of appropriately normalized ratios $Q(e^{\phi} d)/A(e^{\phi} d; Q)$ and $Q(d)/A(d; Q)$ within increasingly broad classes of r.f. culminating in Theorem 5. The crucial finding is that different classes of r.f. involve different weights $A(d; Q)$. However, the variety of possible A's is limited, as follows.

Proposition 1. *If* $f\text{-lim}_{d\to\infty} Q(e^{\phi} d)/A(d; Q)$ *exists and is non-degenerate,* $A(d; Q)$ *is of the form* $d^H L(d)$, *where* $L(d)$ *is a slowly varying function.*

Proof. Proposition 1 follows from Theorem 2 in Lamperti (1962).

Definition. The parameter H, when defined, summarizes as much of the information about $X(t)$ as is reflected in Q. Therefore, this H will be called "exponent of R/S dependence" of $X(t)$.

The first class of main results about $A(d; Q)$ is that $H=1/2$ and $A(d; Q)\sim\sqrt{d}$ holds, *independently of the distribution of* X, as long as X is either iid or stationary with weak (short) dependence. This behavior differs greatly from that of $R(d)$ itself, for which the proper weight $A(d; R)$ is $\sqrt{d}$ when $EX^2<\infty$, but otherwise depends on the distribution of X. The second class of results is that sufficiently strong (long) dependence can lead to $A(d; Q)$ of the form $d^H L(d)$ with $H\neq 0.5$. Again, the exponent H is not characteristic of the distribution of X, but of the intensity of long run dependence in $X(t)$. This is the justification for normalizing R through division by S.

2. On the Use of Q in Practical Statistics

In practice, the question arises, how well one can estimate EQ from a sample of $Q(t,d)$. When $X(t)$ is iid, the r.v. $Q(t+e,d)$ and $Q(t,d)$ are independent when $e \geqq d$, which defines the r.f. $Q(t,d)$ as $d-$dependent and proves that reliable estimation is possible. The following statements, if true, would show estimation to be reliable under wide assumptions.

Conjecture 2. $Q(t,d)$, viewed for fixed d as r.f. of t, is "typically" (under weak conditions to be specified) ergodic and short R/S dependent, the latter term meaning that when the Q function is computed for $Y(t)=Q(t,d)$, the $A(d,Q)$ function is $\sqrt{d}$.

Conjecture 2'. When $w\text{-}\lim_{d\to\infty} Q(e^\phi d)/A(e^\phi d; Q)$ exists and is a stationary r.f. of ϕ, it is also, "typically", ergodic and short dependent.

3. Proof of the Prototype Theorem 1

We assume (without loss of generality) that $EX=0$. We note that $X(t)$, as defined, belongs to the space $D[0,\infty[$ of right continuous real valued functions on $[0,\infty[$ with limits on the left everywhere on $]0,\infty[$; this space is endowed with the Skorohod topology, and in general is just like $D[0,1]$ (the matter has been settled in [11]). Also $X^*(t)$ is a continuous function in $C[0,\infty[$ and the mapping from X to X^* is continuous. From the assumptions, it follows by Donsker's theorem that $w\text{-}\lim_{d\to\infty} X^*(df)/\sqrt{d}$ – with f a fraction between 0 and 1 – is the Brownian motion r.f. $B(f)$, with $B(0)=0$, multiplied by $\sqrt{EX^2}$. By the ergodic theorem on X^2, it follows that the almost sure limit $as\text{-}\lim_{d\to\infty} S(d)$ is $\sqrt{ES^2}=\sqrt{EX^2}$. In other words, $Q(d)$ is only distinguished from R by a numerical factor. The fact that we are interested in $Q(fd)/\sqrt{fd}$ rather than in $Q(fd)/\sqrt{d}$ means it is important to work with $D]0,\infty[$ rather than with $D[0,\infty[$. Classically, applying the continuous mapping theorem in $D]0,\infty[$, we find that $Q(d)/\sqrt{d}$ and $Q(e^\phi d)/\sqrt{e^\phi d}$ are continuous functions of $X^*(fd)/\sqrt{d}$. Hence $d\text{-}\lim_{d\to\infty} Q(d)/\sqrt{d}$ and $w\text{-}\lim_{d\to\infty} Q(e^\phi d)/\sqrt{e^\phi d}$ exist. The former is the unadjusted range of the Brownian bridge; its distribution has been derived by Feller [8]. For the stationarity part of the theorem, we use a representation given in [7] (see also [3, p. 229]) $B(f)=\sqrt{f}\,J(\log f)$, with J a stationary r.f. (namely, the Gauss-Markov r.f.). Hence one has

$$w\text{-}\lim_{d\to\infty} Q(e^\phi d)/\sqrt{e^\phi d}) = (\text{max-min})_{\gamma<0}\,[e^{\gamma/2} J(\phi+\gamma) - e^{\gamma} J(\phi)],$$

which is independent of γ in distribution.

4. The General Case where X^* Lies in the Brownian Domain of Attraction

By taking the first conclusion of the proof of Section 3 as assumption, we obtain the following generalization of Theorem 1.

Theorem 4. *Let X^2 be ergodic and $w\text{-}\lim_{d\to\infty} X^*(fd)/\sqrt{d}$ be Brownian motion. Then $w\text{-}\lim_{d\to\infty} Q(e^\phi d)/\sqrt{e^\phi d}$ is a non-degenerate stationary continuous* r.f. $\Psi(\phi)$. *In particular, $d\text{-}\lim_{d\to\infty} Q(d)/\sqrt{d}$ is a non-degenerate* r.v. $\Psi(0)$. *A fortiori,* $\lim_{d\to\infty} \log Q(d)/\log d = 0.5$.

The ergodicity of X^2 need not follow from the postulated limit behavior of $X^*(fd)/\sqrt{d}$, except in the iid case with $EX^2 < \infty$; in general, it must be assumed separately.

Theorem 4 is easy to state but difficult to apply, except when it reduces to the Prototype Theorem 1. A number of other r.f. that satisfy Theorem 4 are described in [2]. When the r.f. X is Gaussian, with the covariance $C(d)$, the necessary and sufficient condition for weak convergence of X^* to Brownian motion is that $0 < C(0)/2 + \sum_{d=1}^{\infty} C(d) < \infty$, which expresses that the dependence between the X is weak (short). Ergodicity is also satisfied. Examples are the Markov or finite autoregressive r.f. When the r.f. is non Gaussian, the problem of the central limit theorem for dependent r.f. is well known to be complicated, and weak convergence of X^* is even harder. At least it is harder in principle, since it seems that in all specific cases when the central limit theorem holds for X^*, the conditions of Theorem 4 are satisfied.

5. Case where $\{X^*, X^{2*}\}$ Lies in a General Domain of Attraction

Definition of H. Lemma 5. *Suppose one can select the nonrandom functions $A(d; X^*)$ and $B(d; X^{2*})$ in such a way that the vector* r.f. *of coordinates*

$$U(f, d) = \frac{X^*(fd)}{A(d; X^*)} \qquad V(f, d) = \frac{X^{2*}(fd)}{B(d; X^{2*})}$$

converges weakly (in Skorohod topology) to a limit r.f. $\{U(f, \infty), V(f, \infty)\}$, *not identically equal to* 0 *or* ∞, *belonging to the space D. Then $U(f, \infty)$ and $V(f, \infty)$ are both self-similar, in the sense that there exist two constants H' and H'' such that the distributions of $U(fg, \infty)/g^{H'}$ and $V(fg, \infty)/g^{H''}$ are independent of g. Moreover, $A(d; X^*)/d^{H'} = L'(d)$ and $B(d; X^{2*})/d^{H''} = L''(d)$ are slowly varying functions, in the sense of Karamata. Thus, writing $H = 1/2 + H' - H''/2$, the function $A(d; Q) = \sqrt{d}\, A(d; X^*)/\sqrt{B(d; X^{2*})}$ takes the form $d^H L(d)$, with $L(d)$ slowly varying in the sense of Karamata. Finally one must have $H \leqq 1$.*

Theorem 5. *To the conditions of Lemma 5, add either that $H < 1$ or that $H = 1$ but $L(d) \to 0$, and add that $U(f, \infty) \neq f\, U(1, \infty)$. Then $f\text{-}\lim_{d \to \infty} Q(e^{\phi} d)/A(e^{\phi} d; Q)$ in $D]-\infty, \infty[$ is a non-degenerate stationary* r.f. *$\Psi(\phi)$. In particular, $d\text{-}\lim_{d \to \infty} Q(d)/A(d; Q)$ is a non-degenerate* r.v. *$\Psi(0)$. A fortiori, $\lim_{d \to \infty} \log Q/\log d = H$. Also, $H \geqq 0$.*

Conjecture 5. The f-lim in Theorem 5 can be replaced by a w-lim.

Proof of Lemma 5. Note that the mapping from X to S is continuous in $D[0, \infty[$. The self-similarity of $U(d, \infty)$ and $V(f, \infty)$ and the form of A and B were proven by Lamperti (1962), who uses, instead of our term "self-similar", the term "semi-stable". The necessity of $H \leqq 1$ is proven by noting that $H > 1$ would – for sufficiently large d – contradict $X^{2*} > X^{*2}/d$.

Remark that if all the finite dimensional distributions of the vector r.f. $[U(f, d), V(f, d)]$ converge to those of the vector r.f. $[U(f, \infty, V(f, \infty))]$, then the vector r.f. converges weakly iff each of its coordinate scalar r.f.'s converges weakly. This result is proposed in Exercise 6, p. 41, in Billingsley (1968).

Note also that, if Theorem 4 holds, Lemma 5 follows as a Corollary, $U(f, \infty)$ being Brownian motion and $V(f, \infty)$ being degenerate, in the sense that $V(f, \infty)/f$

is a constant (the functional strong law is the same as the ordinary strong law). A second instance where $V(f, \infty)/f$ is a constant will be encountered in Theorem 11. In other cases, both $U(f, \infty)$ and $V(f, \infty)$ count.

Proof of Theorem 5. By elementary manipulation,

$$\begin{aligned} R(e^{\phi}d) &= (\text{max-min})_{0<f<\exp(\phi)}\{X^*(fd) - fe^{-\phi}X^*(e^{\phi}d)\} \\ &= (\text{max-min})_{0<f<\exp(\phi)}\{U(fe^{-\phi}, e^{\phi}d) - fe^{-\phi}U(e^{\phi}, d)\}/A(e^{\phi}d; X^*), \\ S^2(e^{\phi}d) &= [X^{2*}(e^{\phi}d) - e^{-\phi}d^{-1}X^{*2}(e^{\phi}d)]\,d^{-1}e^{-\phi} \\ &= \{V(e^{\phi}, d) - U(e^{\phi}, d)[A^2(e^{\phi}d; Q)/(e^{\phi}d)]\}\,B(e^{\phi}d; X^*)(e^{\phi}d)^{-1}. \end{aligned}$$

Hence,

$$\frac{Q(e^{\phi}d)}{A(e^{\phi}d; Q)} = \frac{(\text{max-min})\{U(fe^{-\phi}, e^{\phi}d) - fe^{-\phi}U(e^{\phi}, d)\}}{\sqrt{V(e^{\phi}, d) - U(e^{\phi}, d)[A^2(e^{\phi}d; Q)/(e^{\phi}d)]}}.$$

The second half of the denominator is asymptotically negligible because of the assumptions made about H and/or $L(d)$. The indeterminate form 0/0 will have a probability tending to 0 with $1/d$, because of the assumptions that $U(f, \infty) \neq fU(1, \infty)$. Further, (R, S), which is a random element in $D[0, \infty[\times D[0, \infty[$, is a continuous function of X in $D[0, \infty[$. Since any finite set of values of $Q(e^{\phi}d)/A(e^{\phi}d; Q)$ is a continuous function of the vector r.f. $[U(f, d), V(f, d)]$, it converges to a limit that is the corresponding function of $[U(f, \infty), V(f, \infty)]$. The stationarity of the limit results from self-similarity of $[U(f, \infty), V(f, \infty)]$; the required generalization of Doob's representation has been proved in [15, p. 64]. Finally, $H \geqq 0$ is necessary to insure that $Q \geqq 1$.

Comment on Conjecture 5. The difficulty here is that we must allow for the possibility of $S(e^{\phi}d)$ and $V(f, \infty)$ being discontinuous, in which case the continuous mapping theorem ceases to be applicable. Nevertheless, the conclusion in Conjecture 5 seems correct. Thus, *even though it has no practical importance*, it is a challenge to the mathematician. If the conditions in Theorem 5 were made stronger, the proof could be carried out (for example, if one adds a condition suggested by Whitt: $V(f, \infty)$ is a.s. continuous and for all a and b such that $0<a<b<1$, one has $\Pr[\inf_{a \leqq f \leqq b} V(f, \infty) > 0\} = 1$.). But such conditions appear as both unnatural and unnecessary. An alternative is to use the first variant of Q described in Section 13, but for practical needs this would be too costly to be considered.

6. Direct Relationship between Independence and the Exponent Value $H = 1/2$

Theorem 6. *If* $EX^2 = \infty$ *and* x *is* iid *and lies in a stable* $(0<\alpha<2)$ *domain of attraction,* $f\text{-}\lim_{d\to\infty} Q(e^{\phi}d)/\sqrt{e^{\phi}d}$ *is a non-degenerate stationary discontinuous* r.f. *of* ϕ.

Proof. Assuming that X is iid with $EX^2 = \infty$, it follows from [33] that the conditions of Theorem 6 are necessary and sufficient for the validity of Lemma 5. By well-known limit theorems, $A(d; X^*) = d^{1/\alpha}L'(d)$, with L' determined by $\Pr[|X| > A(d; X^*)]\,d \to 1$ [10, pp. 314–315]. Also, $B(d; X^{2*}) = A^2(d; X^*) = d^{2/\alpha}[L'(d)]^2$, because $L''(d)$ is determined by $\Pr[X^2 > B(d; X^{2*})]\,d \to 1$. Hence $H = 1/2$ and $L(d) = 1$. (We need not worry about the convergence of the X^* bridge to the stable bridge. This is fortunate, since the latter – though true – is hard to prove; see [16].)

For the applications, the central feature of Theorem 6 is that, in contrast to $A(d; X^*)$, $A(d; Q)$ is independent of α, also of the skewness parameter β of X.

Conjecture 6. (As Conjecture 5.) One can replace f-lim by w-lim.

Comment on Conjecture 6. The numerator and the denominator of Q are both discontinuous, their jumps occurring at the same moment and being dependent. This should be the key factor in the proof.

The case $\alpha=2$. Proposition 6. *If $EX^2=\infty$ and X is* iid *and lies in the Gaussian domain of (necessarily not normal) attraction,* w-$\lim_{d\to\infty} Q(e^\phi d)/\sqrt{e^\phi d}$ *is the same* r.f. *as in Theorem 4.*

Proof. The case $\alpha=2$ is based on Theorem 5 in a case where – like in Theorem 4 – $V(f,\infty)=fV(1,\infty)$ but – contrary to Theorem 4 – the convergence to this limit is weak, not strong. In other words, we need the (not quite familiar) form that the weak law of large numbers takes in the case of iid infinite variance r.v. Using the U and μ notations in [10, p. 236 and pp. 314–315], the equations $dA^{-2}(d; X^*)\,U[A(d; X^*)]\to 1$ and $dB^{-1}(d; X^{2*})\,\mu[B(d; X^{2*})]$ continue to yield $B=A^2$. By [33], w-$\lim_{d\to\infty} U(f,d)=B(t)$ and by [10, p. 236],

$$\Pr\{|V(f,d)/(fd)-1|>\varepsilon\}\to 0.$$

Generalization of Theorem 6 to the Case "$\alpha=0$". The definitions of the stable r.v. and of their domains of attraction exclude the limit value $\alpha=0$. Also, Theorem 5 assumes that the scaling factors $A(d; X^*)$ and $B(d; X^{2*})$ are nonrandom. We shall now allow them to be random, and shall show this may bring a bit of new generality to the study of Q, by allowing the set of values of the index α of the domains of attraction of $Q(d)/\sqrt{d}$ to be closed by adding a limit that can be viewed as corresponding to "$\alpha=0$". This limit is encountered when $X>0$ and $\Pr(|X|>x)$ itself is slowly varying (necessarily, non-increasing). Indeed, Darling [5] has shown that, in this case, the limit of order 1, 1-$\lim_{d\to\infty} X^*(d)/\max_{1\leqq u\leqq d} X(u)=1$, from which it follows that 1-$\lim_{d\to\infty} R(d)/\max_{1\leqq u\leqq d} X(u)=1$; similarly,

$$1\text{-}\lim_{d\to\infty} X^{2*}(d)/\max_{1\leqq u\leqq d} X^2(u)=X^{2*}(d)/\sqrt{[\max_{1\leqq u\leqq d} X(u)]^2}=1$$

from which it follows that 1-$\lim_{d\to\infty} S^2(d)\,d/[\max_{1\leqq u\leqq d} X(u)]=1$. In this case, 1-$\lim_{d\to\infty} R^2/S^2 d=1$, which implies that $A(d; Q)=\sqrt{d}$. To get closer to Theorems 5 and 6, one can view $\max_{1\leqq u\leqq d} X(u)$ both as $A(d; X^*)$ and as $\sqrt{B(d; X^{2*})}$. The resulting r.f. $\{U(f,d), V(f,d)\}$ will converge weakly to $\{U(f,\infty), V(f,\infty)\}$, where $U(f,\infty)=V(,\infty)=0$ for $0\leqq f<f_0$ and $U(f,\infty)=V(f,\infty)=1$ for $f_0<f<1$, f_0 being a r.v. uniformly distributed between 0 and 1.

7. Conjectures Concerning the iid Case for Different Domains of Attraction of X^*

Contrary to [18], we are unable to characterize the limit in Theorem 6, and must be content with the following conjectures. They have been suggested in part by computer simulations, and their proof would justify current practice. For example, conjecture A, combined with conjecture 2, would show Q to be readily estimated from samples.

A) $0 < E\{\lim_{d\to\infty} d^{-0.5} Q(d)\}^h < \infty$ for every $h > 0$.

B) For every β, $\sup_{\alpha\in[0,2]} \Pr\{\lim_{d\to\infty} Q(d)/\sqrt{d} > x\}$ is a non-degenerate distribution.

C) For every β, $\inf_{\alpha\in[0,2]} \Pr\{\lim_{d\to\infty} Q(d)/\sqrt{d} > x\}$ is another non-degenerate distribution.

D) The sup in conjecture B) is effectively attained, being the distribution corresponding to $\alpha = 2$.

E) The inf in conjecture C) is effectively attained, being the distribution corresponding to some $\alpha \in [0, 2]$.

F) $E[\lim_{d\to\infty} Q(d)/\sqrt{d}]$, considered as a function of α, is (for every β) monotonically decreasing with α. Its $\lim_{\alpha\to 0}$ is 1 and its $\lim_{\alpha\to 2}$ is 1.2533.

G) $\operatorname{Var}[\lim_{d\to\infty} Q(d)/\sqrt{d}]$, considered as a function of α, is monotonically decreasing with α. (We know its $\lim_{\alpha\to 0}$ is 0.)

8. Conjectures Concerning the iid Case when X^* Lies in No Domain of Attraction

In this case, $Q(d)/\sqrt{d}$ has no d-lim, and a fortiori $Q(e^{\Phi} d)/\sqrt{e^{\Phi} d}$ has no w-lim[4]. Nevertheless, I postulate the following:

H) The fact that X is iid suffices to establish that

$$\sup_{d\to\infty} \Pr\{d^{-0.5} Q(t, d) < x\} = Q_2(x) < \sup_\alpha \Pr\{\lim_{d\to\infty} Q(d)/\sqrt{d}\},$$

$$\inf_{d\to\infty} \Pr\{d^{-0.5} Q(t, d) < x\} = Q_1(x) > \inf_\alpha \Pr\{\lim_{d\to\infty} Q(d)/\sqrt{d}\}.$$

I) Corollary of H: When X is iid, then $as\text{-}\lim_{d\to\infty} \log Q/\log d = 1/2$.

9. Generalization of the Scope of $H = 0.5$. Lack of Converse Relationship with Either Dependence or Weak Dependence

In broad terms, Section 8 would establish that, even if Theorem 5 fails to apply, the relation $A(d, Q) \sim \sqrt{d}$ continues to hold, with a less demanding interpretation

[4] For example, let $X(t)$ be the symmetric iid r.f. with the following marginal distribution: For $1 \leq x < 10$, $\Pr(|X| > x) = x^{-1}$. For $10^3 \leq x < 10^6$, $\Pr(|X| > x)/\Pr(|X| > 10^3) = x^{-5}$. For $10^{3n} \leq x < 10^{3(n+1)}$, where n is an even integer: $\Pr(|X| > x)/\Pr(|X| > 10^{3n}) = x^{-1}$. For $10^{3n} \leq x \leq 10^{3(n+1)}$, where n is odd: $\Pr(X > x)/\Pr(X > 10^{3n}) = x^{-5}$. The behavor of $X^*(d)$ for this type of r.f. has been documented in an early paper by Paul Lévy. For values of d up to 1000, $X^*(d)\, d^{-1}$ seems headed to converge to a Cauchy motion, but this tendency stops sometime after d exceeds 1000. Later, for values of d up to 10^6, $X^*(d)\, d^{-0.5}$ seems headed to converge to a Brownian motion, but this tendency also eventually stops. The ostensible rate of convergence $A(d; X^*)$ continues to flip thus with no end between the two different analytic forms, d^{-1} and $d^{-0.5}$. The ostensible limit of $X^*/A(d; X^*)$ also flips with no end. This $X(t)$ is an example of r.f. for which there exists no $A(d; Q)$ such that $Q(d)/A(d; Q)$ has a good limit for $d \to \infty$. Nevertheless, it appears that the r.v. $Q(d)/A(d; Q)$ remains positive and finite for all d. This differs from the situation for X^* insofar as $A(d; Q) = d^{-0.5}$ throughout (which is a change in the direction of greater simplicity), and is very similar insofar as $Q(d)\, d^{-0.5}$ first seems destined to tend to the limit corresponding to $\alpha = 1$, then switches to the limit corresponding to $\alpha = 2$ and so continues flipping between different limits with no end.

of $\sim$, as long as X is iid (or is weakly dependent). The converse is, however, false, in the sense that when the conditions of Lemma 5 are denied, $A(d, Q)=\sqrt{d}$ is compatible with very strong dependence.

First Example. In [20, 21], I had introduced a martingale model of certain kinds of price variation, wherein successive increments are very strongly correlated. Also, Theorem 5 is inapplicable. The proper $A(d, Q)$ for the Q of those price increments turns out, however, to be $\sqrt{d}$.

Second Example. Assume that $X(t)=Y(t)-Y(t-1)$, with $Y(t)$ iid. For every t, $X(t)$ and $X(t+2)$ are independent, so, in a sense, X is "finitely dependent". On the other hand, $X^*(t)=Y(t)-Y(0)$, so X^* satisfies no non-trivial central limit theorem in which the limit is somehow "universal". Since the notion of weak (short-range) dependence implies that the limit is Gaussian or stable, one must view the present form of dependence as strong (long range). Note that, for large d, $R(d)\sim(\text{max-min})_{0<u<d}Y(u)$. If ess.min Y (the essential minimum of Y) vanishes, then $R(d)\sim\max_{0<u<d}Y(u)$. Hence, asymptotically for large d, $Q^2/2d=R^2/2S^2d$ is the ratio between the maximum and the sum for d r.v. X^2. Assume further that Y lies in the domain of normal attraction of a stable r.v. of exponent $\alpha<2$. The conditions of a well-known theorem of [5] are satisfied, and it follows that $d\text{-}\lim_{d\to\infty}Q^2/2d$, and hence $d\text{-}\lim_{d\to\infty}Q(d)/\sqrt{d}$ is non-degenerate. The same is true (the details have been worked out by W. Whitt) of $w\text{-}\lim_{d\to\infty}Q(e^{\phi}d)/\sqrt{e^{\phi}d}$. The generalization when $\Pr(Y<0)>0$ is easy. This shows it is possible that $A(d, Q)=\sqrt{d}$ even when X is strongly dependent.

This conclusion in no way contradicts Theorems 5, because in the present case the abscissa r.f. $U(f, d)$ is $Y(t)-Y(0)$. Hence, $\{U(f, \infty), f\geqq 0\}$ is conditionally independent given $Y(0)$, and thus not in D. Incidentally, the requirement that $\{U(f, d), V(f, d)\}$ converge weakly in D, rather than from the viewpoint of finite joint distributions, is shown to be more than a technicality.

10. Some Roles of the Limit Exponent Value $H=1$ with $L(d)+0$

As a first example, assume $X(t)=G(t-t_0)$ where G is a Gaussian r.v. independent of t. In this case, $X^*(u)-(u/d)X^*(d)=Gu(u-d)/2$, and so $R(d)=|G|\,d^2/8$. Also $S^2(d)=G^2d^2/12$. Hence, $Q(d)=dA(d;Q)=d\sqrt{3}/4$. As a second example, let $X(t)$ itself be a Brownian motion or a Lévy motion of exponent $0<\alpha<2$ (stable process) without drift (e.g., such that $EX=0$). Now, $Q(e^{\phi}d)/e^{\phi}d$ is a stationary r.f. of ϕ, dependent on α. When $X(t)$ is Brownian motion with a drift, $Q(d)/d$ behaves for small d as if the drift were absent, and for large d as if the Brownian motion were absent.

Typically, $A(d;Q)\sim d$ extends to other integrals of a stationary r.f. Consequently, if one's purpose is to distinguish between such an integral and a fractional noise (Section 11) with H nearly 1 (a stationary r.f.), R/S testing is useless. R/S estimation is possible but delicate: Finding that $A(d;Q)\sim d$ means little, but can be interpreted by R/S analyzing $X'(t)$, and higher derivatives if necessary, until one reaches H below 1.

11. Relationship between $0<H<1/2$ or $1/2<H<1$ and Strong Dependence. Case of Gaussian $X^*(t)$ Attracted to Fractional Brownian Motions

Assume the limit r.f. $V(f, \infty)$ of Lemma 5 is uniformly constant, so that $B(d; X^{2*})=d$. To have $A(d; Q)\neq\sqrt{d}$, a necessary condition is that $X^*(dh)/\sqrt{d}$ must not be attracted by Brownian motion, in particular $X^*(d)/\sqrt{d}$ must not have a Gaussian limit; such is the case when this last limit is degenerate.

Definition. The "fractional Brownian motion" (fBm) of exponent H $(0<H<1)$ is the r.f. $B_H(t)$ in C and hence in D, such that $B_H(0)=0$ and

$$EB_H(t')\,B_H(t'')=(1/2)[|t'|^{2H}-|t'-t''|^{2H}+|t''|^{2H}].$$

In particular, $EB_H^2(t)=t^{2H}$. The term "fBm" was coined by Mandelbrot and Van Ness [25], because B_H is a "fractional integral" of the Brownian motion $B(t)$, in the sense of Holmgrem, Riemann, Liouville and Weyl; but the function $B_H(t)$ is a very natural one, and had been briefly used by a number of other authors. The case $H=0.5$ is degenerate, in the sense that $B_{0.5}(t)$ is ordinary Brownian motion. The cases $H\neq 0.5$ are therefore referred to as "properly fractional"; the main characteristic of their increments is that they are not strongly mixing in the sense of Rosenblatt [32], but rather strongly dependent. The intensity of long dependence is measured by the single parameter H, to be called "exponent of dependence". If the sign of the dependence is measured by that of the correlation $C(d)$ between $B_H(t)-B_H(t-1)$ and $B_H(t+d)-B_H(t+d-1)$ where d is large, then one can say that the sign of dependence is the same as that of $H-0.5$. Indeed,

$$E\{[B_H(t)-B_H(t-1)][B_H(t+d)-B_H(t+d-1)]\}=(1/2)[|d+1|^{2H}-2|d|^{2H}+|d-1|^{2H}|.$$

The limit of $B_H(t)$ for $H\to 1$ is a r.f. with fully correlated Gaussian increments, i.e., the r.f. $G(t-t_0)$; this limit is in the space D. The limit for $H\to 0$ is $X^*(t)=G(t)-G(0)$; it lies outside of D.

Theorem 11. *One class of* r.f. *such that* $A(d; Q)=d^H L(d)$ *is the class of functions for which* X^2 *is ergodic and* w-$\lim_{d\to\infty} X^*(fd)/d^H L(d)=B_H(t)$.

Proof. This is an immediate corollary of Theorem 5. Observe that the distribution of $\lim_{d\to\infty} Q(e^{\phi}d)/A(e^{\phi}d)$ is, contrary to $A(d)$ itself, independent of $L(d)$.

Lemma 11. *When* $X(t)=B_H(t)-B_H(t-1)$, w-$\lim_{d\to\infty} X^*(fd)/d^H=B_H(t)$.

Clearly, any generalization of Lemma 11 leads to an immediate generalization of Theorem 11.

The first generalization applies to Gaussian processes such that their covariance shares enough of the behavior of the covariance $C(d)$ of $B_H(t)-B_H(t-1)$. Taqqu [35] has shown that w-$\lim_{d\to\infty} X^*(fd)/d^H L(d)=B_H(f)$ when X is a stationary Gaussian r.f. whose covariance $C(d)$ has the property that the function $C(d)\,d^{-2H+2}=L^2(d)$ varies slowly for $d\to\infty$, and, in the case $0<H<0.5$, also satisfies $\lim_{t\to\infty} C(0)+2\sum_{s=1}^{t} C(s)=0$.

A second generalization relies upon the fact that $B_H(t)-B_H(t-1)$ can be written as a moving average. Davydov [6] has proved that w-$\lim_{d\to\infty} X^*(fd)/d^H L(d)-B_H(f)$ for all r.f. of the form $X(t)=\sum_{s=-\infty}^{0} K(t-s)\,Y(s)$, where the $Y(s)$ are iid with

$EY=0$ and $E|Y|^{2h}<\infty(h\geqq 2)$, and where not only $\sum_{-\infty}^{\infty} K(s-t)<\infty$, but $EX^{*2}(t)=t^{2H}L^2(t)$, where $1/(h+2)<H\leqq 1$ and $L(t)$ is a slowly varying function.

Another generalization consists in starting with the Gaussian $X=B_H(t)-B_H(t-1)$ and then studying the nonlinear function $Z(X)$. The case $Z(X)=X^2$ was used briefly as a counterexample in [32]. The most general function Z was studied by Taqqu [35], who has related $A(d; Q)$ to the "Hermite rank" of Z, defined as the order of the first nonvanishing term in the development of $Z(X)$ in Hermite polynomials. When $H>0.5$, $A(d; Q)$ can be the same for $Z(X)$ as for X itself, meaning that the intensity of long-run dependence, as measured by H, can be invariant by the transformation from X to $Z(X)$. The necessary and sufficient condition on $Z(X)$ is, simply, that its Hermite rank must be one, whereas a transformation of higher rank can decrease the value of H. This means that X is not characterized by *one* exponent of dependence, rather by a spectrum of exponents including the original H as a maximum. (As a result, the experimental situation is likely to be confused: a function $Z(X)$ of rank 3 (say), if disturbed slightly, is likely to become a fonction of rank 1, with the result that the analytic form of $A(d; Q)$ will seem to change with d.) When $H<0.5$, on the contrary, one finds, save for exceptional $Z(X)$, that $A(d; Q)=\sqrt{d}$. In other words, the preservation of the long-run dependence expressed by $H<1/2$ hinges very sensitively on the form of Z.

12. Generalization of the Scope of $H\neq 0.5$, and Relationship between the Sign of $H-0.5$ and the Sign of Strong Dependence

The following conjecture is fundamental to the claim of R/S analysis, that Q picks up the rule of long-run dependence in $X(t)$ irrespectively of the marginal distribution.

Conjecture 12. It is conjectured that Theorem 5, with $A(d; Q)=d^H$ independent of α, also applies to the fractional integral of order $H-0.5$ of Lévy's stable process of exponent α. In particular (conjectural lemma) $B(d, X^{2*})=d^{\alpha/2}$ independently of H. The range of admissible H depends on α, and an elaboration of this conjecture would take us too far away.

In a different vein, it may happen that Theorem 5 fails to hold, but, in some weakened sense, $Q(d)/A(d; Q)$ should continue to have a limit, with $A(d; Q)=d^H L(d)$ and $L(d)$ slowly varying, or at least that one has *a.s.* $\lim_{d\to\infty} \log Q(t, d)/\log d=H$. In either case, it is tempting to say that long run dependence of $X(t)$ is "regular" and to take $H-0.5$ as measure of its intensity. However, we know from Section 9 that $H=0.5$ is quite compatible with very strong dependence. To illuminate the issue, it is good to push the second example of Section 9, namely $X(t)=Y(t-1)$ with $Y(t)$ iid.

First, let $|Y(t)|$ be bounded. Then $\lim_{d\to\infty} R(d)=\text{ess max } Y(t)-\text{ess min } Y(t)$, which is finite. So is $\lim_{d\to\infty} S$. Hence $1\leqq\lim_{d\to\infty}(R/S)<\infty$, and in particular $H=0$ with $L(d)=1$. This example is in conformity with the notion that for $X(t)$ the dependence is negative and as extreme as can be. So is the case when Y is Gaussian. Then, for large d, $R(d)$ is well known [4] to behave like $\sqrt{\log d}$. Since $S(t, d)$ is practically nonrandom, $H=0$ with $L(d)=\sqrt{\log d}$.

Next, let $Y>1$, and $\Pr(Y>y)=y^{-\alpha}$ with $\alpha>2$. Here, $EX^2=2EY^2<\infty$ and therefore $S^2\to 2\,\text{Var}\,X$, so it is a numerical factor of no consequence. On the other hand, it is clear that, for large d, $R(d) \sim \max_{0 \leqq u \leqq d} X(u)$. Using the theory of maxima of iid, one finds that for $d \to \infty$, $R(d)/d^{1/\alpha}$ converges to the Fréchet r.v. of exponent α, Φ_α, which is defined as being such that $\Pr(\Phi_\alpha < x) = \exp(-x^{-\alpha})$. As a result, $Q(d)/d^{1/\alpha}$ converges towards a Fréchet r.v., and $A(d, Q) = d^{1/\alpha}$. Thus, although intuitively the dependence in X has the same strength for all α, the behavior of $Q(d)/A(d, Q)$ mimics a strength dependent upon α.

13. Some Transforms of the Cumulated Centered Deviations Other than Q

The variants of Q to be discussed have different purposes. The reason for looking at the first is technical: it avoids – at a price – the gap in Theorem 5 that has led to Conjecture 5. The reasons for looking at the others were esthetic: The first proves disappointing, the second indifferent, and the third is probably an improvement over Q.

The Whitt Transform. Define $\tilde{Q}(d) = R(d)/\tilde{S}(d)$, with $\tilde{S}^2(d) = d^{-1} \int_0^d S^2(u)\, du$. There is no longer a need for a counterpart to Conjecture 5, because the counterpart of Theorem 5, with $A(d, \tilde{Q}) = A(d, Q)$ and $V(f, \infty)$ replaced by $f^{-1} \int_0^f V(u, \infty)\, du$, can be strengthened from f-lim to w-lim. Indeed, division on $D(0, \infty) \times D(0, \infty)$ will have become a continuous function because Whitt's conditions (as stated in the comments on Conjecture 5) are satisfied. In practice, the continuity of the denominator $\tilde{S}$ is paid for dearly. In $Q(d)$, the jumps in $R(d)$ are to a large extent counterbalanced by the simultaneous jumps in $S(d)$, but they cannot be counterbalanced by the nonexisting jumps of $\tilde{S}(d)$. Hence the behavior of $\tilde{Q}(d)$ is likely to look very wild in comparison with the behavior of $Q(d)$, as exemplified in the simulations found in [23].

The R/S_p Transform. It is defined as

$$Q_p(d) = R(d)/S_p(d),$$

where

$$S_p^p(d) = d^{-1} \sum_{u=1}^{d} |X(u) - d^{-1} X^*(d)|^p.$$

Thus R/S_p generalizes R/S_2, which is its special case R/S. The case R/S_p with $p \neq 2$ has no known virtue that is not shared by R/S, but may serve to make the virtues of R/S glow more brightly. It suffices to make the comparison when X is iid. When X is Gaussian, or more generally has finite absolute moments of every order, h, then, as $d \to \infty$, $S_p^p(d)$ a.s. converges to a nonrandom positive finite constant. The statistics R, R/S or R/S_p are then identical, except for numerical factors. Next, when $X(t)$ are iid with $\Pr(X > x) = x^{-\alpha}$, it is easy to see that $A(d; Q_p) = \sqrt{d}$ if "either $\alpha > \max(2, p)$ or $\alpha < \min(2, p)$," and that, otherwise, $A(d; Q_p)$ depends upon both α and p. In other words, the most robust value of p is $p = 2$.

The R/R_* Transform. The definition of the ratio R/S suffers from a lack of symmetry, in the sense that its numerator and denominator are obtained by different operations. A first symmetric alternative to R/S is obtained by taking as denominator, instead of S, some permutation-invariant range linked to X. For example, the range of X^{**}, obtained by reordering $X(1) \ldots X(d)$ at random, is permutation invariant in distribution, and MR_*, defined as its mean value over all permutations, is permutation invariant. Either can be used as denominator. They have not been studied fully, but it seems that R/R_* and R/MR_* would, by and large, serve the same purpose as R/S.

The S_*/S Transform. This is a variant of R/S in which the numerator is also evaluated as a moment rather than a range. Though it too was originally designed for the sake of symmetry, it happens in addition to have a broader range of applicability than R/S. Indeed, replace R by the square root of

$$\begin{aligned} S_*^2 &= d^{-1} \sum_{u=1}^{d} [X^*(u) - (u/d) X^*(d)]^2 \\ &= d^{-1} X^{*2*}(d) - 2d^{-2} X^*(d) [uX^*(d)]^* + d^{-3} u^{2*} X^*(d) \\ &= d^{-1} X^{*2*}(d) - 2(d+1) d^{-2} X^{*2}(d) + 2d^{-2} X^*(d) X^{**}(d) \\ &\quad + d^{-3} u^{2*} X^*(d). \end{aligned}$$

In general, the behavior of S_*/S is indistinguishable from that of R/S. However, in the example where $X(t) = Y(t) - Y(t-1)$, with the $Y(t)$ iid, the S_*/S behavior is prevented from mimicking independence.

14. Conclusion

Statistical expressions tend to become more complicated with increase of the range of possibilities between which one has to discriminate. Thus far, nearly all statistical techniques relative to r.f. have assumed dependence to be weak. Now, Hurst's phenomenon (as I interpret it) forces us to face the possibility of strong dependence; most conveniently, the statistic Q turns out to be an excellent tool to study the new possiblities.

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(Received May 3, 1974; in revised form January 7, 1975)