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A New Method to Prove Strassen Type Laws of Invariance Principle. II

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1. Introduction

Let $X_1, X_2, \ldots$ be a sequence of i.i.d.r.v.'s with

$$P(X_i < t) = \begin{cases} 0 & \text{if } t < 0 \\ t & \text{if } 0 \leqq t \leqq 1 \\ 1 & \text{if } t > 1 \end{cases}$$

and let $F_n(s)$ be the empirical distribution function based on the sample $X_1, X_2, \ldots, X_n$. The linear interpolation between $n F_n(s)$ and $(n+1) F_{n+1}(s)$ will be denoted by $t F_t(s)$ and put

$$\xi(s, t) = t(F_t(s) - s) \qquad (0 \leqq t < \infty;\ 0 \leqq s \leqq 1).$$

A separable Gaussian process $K(s, t)$ defined on $[0, 1] \times [0, \infty)$ will be called a Kiefer-process (of first order) if

$$E(K(s_1, t_1)\, K(s_2, t_2)) = \min(t_1, t_2)[\min(s_1, s_2) - s_1 s_2],$$
$$E(K(s, t)) = 0.$$

Let us mention that a Kiefer process can be generated by the Wiener process $W(s, t)$ of two variables as follows:

$$W(s, t) - s\, W(1, t) = K(s, t) \qquad (0 \leqq t < \infty,\ 0 \leqq s \leqq 1),$$

where $W(s, t)$ has zero expectation and covariance function $\min(s_1, s_2) \min(t_1, t_2)$. In [2] the following is proved

Theorem A ([2]). *One can define a probability space $(\Omega, \mathscr{S}, P)$ and processes $\tilde{\xi}(s, t)$ and $K(s, t)$ (on Ω) such that*

(i) *$\tilde{\xi}(s, t)$ has the same joint law as $\xi(s, t)$,*

(ii) *$K(s, t)$ is a Kiefer process,*

(iii) $\sup\limits_{0 \leqq s \leqq 1} |\tilde{\xi}(s, t) - K(s, t)| = O(t^{\frac{1}{3}} (\log t)^{\frac{2}{3}})$ $(t \to \infty)$ *with probability* 1.

From now on let $X_1, X_2, \ldots$ be a sequence of i.i.d.r.v.'s uniformly distributed on the unit cube of the d-dimensional Euclidean space, i.e. the common density function $f(s_1, s_2, \dots, s_d)$ of $X_1, X_2, \dots$ is

$$f(s_1, s_2, \dots, s_d) = \begin{cases} 1 & \text{if } 0 \leqq s_i \leqq 1 \quad (i=1,2,\dots,d) \\ 0 & \text{otherwise} \end{cases}$$

and let $F_n(s) = F_n(s_1, s_2, \dots, s_d)$ be the empirical distribution function based on the sample $X_1, X_2, \dots, X_n$. The linear interpolation between $nF_n(s)$ and $(n+1)F_n(s)$ will be denoted by $tF_t(s)$ and put

$$\hat{\xi}(s_1, s_2, \dots, s_d; t) = \hat{\xi}(s, t) = t\big(F_t(s_1, s_2, \dots, s_d) - s_1 s_2 \dots s_d\big).$$

We give the following:

Definition. The separable Gaussian process

$$K(s,t) = K(s_1, s_2, \dots, s_d; t) \qquad (0 \leqq s_i \leqq 1;\ i=1,2,\dots,d;\ 0 \leqq t < \infty)$$

will be called a Kiefer process (of d-th order) if

$$\begin{aligned} &E\big(K(s_{11}, s_{12}, \dots, s_{1d}; t_1)\, K(s_{21}, s_{22}, \dots, s_{2d}; t_2)\big) \\ &\quad = \min(t_1, t_2)[\min(s_{11}, s_{21}) \min(s_{12}, s_{22}), \dots, \min(s_{1d}, s_{2d}) \\ &\qquad - s_{11} s_{12} \dots s_{1d} s_{21} s_{22} \dots s_{2d}], \end{aligned}$$

$$E\big(K(s,t)\big) = 0.$$

A Kiefer process (of d-th order) can be generated by the Wiener process $W(s_1, s_2, \dots, s_d; t)$ of $d+1$ variables as follows:

$$W(s_1, s_2, \dots, s_d; t) - s_1 s_2 \dots s_d W(1, 1, \dots, 1; t) = K(s_1, s_2, \dots, s_d; t),$$
$$(0 \leqq t < \infty, 0 \leqq s_i \leqq 1;\ i=1,2,\dots,d),$$

where $W(s_1, \dots, s_d; t)$ has zero expectation and covariance function

$$\min(t_1, t_2) \min(s_{11}, s_{21}) \min(s_{12}, s_{22}) \dots \min(s_{1d}, s_{2d}).$$

Now we can formulate our

Theorem. *Suppose that the r.v.'s $X_1, X_2, \dots$ are defined on a probability space $(\Omega, \mathscr{S}, P)$ which is rich enough to define a sequence $W_1(s), W_2(s), \dots$ of independent Wiener processes with zero expectation and covariance function*

$$\min(s_{11}, s_{21}) \min(s_{12}, s_{22}) \dots \min(s_{1d}, s_{2d})$$

and a sequence $\Pi_1, \Pi_2 \dots$ of independent r.v.'s such that

(i) $P(\Pi_n = k) = \frac{n^k}{k!} e^{-n}$,

(ii) *the sequences $\{X_n\}$, $\{W_n\}$, $\{\Pi_n\}$ are also independent.*

Then one can define a Kiefer process $K(s,t)$ on Ω such that

$$\sup_{\substack{0\leq s_i\leq 1\\(i=1,2,\ldots,d)}} |K(s,t)-\hat{\xi}(s,t)|=O\left(t^{\frac{d+1}{2(d+2)}}(\log t)^2\right)\qquad (t\to\infty)$$

with probability 1.

A main tool of our proof will be Lemma 8 which is practically the same result as that of Brillinger ([3]). Hence one can say that our proof shows that Kiefer's result can be obtained as a consequence of a Brillinger type theorem.

For the sake of simplicity the proof of our Theorem will be prepared only in case $d=2$. The general case can be treated in a quite similar way.

2. Lemmas

Lemma 1. *Let Π be a r.v. with distribution*

$$P(\Pi=k)=\frac{\lambda^k}{k!}e^{-\lambda}\qquad (k=0,1,2,\ldots;\lambda>1).$$

Then for any constant $A>1$ there exists a polynomial $B(x)$ of second order (depending only on A) such that for any x ($|x|\leq A\sqrt{\log\lambda}$) we have

$$F_\lambda(x)=P\left(\frac{\Pi-\lambda}{\sqrt{\lambda}}\leq x\right)=\Phi(x)+\frac{\exp\left(-\frac{x^2}{2}\right)}{\sqrt{\lambda}}f(x,\lambda)\tag{1}$$

where $|f(x,\lambda)|\leq|B(x)|$.

Consequence. Suppose $|x_\lambda|\leq A\sqrt{\log\lambda}$. Then

$$F_\lambda(x_\lambda)\sim\Phi(x_\lambda)\quad\text{and}\quad 1-F_\lambda(x_\lambda)\sim 1-\Phi(x_\lambda).$$

Proof of Lemma 1. Put $[\lambda+x\sqrt{\lambda}]=\nu$, then it is well-known

$$\begin{aligned}F_\lambda(x)&=\int_\lambda^\infty\frac{t^\nu}{\nu!}e^{-t}\,dt=\exp\left(-\frac{\theta_\nu}{12\nu}\right)\sqrt{\frac{\nu}{2\pi}}\int_{-\infty}^{1-\lambda/\nu}((1-t)\,e^t)^\nu\,dt\\&=\exp\left(-\frac{\theta_\nu}{12\nu}\right)\sqrt{\frac{\nu}{2\nu}}\left[\int_{-\infty}^{-2A\sqrt{\frac{\log\lambda}{\lambda}}}((1-t)\,e^t)^\nu\,dt+\int_{-2A\sqrt{\frac{\log\lambda}{\lambda}}}^{1-\lambda/\nu}((1-t)\,e^t)^\nu\,dt\right],\end{aligned}\tag{2}$$

where $0<\theta_\nu<1$.

The first member of the right hand side clearly can be estimated by

$$\exp\left(-\frac{\theta_\nu}{12\nu}\right)\sqrt{\frac{\nu}{2\pi}}\int_{-\infty}^{-2A\sqrt{\frac{\log\lambda}{\lambda}}}((1-t)\,e^t)^\nu\,dt=O\left(\frac{1}{\lambda^{4A^2/3}}\right).\tag{3}$$

The second integral can be evaluated, making use of the simple formula $(1-t)\,e^t=\exp\left(-\frac{t^2}{2}+g(t)\right)$ where $|g(t)|\leqq|t|^3$ if $|t|\leqq 1$, as follows

$$\sqrt{\frac{\nu}{2\pi}}\int_{-2A\sqrt{\frac{\log\lambda}{\lambda}}}^{1-\lambda/\nu}((1-t)e^t)^\nu\,dt=\sqrt{\frac{\nu}{2\pi}}\int_{-2A\sqrt{\frac{\log\lambda}{\lambda}}}^{1-\lambda/\nu}\exp\left(-\frac{t^2}{2}\nu+g(t)\,\nu\right)dt$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-2A\sqrt{\frac{\log\lambda}{\lambda}}\sqrt{\nu}}^{\frac{\nu-\lambda}{\sqrt{\nu}}}\exp\left(-\frac{s^2}{2}+g\left(\frac{s}{\sqrt{\nu}}\right)\nu\right)ds \tag{4}$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\frac{\nu-\lambda}{\sqrt{\nu}}}\exp\left(-\frac{s^2}{2}+g\left(\frac{s}{\sqrt{\nu}}\right)\nu\right)ds+h(\lambda)$$

where $h(\lambda)=O\left(\frac{1}{\lambda^{4A^2/3}}\right)$.

Now a very simple calculation shows:

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\frac{\nu-\lambda}{\sqrt{\nu}}}\exp\left(-\frac{s^2}{2}+g\left(\frac{s}{\sqrt{\nu}}\right)\nu\right)ds=\Phi(x)+\exp\left(-\frac{x^2}{2}\right)\frac{\chi(x,\lambda)}{\sqrt{\lambda}} \tag{5}$$

where $|\chi(x,\lambda)|\leqq\Pi_2(x)$ and $\Pi_2(x)$ is a polynomial of second order. (2), (3), (4) and (5) imply (1).

Lemma 2. *Let $f_\lambda(t)=\Phi^{-1}(F_\lambda(t))$. Then*

$$|f_\lambda(t)-t|=O\left(\frac{\log\lambda}{\sqrt{\lambda}}\right)\qquad(\lambda\to\infty)$$

uniformly in t provided that $|t|\leqq A\sqrt{\log\lambda}$, where the capital O depends only on A.

Proof is practically the same as that of Lemma 3 of [1], so it will be omitted.

Consider the lattice points $\left(\frac{i}{r},\frac{j}{r}\right)$ $(i=0,1,2,\dots,r;\ j=0,1,2,\dots,r)$ of the unit square and let $A_{ij}=\left[\frac{i}{r},\frac{(i+1)}{r}\right)\times\left[\frac{j}{r},\frac{(j+1)}{r}\right)$ $(i,j=0,1,2,\dots,r-1)$. Let α_{ij} be the number of the elements of the sample $X_1,X_2,\dots,X_n$ lying in the square A_{ij}. Further let Π be a r.v. of Poisson distribution with parameter n, and independent of the $\{X_i\}$. Finally let β_{ij} be the number of elements of the sample $X_1,X_2,\dots,X_\Pi$ lying in A_{ij}. Introduce the notations

$$\mathfrak{A}_{IJ}=\sum_{\substack{i\leqq I\\ j\leqq J}}\frac{\alpha_{ij}-n/r^2}{\sqrt{n}},$$

$$\mathfrak{B}_{IJ}=\sum_{\substack{i\leqq I\\ j\leqq J}}\frac{\beta_{ij}-\Pi/r^2}{\sqrt{n}}$$

$(1\leqq I,J\leqq r)$.

Now we can formulate our

Lemma 3. *We have*

(i) $$P\left\{\sup_{1\leqq I,J\leqq r}|\mathfrak{A}_{IJ}-\mathfrak{B}_{IJ}|\geqq\frac{K(\log n)^{\frac{3}{4}}}{n^{\frac{1}{4}}}\right\}=O\left(\frac{1}{n^2}\right)$$

if K is a big enough positive constant and $r^2=o(n)$,

(ii) *the r.v.'s β_{ij} are independent obeying the Poisson law of parameter n/r^2.*

Proof. Our second statement is well-known, so we have only to prove the first one. Introduce the notations:

$$|\alpha_{ij}-\beta_{ij}|=\gamma_{ij},$$

$$C_{IJ}=|\mathfrak{A}_{IJ}-\mathfrak{B}_{IJ}|=\left|\sum_{\substack{i\leqq I\\ j\leqq J}}\frac{\gamma_{ij}-\frac{|n-\Pi|}{r^2}}{\sqrt{n}}\right|.$$

Then we have

$$P\left(C_{IJ}\geqq\frac{K(\log n)^{\frac{3}{4}}}{n^{\frac{1}{4}}}\right)=\sum_{k=0}^{\infty}P\left(C_{IJ}\geqq\frac{K(\log n)^{\frac{3}{4}}}{n^{\frac{1}{4}}}\middle|\Pi=k\right)P(\Pi=k)$$

$$\leqq\sum_{\left|\frac{k-n}{\sqrt{n}}\right|\geqq K\sqrt{\log n}}P(\Pi=k)+\sum_{\left|\frac{k-n}{\sqrt{n}}\right|<K\sqrt{\log n}}P\left(C_{IJ}\geqq\frac{K(\log n)^{\frac{3}{4}}}{n^{\frac{1}{4}}}\middle|\Pi=k\right)$$

$$\leqq P\left(\left|\frac{\Pi-n}{\sqrt{n}}\right|\geqq K\sqrt{\log n}\right)+\max_{|j|\leqq K\sqrt{n\log n}}P\left(C_{IJ}\geqq\frac{K(\log n)^{\frac{3}{4}}}{n^{\frac{1}{4}}}\middle|\Pi-n=j\right)$$

$$=O\left(\frac{1}{n^{K^2/2}}\right)+O\left(\frac{1}{n^{K/2}}\right).$$

Since

$$P\left\{\sup_{1\leqq I,J\leqq r}C_{IJ}\geqq\frac{K(\log n)^{\frac{3}{4}}}{n^{\frac{1}{4}}}\right\}\leqq r^2\sup_{1\leqq I,J\leqq r}P\left(C_{IJ}\geqq\frac{K(\log n)^{\frac{3}{4}}}{n^{\frac{1}{4}}}\right)$$

we have our statement.

The following is well-known:

Lemma 4. *Making use of the notations of Lemma 3 let $r=[n^{\alpha}]$ $(0<\alpha<\frac{1}{2})$, $G_n(x,y)=\sqrt{n}(F_n(x,y)-xy)$ and let $B(x,y)$ be a Brownian bridge with zero expectation and covariance function $\min(x_1,x_2)\min(y_1,y_2)-x_1x_2y_1y_2$. Then for any $A>0$ there exists a $B=B(A)>0$ such that*

$$P\left(\sup_{i,j}\sup_{\substack{(x_1,y_1)\in A_{ij}\\(x_2,y_2)\in A_{ij}}}|B(x_1,y_1)-B(x_2,y_2)|\geqq B\frac{\log r}{\sqrt{r}}\right)=O\left(\frac{1}{r^A}\right)$$

and

$$P\left(\sup_{i,j}\sup_{\substack{(x_1,y_1)\in A_{ij}\\(x_2,y_2)\in A_{ij}}}|G_n(x_1,y_1)-G_n(x_2,y_2)|\geqq B\frac{\log r}{\sqrt{r}}\right)=O\left(\frac{1}{r^A}\right).$$

Our second statement easily follows, for example, from Theorem 2 of [4]. As to our Brownian bridge on the unit square $B(x, y)$, it can be generated by the Wiener process $W(x, y)$ of two variables as follows:

$$B(x, y) = W(x, y) - x\,y\,W(1, 1), \qquad (0 \leqq x, y \leqq 1),$$

where $W(x, y)$ has zero expectation and covariance function $\min(x_1, x_2)\min(y_1, y_2)$.

Our next lemma can be easily proved using well-known technics of Gaussian processes.

Lemma 5. *Let $K(s_1, s_2; t)$ be a Kiefer process. Then for any $A > 0$ there exists a $B = B(A) > 0$ such that*

$$P\left\{\sup_{\substack{0 \leqq t \leqq T \\ 0 \leqq s_1, s_2 \leqq 1}} |K(s_1, s_2; t)| \geqq B\sqrt{T \log T}\right\} \leqq \frac{1}{T^A}$$

if T is large enough.

Lemma 6. *Let r be an integer and let $\{\Omega, \mathscr{S}, P\}$ be a probability space. Suppose that on Ω there exist a double array $\{N_{ij}\}$ $(i, j = 1, 2, \ldots, r)$ of independent r.v.'s of standard normal law and a double array $W_{ij}(s_1, s_2)$ $(i, j = 1, 2, \ldots, r)$ of independent Wiener processes such that the arrays $\{N_{ij}\}$ and $\{W_{ij}\}$ are also independent. Then there exists a Wiener process $W(s_1, s_2)$ $(0 \leqq s_1, s_2 \leqq 1)$ on Ω such that*

$$W(i/r, j/r) = \frac{1}{r} \sum_{\substack{\alpha \leqq i \\ \beta \leqq j}} N_{\alpha\beta}. \tag{6}$$

Proof. Let

$$W_{ij}^{(1)}(s_1, s_2) = W_{ij}(s_1, s_2) - s_1 s_2 W_{ij}(1, 1) + s_1 s_2 N_{ij}, \qquad (0 \leqq s_1, s_2 \leqq 1),$$

$$W_{ij}^{(2)}(s_1, s_2) = \frac{1}{r} W_{ij}^{(1)}(r s_1, r s_2), \qquad (0 \leqq s_1, s_2 \leqq 1/r),$$

$$W_{ij}^{(3)}(s_1, s_2) = W_{ij}^{(2)}\left(s_1 - \frac{i-1}{r}, s_2 - \frac{j-1}{r}\right), \left(\frac{i-1}{r} \leqq s_1 \leqq \frac{i}{r}; \frac{j-1}{r} \leqq s_2 \leqq \frac{j}{r}\right),$$

$$\begin{aligned} W_j(s_1, s_2) &= W_{1j}(1/r, s_2) + W_{2j}(2/r, s_2) + \cdots + W_{i-1,j}\left(\frac{i-1}{r}, s_2\right) + W_{ij}(s_1, s_2) \\ &\quad \text{if } \frac{i-1}{r} \leqq s_1 \leqq \frac{i}{r}, \frac{j-1}{r} \leqq s_2 \leqq \frac{j}{r}, \\ W(s_1, s_2) &= W_1(s_1, 1/r) + W_2(s_1, 2/r) + \cdots + W_{i-1}\left(s_1, \frac{i-1}{r}\right) + W_i(s_1, s_2) \\ &\quad \left(0 \leqq s_1 \leqq 1; \frac{i-1}{\mathrm{r}} \leqq s_2 \leqq \frac{i}{\mathrm{r}}\right). \end{aligned} \tag{7}$$

It is quite clear that $W(s_1, s_2)$ of (7) is obeying condition (6). We only have to check that $W(s_1, s_2)$ is really a Wiener process, and this is rather simple. The Wiener processes of this lemma are, of course, defined as the one generating $B(x, y)$ of Lemma 4.

Lemma 7. *Let $0=t_0<t_1<t_2<\dots$ be a sequence of real numbers and let $\{\Omega, \mathcal{S}, P\}$ be a probability space. Suppose that there exist a sequence $\{B_i(s_1, s_2)\}$ of independent Brownian bridges and a sequence $\{K_i(s_1, s_2; t)\}$ of independent Kiefer processes (both of them defined on Ω) such the sequences $\{B_i\}$ and $\{K_i\}$ are mutually independent. Then there exists a Kiefer process $K(s_1, s_2, t)$ such that*

$$K(s_1, s_2, t_i)=\sqrt{t_1}\, B_2(s_1, s_2)+\sqrt{t_2-t_1}\, B_2(s_1, s_2)+\cdots+\sqrt{t_i-t_{i-1}}\, B_i(s_1, s_2). \quad (8)$$

Proof. Let

$$K_i^{(1)}(s_1, s_2; t)=K_i(s_1, s_2; t-t_{i-1})-\frac{t-t_{i-1}}{t_i-t_{i-1}} K_i(s_1, s_2; t_i-t_{i-1}) + \frac{t-t_{i-1}}{\sqrt{t_i-t_{i-1}}} B_i(s_1, s_2) \quad (t_{i-1}\leqq t\leqq t_i)$$

and let

$$\begin{aligned} K(s_1, s_2; t)=&\sqrt{t_1}\, B_1(s_1, s_2)+\sqrt{t_2-t_1}\, B_2(s_1, s_2)+\cdots \\ &+\sqrt{t_{i-1}-t_{i-2}}\, B_{i-1}(s_1, s_2)+K_i^{(1)}(s_1, s_2; t) \qquad t_{i-1}\leqq t\leqq t_i. \end{aligned} \quad (9)$$

Now condition (8) clearly holds and it is easy to see that $K(s_1, s_2; t)$ (defined by (9)) is a Kiefer process. Here the Brownian bridges $\{B_i(s_1, s_2)\}$ are defined as that of Lemma 4.

Lemma 8. *Suppose that the conditions of the Theorem hold. Then for each n one can construct a Brownian bridge $B(s_1, s_2)=B_n(s_1, s_2)$ such that*

$$P\left\{\sup_{0\leqq s_1, s_2\leqq 1} |G_n(s_1, s_2)-B(s_1, s_2)|\cdot\frac{n^{\frac{1}{6}}}{(\log n)^{\frac{3}{2}}}\geqq C\right\}=O\left(\frac{1}{n^2}\right) \quad (10)$$

where $G_n(x, y)=\sqrt{n}(F_n(x, y)-xy)$, and C is large enough.

Remark. We get the d-dimensional version of this lemma if we replace $n^{\frac{1}{6}}$ by $n^{\frac{1}{2}\frac{(d+1)}{(d+1)}}$.

Proof. Let [1] $r=n^{\frac{1}{3}}$ and define the process $B(x, y)$ as follows: let

$$N_{ij}=f_{n/r^2}\left(\frac{\beta_{ij}-n/r^2}{\sqrt{n/r^2}}\right)$$

where f_n resp. β_{ij} were defined in Lemmas 2. resp. 3. Then by Lemma 5 there exists a Wiener process $W(x, y)$ such that

$$W\left(\frac{i}{r}, \frac{j}{r}\right)=\frac{1}{r}\sum_{\substack{0\leqq\alpha\leqq i\\ 0\leqq\beta\leqq j}} N_{\alpha\beta}$$

and let $B(x, y)=W(x, y)-xy\, W(1, 1)$.

We have to show that B obeys (10).

In fact by Lemma 2

$$\left|N_{ij}-\frac{\beta_{ij}-n/r^2}{\sqrt{n/r^2}}\right|=O\left(\frac{\log n}{\sqrt{n/r^2}}\right) \qquad (i, j=1, 2, \dots, r)$$

provided that $|(\beta_{ij}-n/r^2)/\sqrt{n/r^2}|\leqq A\sqrt{\log n}$.

[1] In the d-dimensional case $r=n^{1/d+1}$.

Since

$$P\left\{\sup_{i,j}\left|\frac{\beta_{ij}-n/r^2}{\sqrt{n/r^2}}\right|>A\sqrt{\log n}\right\}\leqq 1/n^2$$

if A is large enough, we have

$$P\left\{\sup_{i,j}\left|N_{ij}-\frac{\beta_{ij}-n/r^2}{\sqrt{n/r^2}}\right|>C\frac{\log n}{\sqrt{n/r^2}}\right\}\leqq 1/n^2 \tag{11}$$

if C is large enough.

Let

$$\varepsilon_{ij}=N_{ij}-\frac{\beta_{ij}-n/r^2}{\sqrt{n/r^2}}$$

and

$$\varepsilon_{ij}^*=\begin{cases}\varepsilon_{ij} & \text{if } |\varepsilon_{ij}|\leqq C\dfrac{\log n}{\sqrt{n/r^2}}\\ 0 & \text{otherwise.}\end{cases}$$

Then

(i) the r.v.'s ε_{ij}^* are independent,

(ii) $|E\varepsilon_{ij}^*|=O\left(\frac{1}{n}\right)$;

hence

$$P\left\{\sup_{I,J}\left|\sum_{\substack{i\leqq I\\ j\leqq J}}(\varepsilon_{ij}^*-E(\varepsilon_{ij}^*))\right|\frac{\sqrt{n}}{r\log n}\geqq \mathrm{Ar}\sqrt{\log r}\right\}\leqq 1/n^2,$$

and also

$$P\left\{\sup_{I,J}\left|\sum_{\substack{i\leqq I\\ j\leqq J}}e_{ij}^*\right|\frac{\sqrt{n}}{r\log n}\geqq \mathrm{Ar}\sqrt{\log r}\right\}\leqq 1/n^2$$

if A is large enough, which, by (11), implies:

$$P\left\{\sup_{I,J}\left|\sum_{\substack{i\leqq I\\ j\leqq J}}\varepsilon_{ij}\right|\geqq \mathrm{Ar}^2\frac{1}{\sqrt{n}}(\log n)^{\frac{3}{2}}\right\}=O\left(\frac{1}{n^2}\right),$$

i.e.

$$P\left\{\sup_{I,J}\left|\frac{1}{r}\sum_{\substack{i\leqq I\\ j\leqq J}}N_{ij}-\sum_{\substack{i\leqq I\\ j\leqq J}}\frac{\beta_{ij}-n/r^2}{\sqrt{n}}\right|\geqq A\frac{r}{\sqrt{n}}(\log n)^{\frac{3}{2}}\right\}=O\left(\frac{1}{n^2}\right). \tag{12}$$

Especially if $I=J=r$ we have

$$P\left\{\left|W(1,1)-\frac{\Pi-n}{\sqrt{n}}\right|\geqq A\frac{r}{\sqrt{n}}(\log n)^{\frac{3}{2}}\right\}=O\left(\frac{1}{n^2}\right). \tag{13}$$

(12) and (13) together imply

$$P\left\{\sup_{I,J}\left|B\left(\frac{I}{r},\frac{J}{r}\right)-\mathfrak{B}_{IJ}\right|\geqq 2A\frac{r}{\sqrt{n}}(\log n)^{\frac{3}{2}}\right\}=O\left(\frac{1}{n^2}\right).$$

Now making use of Lemmas 3 and 4 one has Lemma 8.

3. Proof of the Theorem

Consider the sequence $0=n_0<n_1<n_2<\cdots$ of integers where $n_k=k^4$ and let $n_k-n_{k-1}=m_k$. Denote by $H_k(x,y)$ the empirical distribution function based on the sample

$$X_{n_{k-1}+1},\ldots,X_{n_k} \quad \text{and let} \quad \sqrt{m_k}\,\{H_k(x,y)-xy\}=\tilde{H}_k(x,y);$$

further let $B_k(x,y)$ be a Brownian bridge for which

$$P\left\{\sup_{x,y}|B_k(x,y)-\tilde{H}_k(x,y)|\,\frac{m_k^{\frac{1}{6}}}{(\log m_k)^{\frac{3}{2}}}\geqq C\right\}=O\left(\frac{1}{m_k^2}\right).$$

By Lemma 7 there exists a Kiefer process $K(s_1,s_2,t)$ for which

$$K(s_1,s_2;n_i)=\sum_{j=1}^{i}\sqrt{m_j}\,B_j(s_1,s_2).$$

It will be shown that this Kiefer process is obeying statement of our Theorem. Since

$$\sum_{k=1}^{K}\sqrt{m_k}\,H_k(x,y)=n_K\big(F_{n_K}(x,y)-xy\big),$$

one can get by Lemma 8:

$$P\left\{\sup_{x,y}\left|n_K\big(F_{n_K}(x,y)-xy\big)-K(x,y;n_K)\right|\geqq\left(A\log K\sum_{k=1}^{K}m_k^{\frac{2}{3}}(\log m_k)^3\right)^{\frac{1}{2}}\right\}=O\left(\frac{1}{K^2}\right)$$

hence our Theorem follows from Lemma 5 and the Borel-Cantelli theorem.

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