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# A New Method to Prove Strassen Type Laws of Invariance Principle. I

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## 1. Summary

A new method is developed to produce strong laws of invariance principle without making use of the Skorohod representation. As an example, it will be proved that $\lim_{n\to\infty}(S_n - W(n))/n^{1/6+\varepsilon}=0$ with probability 1, for any $\varepsilon>0$, where $S_n = X_1 + X_2 + \cdots + X_n$, $\{X_i\}$ is a sequence of i.i.d.r.v.'s with $P(X_i<t)=F(t)$, and $F(t)$ is a distribution function obeying (i), (ii) and $W(n)$ is a suitable Wiener-process. Strassen in [1], proved (under weaker conditions):

$$S_n - W(n) = O(\sqrt[4]{n \log\log n}\,\sqrt{\log n})$$

with probability one. He conjectured that if

$$S_n - W(n) = o(\sqrt[4]{n \log\log n}\,\sqrt{\log n})$$

then $F(x)=\phi(x)$ where $\phi(.)$ is the unit normal distribution function. (See also [2], [6] and [7].) Our result above is a negative answer to this question.

## 2. Introduction

In this paper we prove the following.

**Theorem.** *Let $F(x)$ be a continuous distribution function satisfying the following conditions:*

(i)
$$\int_{-\infty}^{+\infty} x\,dF(x) = \int_{-\infty}^{+\infty} x^3\,dF(x) = 0,$$
$$\int_{-\infty}^{+\infty} x^2\,dF(x) = 1, \quad \int_{-\infty}^{+\infty} x^8\,dF(x) < \infty,$$

(ii) $\limsup_{|t|\to\infty} |f(t)| < 1,$

*where $f(t)=\int_{-\infty}^{+\infty} e^{itx}\,dF(x)$ is the characteristic function of $F(x)$.*

*Then there exists a probability space $\{\Omega, \mathcal{S}, P\}$, a sequence $\{X_i\}$ of i.i.d.r.v.'s and a Wiener process $W(t)$ (both of them are defined on $\Omega$) such that*

$$P(X_1<t)=F(t)$$

*and*

$$\frac{S_n - W(n)}{n^{\frac{1}{6}+\varepsilon}} \to 0 \qquad (n\to\infty) \tag{1}$$

*with probability 1 for any $\varepsilon>0$, where $S_n = X_1 + X_2 + \cdots + X_n$.*

In this connection Berkes has also remarked that if the first $k>3$ moments of $F(x)$ agree to the corresponding first $k$ moments of $\phi(x)$ then, practically the same proof shows that the power $\frac{1}{6}+\varepsilon$ in (1) can be replaced by a smaller one.

In paragraph two some lemmas will be given while in three we prove the Theorem.

### 3. Lemmas

**Lemma A** ([3] p. 82, Theorem 25, or [4] p. 220). *Suppose that the conditions of the Theorem are fulfilled and let $X_1, X_2, \dots$ be a sequence of i.i.d.r.v.'s with $P(X_i<t)=F(t)$ and*

$$P\left(\frac{X_1+X_2+\cdots X_n}{\sqrt{n}}<t\right)=F_n(t).$$

*Then*

$$F_n(x)-\phi(x)=\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}\sum_{i=1}^{6}\frac{Q_i(x)}{n^{i/2}}+o\left(\frac{1}{n^3}\right)$$

$$=\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}\left[\frac{Q_2(x)}{n}+\frac{Q_3(x)}{n^{\frac{3}{2}}}+\frac{Q_4(x)}{n^2}+\frac{Q_5(x)}{n^{\frac{5}{2}}}+\frac{Q_6(x)}{n^3}\right]+o\left(\frac{1}{n^3}\right)$$

*uniformly in $x$, where $Q_i(x)$ $(i=1,2,\dots 6)$ is a polynomial of degree $i+3$ with coefficients depending only on the first eight moments of $F(x)$ and $Q_1(x)=0$ (since $EX_1^3$ is assumed to be 0).*

**Lemma 1.** *Under the conditions of the Theorem we have*

$$\Phi(x_n)\sim F_n(x_n) \quad \text{and} \quad 1-F_n(x_n)\sim 1-\phi(x_n)$$

*provided that $\{x_n\}$ is a sequence of real numbers for which $|x_n|\leqq c\sqrt{\log n}$, where $0<c<\sqrt{6}$ and the sign $\sim$ means asymptotic equality.*

*Proof.* This Lemma is a simple consequence of Lemma A (see also [5], Theorem 4).

**Lemma 2.** *Let $\{a_n\}$ and $\{b_n\}$ be two sequences of real numbers for which*

$$0<a_n<1, \quad 0<b_n<1,$$

$$a_n\sim b_n, \quad 1-a_n\sim 1-b_n.$$

*Then*

$$(\phi^{-1}(a_n))^2-(\phi^{-1}(b_n))^2\to 0 \quad (n\to\infty).$$

The *proof* of this statement can easily be seen via elementary calculations.

**Lemma 3.** *Suppose that the conditions of the Theorem are fulfilled and put $f_n(t)=\phi^{-1}(F_n(t))$. Then*

$$|f_n(t)-t|=O\left(\frac{(\log n)^{\frac{5}{2}}}{n}\right) \tag{2}$$

*provided that*

$$|t|\leqq c\sqrt{\log n}.$$

*where $0<c<\sqrt{6}$.*

*Proof.* By Lagrange's mean value theorem we have

$$\begin{aligned}|f_n(t)-t| &= |\phi^{-1}(F_n(t))-\phi^{-1}(\phi(t))| \\ &= |F_n(t)-\phi(t)|\cdot\left.\frac{d\phi^{-1}(y)}{dy}\right|_{y=\xi_t} \\ &= |F_n(t)-\phi(t)|\frac{1}{\phi'(\phi^{-1}(\xi_t))},\end{aligned}$$

where $\min(F_n(t),\phi(t))\leqq\xi_t\leqq\max(F_n(t),\phi(t))$. Hence, by Lemma 1, $\xi_t\approx\phi(t)$ and $1-\xi_t\approx 1-\phi(t)$. By Lemma 2 $(\phi^{-1}(\xi_t))^2-t^2\to 0$, i.e. by Lemma A

$$|f_n(t)-t|\leqq\left[\frac{e^{-t^2/2}}{\sqrt{2\pi}}\left(\frac{C(\log n)^{\frac{5}{2}}}{n}\right)+o\left(\frac{1}{n^3}\right)\right]\frac{1}{\frac{1}{\sqrt{2\pi}}e^{-t^2/2}}=O\left(\frac{(\log n)^{\frac{5}{2}}}{n}\right),$$

which proves our Lemma.

**Lemma 4.** *Let $y_0=0, y_1, y_2, \dots$ be a sequence of independent r.v.'s with*

$$P(y_i<x)=\frac{1}{\sqrt{2\pi S_i}}\int_{-\infty}^{x}e^{-u^2/2S_i}\,du$$

*where $\{S_i\}$ is a sequence of positive numbers. Further put $t_0=0$, and*

$$t_i=\sum_{j=1}^{i}S_j\qquad(i=1,2,\dots).$$

*Finally let $W_1(t), W_2(t), \dots$ be a sequence of mutually independent standard Wiener processes on the positive half line which are also independent of the sequence $\{y_i\}$.*

*Put*

$$y_0+y_1+y_2+\cdots+y_n=Z_n\qquad(n=0,1,\dots),$$

$$B_i(t)=W_i(t)-\frac{t}{S_i}W_i(S_i)\qquad(i=1,2,\dots;\ 0\leqq t\leqq S_i),$$

$$\bar{B}_i(t)=B_i(t-t_{i-1})\qquad(i=1,2,\dots;\ t_i\leqq t\leqq t_{i+1})$$

$$\xi(t)=\bar{B}_i(t)+Z_{i-1}+\frac{Z_i-Z_{i-1}}{S_i}(t-t_{i-1})$$

$$\text{if } t_{i-1}\leqq t\leqq t_i\qquad(i=1,2,\dots).$$

*Then $\xi(t)$ is a standard Wiener process.*

*Proof* is trivial.

## 4. Proof of the Theorem

Let $\{\Omega,\mathscr{S},P\}$ be a probability space which is rich enough to define a sequence $\{X_n\}$ of i.i.d.r.v.'s and a sequence $\{W_n(t)\}$ of independent Wiener processes on it such that $\{X_n\}$ and $\{W_n(t)\}$ are also independent and $P(X_n<t)=F(t)$. Further let $n_k=[k^\alpha]$ $(\frac{3}{2}<\alpha<2;\ k=0,1,2,\dots)$ and $n_j-n_{j-1}=m_j\approx\alpha j^{\alpha-1}$ $(j=1,2,\dots)$. Introduce the following notations:

$$X_{n_k+1}+X_{n_k+2}+\cdots+X_{n_{k+1}}=y_{k+1},$$

$$X_1+X_2+\cdots+X_n=S_n,$$

$$\frac{y_k}{\sqrt{m_k}}=Z_k,\qquad P(Z_k<t)=F_k(t),$$

$$f_k(t)=\phi^{-1}(F_k(t)),\qquad f_k(Z_k)=R_k,$$

$$\sum_{k=1}^{K}\sqrt{m_k}\,R_k=T_{n_K},\qquad R_k-Z_k=e_k$$

and define the event $\mathscr{F}$ as follows:

$$\mathscr{F}=\{|Z_k|>c\sqrt{\log m_k}\text{ infinitely often}\},$$

where $\sqrt{\frac{32}{7}}<c<\sqrt{6}$.

Clearly we have

$$P(R_k<t)=\phi(t)\qquad (k=1,2,\ldots)$$

and by Lemma 1 and by the Borel-Cantelli lemma $P(\mathscr{F})=0$.

By Lemma 3

$$|e_k|=O\left(\frac{(\log m_k)^{\frac{5}{2}}}{m_k}\right)\tag{3}$$

provided that $|Z_k|\leqq c\sqrt{\log m_k}$, i.e. (3) holds with probability 1, except for finitely many $k$.

Let

$$e_k^*=\begin{cases}e_k & \text{if } |Z_k|\leqq c\sqrt{\log m_k}\\ 0 & \text{otherwise,}\end{cases}$$

and

$$\alpha_k=\begin{cases}1 & \text{if } |Z_k|>c\sqrt{\log m_k}\\ 0 & \text{otherwise.}\end{cases}$$

Then

$$E(e_k^*)=E(\alpha_k\,e_k)\leqq\left(E(e_k^8)\right)^{\frac{1}{8}}\left(E(\alpha_k^{\frac{8}{7}})\right)^{\frac{7}{8}}=O\left(\frac{1}{m_k^{\frac{7c^2}{16}}}\right),$$

and, by the law of iterated logarithm,

$$\lim_{K\to\infty}\frac{\sum_{k=1}^{K}\sqrt{m_k}\,e_k}{K^{\frac{2-\alpha}{2}}(\log K)^5}=\lim_{K\to\infty}\frac{\sum_{k=1}^{K}\sqrt{m_k}\,e_k^*}{K^{\frac{2-\alpha}{2}}(\log K)^5}$$

$$=\lim_{K\to\infty}\frac{\sum_{k=1}^{K}\sqrt{m_k}(e_k^*-E\,e_k^*)}{n_K^{\frac{2-\alpha}{2\alpha}}(\log n_K)^5}+\lim_{K\to\infty}\frac{\sum_{k=1}^{K}\sqrt{m_k}\,E\,e_k^*}{K^{\frac{2-\alpha}{2}}(\log K)^5}=0;$$

hence

$$\lim_{K\to\infty} \frac{S_{n_K} - T_{n_K}}{n_K^{\frac{2-\alpha}{2\alpha}} (\log n_K)^5} = 0.$$

By Lemma 4 there exists a Wiener process $\xi(t)$ such that

$$T_{n_K} = \xi(n_K) \qquad (K = 1, 2, \ldots).$$

Then our statement follows from the following two simple relations:

$$\lim_{k\to\infty} \frac{\sup\limits_{n_k \leqq t < n_{k+1}} (\xi(t) - \xi(n_k))}{m_k^{\frac{1}{2}+\varepsilon}} = 0,$$

$$\lim_{k\to\infty} \frac{\sup\limits_{n_k \leqq n < n_{k+1}} (S_n - S_{n_k})}{m_k^{\frac{1}{2}+\varepsilon}} = 0,$$

on choosing $\alpha = \frac{3}{2} + \delta$, $\delta > 0$.

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