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Reciprocal Processes

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Introduction

The concept of a reciprocal process was first formulated by Bernstein in 1932 [1]. In 1961 Slepian exploited the reciprocal property of a particular Gaussian process to obtain explicitly a first passage time density for the process [13]. The real-valued reciprocal processes which are stationary and Gaussian are classified in [8]. The first two sections of this paper are devoted to a systematic study of reciprocal processes whose time parameter is a finite closed interval. In the second section, we define the notion of a reciprocal transition probability function. The main result is that given any reciprocal transition probability function there is a probability space supporting a reciprocal process whose transitions are governed by the given transition function. In the third section we give a method of constructing reciprocal processes from Markov processes. Given a Markov process $\{Y_t, a \leqq t \leqq b\}$ with state space (S, Σ) whose transition function has with respect to some measure λ on Σ an everywhere positive transition density $q(s, x, t, y)$, $a \leqq s < t \leqq b$, x, y in S, we obtain a reciprocal process $\{X_t, a \leqq t \leqq b\}$ by first tying down $\{Y_t, a \leqq t \leqq b\}$ at $Y_a = x$ and $Y_b = y$ and then giving (x, y) an arbitrary probability distribution on $\Sigma \times \Sigma$. This method is a generalization of one due to Schrödinger ([11, 12]) and discussed by Bernstein [1] (see also Miller's appendix on p. 202–223 of [10]). Since any Markov process is a reciprocal process, a question arises as to whether all of the processes which are constructed by this method are not only reciprocal but Markovian. We prove the following result: An endpoint distribution μ gives rise to a Markov process $\{X_t, a \leqq t \leqq b\}$ if and only if there is a product measure π on $\Sigma \times \Sigma$ for which $d\mu/d\pi = q$, where $q(x, y) = q(a, x; b, y)$. For example, it is easy to see that if we reproduce the original process by taking for μ the original joint distribution of Y_a and Y_b, it is of this form (as indeed it must be if the result is at all valid). Two questions arise. First, are there any other probability distributions on $\Sigma \times \Sigma$ which are of this form? (If not, the original process $\{Y_t, a \leqq t \leqq b\}$ is the only one of the derived processes $\{X_t, a \leqq t \leqq b\}$ which is Markov.) We show that under quite general conditions, the answer is yes: In fact, given any probability measures μ_1 and μ_2 on Σ there is a measure μ having μ_1 and μ_2 for marginals for which $d\mu/d\pi = q$ for some product measure π on $\Sigma \times \Sigma$. Thus our construction yields a Markov process $\{X_t, a \leqq t \leqq b\}$ with prescribed distributions for X_a and X_b. If μ_1 and μ_2 are absolutely continuous with respect to λ, finding such a μ amounts to solving a pair of nonlinear functional equations first derived by Schrödinger ([11] and [12]) in a completely different way based on considerations partly physical and partly probabilistic, which seem to have no

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connection with the Markovian or non-markovian nature of the process so constructed. The problem of the existence and uniqueness of solutions to Schrödinger's functional equations was first treated systematically by Fortet [7]. Beurling [2] has formulated and analyzed a more general problem which includes ours as a special case. He obtains not only existence but uniqueness of the solution in case S is locally compact, q is bounded and continuous, and $\iint \log q(x, y)\,\mu_1(dx)\,\mu_2(dy)$ is finite. We are able to remove this last condition. The uniqueness part of the result answers a second question which arises, namely, do perhaps all probability measures μ on $\Sigma \times \Sigma$ satisfy $d\mu/d\pi = q$ for some product measure π on $\Sigma \times \Sigma$? If this were so, our construction would not yield any reciprocal processes which are not Markov. (We remark that Bernstein [1] seemed unaware that Schrödinger's construction, with endpoint measures obtained via his functional equations, yields only Markov processes.) However, if we are given probability measures μ_1 and μ_2 on Σ, exactly one of the processes $\{X_t, a \leqq t \leqq b\}$ with the distributions of X_a and X_b given by μ_1 and μ_2 respectively is Markov, all the rest being reciprocal but not Markov. (There are as many processes constructed with the distributions of X_a and X_b so prescribed as there are probability measures μ in $\Sigma \times \Sigma$ with marginals μ_1 and μ_2.)

The reciprocal processes constructed from Markov processes by the method of the third section have transition functions which are absolutely continuous with respect to the reference measure λ. In the last section we examine the question of whether the converse holds: that is, given a reciprocal process whose transition function is absolutely continuous with respect to λ, is there a Markov process from which it can be constructed by our method? Our answer is in the partial affirmative.

There are a number of equations in this paper in which strict equality is indicated, but which actually hold almost everywhere with respect to some measure. The necessity for such a qualification will in each case be clear from the context.

§ 1

We begin by defining our basic notion. (S, Σ) is an arbitrary measurable space.

Definition. Let $\{X_t, a \leqq t \leqq b\}$ be an (S, Σ)-valued stochastic process on the finite closed interval $[a, b]$ with underlying probability space $(\Omega, \mathscr{A}, P)$. We say that $\{X_t, a \leqq t \leqq b\}$ is a *reciprocal process* if, for each $a \leqq s < t \leqq b$,

$$P(AB|X_s, X_t) = P(A|X_2, X_t)\,P(B|X_s, X_t)$$

whenever A belongs to the σ-field generated by the random variables $\{X_r\colon a \leqq r < s$ or $t < r \leqq b\}$ and B to the σ-field generated by $\{X_r\colon s < r < t\}$.

The following two lemmas are proved in [8].

Lemma 1.1. *The process $X_t, a \leqq t \leqq b$ is reciprocal if and only if*

$$E\{f(X_n)|X_{s_1}, \ldots, X_{s_n}, X_t, X_v\} = E\{f(X_n)|X_t, X_v\} \tag{1.1}$$

for each $a \leqq t < u < v \leqq b$, $\{s_1, \ldots, s_n\} \subset [a, b] - (t, v)$. and bounded Borel-measurable f.

Lemma 1.2. *If $\{X_t, a \leqq t \leqq b\}$ is a Markov process, then it is a reciprocal process.*

The following lemma is referred to in the next section.

Lemma 1.3. *Suppose $\{X_t, a \leqq t \leqq b\}$ is a reciprocal process, that $a \leqq s < t < u < v \leqq b$, and that f and g are bounded Borel functions. Then*

$$E\{f(X_t)\, E\{g(X_u)|X_t, X_v\}\,|X_s, X_v\} = E\{g(X_u)\, E\{f(X_t)|X_s, X_u\}\,|X_s, X_v\}. \quad (1.2)$$

Proof. Using the reciprocal property, we have

$$\begin{aligned} & E\{f(X_t)\, E\{g(X_u)|X_t, X_v\}\,|X_s, X_v\} \\ &= E\{f(X_t)\, E\{g(X_u)|X_s, X_t, X_v\}\,|X_s, X_v\} \\ &= E\{E\{f(X_t)\, g(X_u)|X_s, X_t, X_v\}\,|X_s, X_v\} \\ &= E\{f(X_t)\, g(X_u)|X_s, X_v\} \\ &= E\{E\{f(X_t)\, g(X_u)|X_s, X_u, X_v\}\,|X_s, X_v\} \\ &= E\{g(X_u)\, E\{f(X_t)|X_s, X_u, X_v\}\,|X_s, X_v\} \\ &= E\{g(X_u)\, E\{f(X_t)|X_s, X_u\}\,|X_s, X_v\}. \end{aligned}$$

Lemma 1.4. *If $\{X_t, a \leqq t \leqq b\}$ is a reciprocal process, and either X_a or X_b is a.s. constant, then it is a Markov process.*

Proof. First, suppose X_b is constant a.s. Then, if $a \leqq t_1, < \cdots < t_n < u \leqq b$, and if f is bounded measurable,

$$\begin{aligned} E\{f(X_u)|X_{t_1}, \ldots, X_{t_n}\} &= E\{f(X_u)|X_{t_1}, \ldots, X_{t_n}, X_b\} \\ &= E\{f(X_u)|X_{t_n}, X_b\} \\ &= E\{f(X_n)|X_{t_n}\}. \end{aligned}$$

Thus $\{X_t, a \leqq t \leqq b\}$ is Markov. Since the Markov and reciprocal properties are both preserved under reversal of the time direction, the conclusion also holds if X_a is constant a.s.

§ 2

We begin by defining axiomatically a class of reciprocal transition probability functions which are to reciprocal processes what transition probability functions are to Markov processes (for the latter, see [9], Section 38.2). First let $I = [a, b]$ be a closed interval of real numbers. Let (S, Σ) be a measurable space. We use $\mathscr{D}$ to denote the set of all ordered sextuples (s, x, t, E, u, y) for which x and y are in S, $a \leqq s < t < u \leqq b$, and $E \in \Sigma$. A real valued function P on $\mathscr{D}$ is called a *reciprocal transition probability function* if the following three conditions are satisfied:

A 1. For each x and y in S and $a \leqq s < t < u \leqq b$, the map

$$E \to P(s, x, t, E, u, y), \qquad E \in \Sigma$$

defines a probability measure on Σ.

A 2. For each $E \in \Sigma$ and $a \leqq s < t < u < v \leqq b$, the map

$$(x, y) \to P(s, x, t, E, u, y)$$

is $\Sigma \times \Sigma$-measurable.

A 3. For each $a \leqq s < t < u < v \leqq b$, $C \in \Sigma$, $D \in \Sigma$, $x \in S$, and $y \in S$,

$$\int_D P(s, x, u, d\xi, v, y)\, P(s, x, t, C, u, \xi) = \int_C P(s, x, t, d\eta, v, y)\, P(t, \eta, u, D, v, y).$$

Intuitively, $P(s, x, t, E, u, y)$ is the probability that a particle located at x at time s and at y at time u is in the set E at time t. To help keep this in mind, we write $P(s, x; t, E; u, y)$ for $P(s, x, t, E, u, y)$. The following are two consequences of A 1–A 3. The first is obtained by setting $C = A$ and $D = S$ in A 3 and applying A 1, the second by setting $C = S$ and $D = A$ in A 3.

For each $a \leqq s < t < u < v \leqq b$, $A \in \Sigma$, $x \in S$, and $y \in S$,

A 4. $$\int P(s, x; t, d\eta; v, y)\, P(t, \eta; u, A; v, y) = P(s, x; u, A; v, y)$$

and

A 5. $$\int P(s, x; u, d\xi; v, y)\, P(s, x; t, A; u, \xi) = P(s, x; t, A; v, y).$$

Let Ω be the set of all S-valued functions on $[a, b]$. For each $t \in [a, b]$, we denote by X_t the function on Ω for which $X_t(\omega) = \omega(t)$, $\omega \in \Omega$. The smallest σ-field $\mathscr{G}$ on Ω relative to which X_t is $\mathscr{G} - \Sigma$ measurable for each $t \in [a, b]$ is denoted by $\mathscr{I}$.

Theorem 2.1. *Assume that S is a σ-compact Hausdorff space, with Σ the σ-field generated by the open sets. Let $P(s, x; t, E; u, y)$ be a reciprocal transition probability function as defined above, and let μ be a probability measure on $\Sigma \times \Sigma$. Then there is a probability measure P_μ on $\mathscr{I}$ such that, relative to the probability space $(\Omega, \mathscr{I}, P_\mu)$, $\{X_t, a \leqq t \leqq b\}$ is a reciprocal process for which*

(i) $$P_\mu\{X_a \in A, X_b \in B\} = \mu(A \times B), \qquad A \in \Sigma,\ B \in \Sigma,$$

and

(ii) *for all $a \leqq s < t < u \leqq b$ and $A \in \Sigma$,*

$$P_\mu(X_t \in A \mid X_s, X_u\} = P(s, X_s; t, A; u, X_u).$$

There is only one such measure, and its finite-dimensional distributions are given as follows. Suppose $a < t_1 < \cdots < t_n < b$, $A \in \Sigma$, $B \in \Sigma$, and $E_i \in \Sigma$, $i = 1, \ldots, n$. Let

$$\Lambda = \{X_a \in A, X_{t_1} \in E_1, \ldots, X_{t_n} \in E_n, X_b \in B\}. \tag{2.1}$$

Then $P_\mu(\Lambda)$ is equal to

$$\int_{A \times B} d\mu(x, y) \int_{E_1} P(a, x; t_1, dz_1; b, y) \ldots \int_{E_{n-1}} P(t_{n-2}, z_{n-2}; t_{n-1}, dz_{n-1}; b, y)\, P(t_{n-1}, z_{n-1}; t, E_n; b, y). \tag{2.2}$$

Proof. We begin by showing that if $\{X_t, a \leqq t \leqq b\}$ is a reciprocal process on $(\Omega, \mathscr{I}, P)$ relative to which (i) and (ii) hold (with "P_μ" replaced by "P"), then, if Λ is given by (2.1), $P(\Lambda)$ is given by (2.2). (This will, of course, establish the uniqueness asserted by the theorem.) To this end let $a \leqq t_1 < \cdots < t_n < b$, and $E_i \in \Sigma$, $i = 1, \ldots, n$. I claim

$$\begin{aligned} &P\{X_{t_2} \in E_2, \ldots, X_{t_n} \in E_n | X_{t_1}, X_b\} \\ &\quad = \int_{E_2} P(t_1, X_{t_1}; t_2, dz_2; b, X_b) \int_{E_3} P(t_2, x_2; t_3, dx_3; b, X_b) \ldots \\ &\quad \int_{E_{n-1}} P(t_{n-2}, x_{n-2}; t_{n-1}, dx_{n-1}; b, X_b) \, P(t_{n-1}, x_{n-1}; t_n, E_n; b, X_b. \end{aligned} \tag{2.3}$$

We prove (2.3) by induction on n. For $n = 1$, it reduces to (ii), which we are assuming. Note that (ii) also implies that

$$E\{f(X_{t_1}, X_b) | X_a, X_b\} = \int P(a, X_a; t_1, dz; b, X_b) f(x, X_b)$$

for any bounded $\Sigma \times \Sigma$-measurable f on $S \times S$, and that there is such an f for which the right hand side of (2.3) is $f(X_{t_1}, X_b)$. Assuming that (2.3) holds as it stands, we have, if $a \leqq t_0 < t_1 < \cdots < t_n < b$

$$\begin{aligned} &P\{X_{t_1} \in E_1, \ldots, X_{t_n} \in E_n | X_{t_0}, X_b\} \\ &\quad = E\{I_{E_1}(X_{t_1}) P\{X_{t_2} \in E_2, \ldots, X_{t_n} \in E_n | X_{t_0}, X_{t_1}, X_b\} | X_{t_0}, X_b\} \\ &\quad = E\{I_{E_1}(X_{t_1}) P\{X_{t_2} \in E_2, \ldots, X_{t_n} \in E_n | X_{t_1}, X_b\} | X_{t_0}, X_b\} \\ &\quad = E\{I_{E_1}(X_{t_1}) f(X_{t_1}, X_b) | X_{t_0}, X_b\} \\ &\quad = \int_{E_1} P(t_0, X_{t_0}; t_1, dz_1; b, X_b) f(X_{t_1}, X_b) \\ &\quad = \int_{E_1} P(t_0, x_{t_0}; t_1, z_1; b, X_b) \int_{E_2} P(t_1, X_{t_1}; t_2, dz_1; b, X_b) \ldots \\ &\quad \int_{E_{n-1}} P(t_{n-2}, X_{n-2}; t_{n-1}, dx_{n-1}; b, X_b) \, P(t_{n-1}, X_{n-1}; t, E_n, b, X_b). \end{aligned} \tag{2.4}$$

This shows that (2.3) holds for all n. Using (2.4) for $t_0 = a$, the fact that

$$P(\Lambda) = E\{I_A(X_a) \, I_B(X_b) \, P\{X_{t_1} \in E_1, \ldots, X_{t_n} \in E_n | X_a, X_b\}\},$$

and (i), we conclude that $P(\Lambda)$ is equal to expression (2.2).

Next, we construct P_μ. Rather than using a consistency argument to extend the set function defined by (2.2) to $\mathscr{I}$, we proceed indirectly. Fix $y \in S$. For each $a \leqq s < t < b$, $z \in S$, and $E \in \Sigma$, set

$$Q_y(s, z; t, E) = P(s, z; t, E; b, y).$$

Then, if $a \leqq s < t < u < b$, we have, using (A 4),

$$\begin{aligned} \int Q_y(s, z; t, d\eta) \, Q_y(t, \eta; u, E) &= \int P(s, z; t, d\eta; b, y) \, P(t, \eta; u, E; b, y) \\ &= P(s, z; u, E; b, y) \\ &= Q_y(s, z; u, E). \end{aligned}$$

It follows that $Q_y(s, z; t, E)$ is a (Markov) transition probability function. Let Ω_0 be the set of all functions from $[a, b)$ into S, and $\mathscr{I}_0$ the smallest σ-field on Ω_0 rendering measurable all the coordinate functions $X_t, a \leqq t < b$. Because of our assumptions on (S, Σ) it follows ([4], p. 16) that given any probability measure γ

on Σ there is a measure $\tilde{Q}_{y,\gamma}$ on $\mathscr{I}_0$ such that, relative to $(\Omega_0, \mathscr{I}_0, \tilde{Q}_{y,\gamma})$, $\{X_t, a \leq t < b\}$ is a Markov process with γ as initial measure and $Q_y(s, z; t, E)$ as transition probability function ([9], p. 569). Now, $(S \times S, \Sigma \times \Sigma, \mu)$ is a probability space. Let X and Y be the random variables defined thereon by $X(x, y) = x$ and $Y(x, y) = y$ for all $(x, y) \in S \times S$. Let ν be the conditional distribution of X given Y ([9], p. 359). Then ν is defined on $S \times \Sigma$, $\nu(y, \cdot)$ is a probability measure on Σ for each $y \in S$ and $\nu(\cdot, E)$ is Σ-measurable for each $E \in \Sigma$. Using $\nu(y, \cdot)$ as the initial measure we define $\tilde{Q}_y$ on $\mathscr{I}_0$ as above. Checking first the case where Δ is a cylinder with finite-dimensional base, we see that $\tilde{Q}_y(\Delta)$ is a Σ-measurable function of y for each $\Delta \in \mathscr{I}_0$. Let η be the distribution of Y; that is, $\eta(F) = \mu(S \times F)$ for each $F \in \Sigma$. We define P_μ on $\mathscr{I}_0 \times \Sigma$ by

$$P_\mu(\Delta \times F) = \int_F \eta(dy)\, \tilde{Q}_y(\Delta) \qquad \Delta \in \mathscr{I}_0,\ F \in \Sigma. \tag{2.5}$$

It is observed on p. 359 of [9] that this indeed defines a measure on $\mathscr{I}_0 \times \Sigma$. The measure P_μ is not yet defined on $\mathscr{I}$ as promised. But the correspondence $\omega \leftrightarrow (\omega_0, \omega(b))$ between Ω and $\Omega_0 \times S$, where ω_0 is the restriction to $[a, b)$ of $\omega \in \Omega$, is one-to-one and $\mathscr{I} - \mathscr{I}_0 \times \Sigma$ bimeasurable, permitting us to identify the measurable spaces $(\Omega, \mathscr{I})$ and $(\Omega_0 \times S, \mathscr{I}_0 \times \Sigma)$. Accordingly, (2.5) does define a probability measure on $\mathscr{I}$.

Next, we verify that if Λ is as in (2.1), then $P_\mu(\Lambda)$ is given by (2.2). First, suppose that f is a bounded $\Sigma \times \Sigma$-measurable function on $S \times S$. Then the definitions of γ and ν easily yield $\int \gamma(dy) \int \nu(y, dx) f(x, y) = \int f\, d\mu$; consequently,

$$\int_B \gamma(dy) \int_A \nu(y, dx) f(x, y) = \int_{A \times B} f\, d\mu \tag{2.6}$$

for any $A \in \Sigma$, $B \in \Sigma$. Now let

$$f(x, y) = \int_{E_1} Q_y(a, x; t_1, dz_1) \ldots \int_{E_{n-1}} Q_y(t_{n-1}, dx_{n-1}; t_n, A_n)$$

and observe ([9], p. 569) that if

$$\Delta = \{\omega \in \Omega_0 \colon X_a(\omega) \in A, X_{t_1}(\omega) \in E_1, \ldots, X_{t_n}(\omega) \in E_n\},$$

then

$$\tilde{Q}_y(\Delta) = \int_A \nu(y, dx) f(x, y). \tag{2.7}$$

If Λ is given by (2.1), we identify Λ with $\Delta \times B$, so combining (2.5), (2.6) and (2.7) we see that $P_\mu(\Lambda)$ is indeed given by (2.2). It is evident from (2.2) that (i) holds.

We next show that (ii) holds. Suppose $a < t < u < v < b$. It is easy to see from the form (2.2) of the finite dimensional distributions that

$$\int h(X_t, X_v)\, dP_\mu = \int d\mu(x, y)\, P(a, x; t, dw; b, y) \int P(t, w; v, dz; b, y)\, h(w, z) \tag{2.8}$$

for all bounded $\Sigma \times \Sigma$-measurable functions h on $S \times S$. Let $B \in \Sigma$, $C \in \Sigma$, $D \in \Sigma$. Let

$$h(w, z) = P(t, w; u, C; v, z)\, I_B(w)\, I_D(z),$$

and apply (2.8) to obtain

$$\int_{\{X_t\in B,\, X_v\in D\}} P(t, X_t; u, C; v; X_v)\, dP_\mu = \int d\mu(x,y) \int_B P(a,x;t,dw;b,y) \int_D P(t,w;v,dz;b,y)\, P(t,w;u,C;v,z). \tag{2.9}$$

By (A 3), however,

$$\int_D P(t,w;v,dz;b,y)\, P(t,w;u,C;v,z) = \int_C P(t,w;u,d\eta;b,y)\, P(u,\eta;v,D;b,y).$$

Substituting the right hand side of this last expression into the right hand side of (2.9), and referring to (2.2), we see that

$$\int_{\{X_t\in B,\, X_v\in D\}} P(t, X_t; u, C; v, X_v)\, dP_\mu = P_\mu\{X_t\in B, X_u\in C, X_v\in D\}.$$

Since this holds for all B, D in Σ it follows that

$$P_\mu\{X_u\in C \mid X_t, X_v\} = P_\mu(t, X_t; u, C; v, X_v).$$

A similar argument shows that this last also holds if $t=a$ or $v=b$. Thus (ii) is proved.

We complete the proof of the theorem by establishing the reciprocal property of $\{X_t, a\leqq t\leqq b\}$ relative to $(\Omega, \mathscr{I}, P_\mu)$. Suppose that $a<t_n<\cdots<t_1<t<u<v<v_1<\cdots<v_m<b$, and that $C\in\Sigma$. We will show that

$$P_\mu\{X_u\in C \mid X_a, X_{t_n}, \ldots, X_{t_1}, X_t, X_v, X_{v_1}, \ldots, X_{v_m}, X_b\} = P(t, X_t; u, C; v, X_v).$$

To do this, we must show that

$$\int_{\Lambda\Delta} P(t, X_t; u, C; v, X_v)\, dP_\mu = P_\mu(\Lambda\Delta\{X_u\in C\}), \tag{2.10}$$

whenever

$$\Lambda = \{X_a\in A, X_{t_n}\in D_n, \ldots, X_{t_1}\in D_1, X_t\in D\}$$

and

$$\Delta = \{X_v\in E, X_{v_1}\in E_1, \ldots, X_{v_m}\in E_m, X_b\in B\}$$

with $A, D_n, \ldots, D_1, D, E, E_1, \ldots, E_m$ all in Σ.

To this end, let

$$K(y,z) = \int_{E_2} P(v,y;v_1,dy_1;b,z) \ldots \int_{E_{m-1}} P(v_{m-2}, y_{m-2}; v_{m-1}, dy_{m-1}; b, z)\, P(v_{m-1}, y_{m-1}; v_m, E_m; b, z).$$

It follows from (2.2) that if f is any bounded $\Sigma\times\Sigma$-measurable function on $S\times S$, then

$$\int_{\Lambda\Delta} f(X_t, X_v)\, dP_\mu = \int_{A\times B} d\mu(w,z) \int_{D_n} P(a,w;t_n,dx_n;b,z) \ldots \int_{D_n} P(t_n, x_n; t_{n-1}, dx_{n-1}; b, z) \int_D P(t_1, x_1; t, dx; b, z)\, F(x,z), \tag{2.11}$$

where

$$F(x,z)=\int_D P(t,x;v,dy;b,z)f(x,y)K(y,z).$$

In particular, if $f(x,y)=P(t,x;u,C;v,y)$, the left hand side of (2.10) is equal to the right hand side of (2.11) with

$$F(x,z)=\int_D P(t,x;v,dy;b,z)P(t,x;u,C;v,y)K(x,z). \tag{2.12}$$

Ba (A 3),

$$\int_D P(t,x;v,dy;b,z)P(t,x;u,C;v,y)=\int_C P(t,x;u,d\eta;b,z)P(u,\eta;v,D;b,z),$$

and it easily follows that

$$\begin{aligned}\int_D &P(t,x;v,dy;b,z)P(t,x;u,C;v,y)K(y,z)\\ &=\int_C P(t,x;u,d\eta;b,z)\int_D P(u,\eta;v,dy;b,z)K(y,z).\end{aligned} \tag{2.13}$$

Substituting the right hand side of (2.13) for $F(x,z)$ into the right hand side of (2.11), and referring to (2.2) again, we obtain (2.10). Thus $\{X_t, a\leqq t\leqq b\}$ (as a process on $(\Omega,\mathscr{I},P_\mu)$) is reciprocal, and the proof of the theorem is complete.

If $\{X_t, a\leqq t\leqq b\}$ is a reciprocal process, and if we define $P(s,x;t,E;u,y)$ to be a conditional distribution satisfying (ii) of the theorem with appropriate almost everywhere qualifications, A 3 must hold (with similar qualifications), as is seen by setting $f=I_C$ and $g=I_D$ in Lemma 1.3. This shows that A 3 is not too strong a condition to impose on reciprocal transition functions.

§ 3

Suppose $\{Y_t, a\leqq t\leqq b\}$ is a Markov process with Markov transition probability function $Q(s,x,t,E)$, $a\leqq s<t\leqq b$, $x\in S$, $E\in\Sigma$. We assume that Q is given by a positive density relative to some σ-finite measure λ on Σ; that is, there is a strictly positive function $q(s,x;t,y)$ defined for $a\leqq s<t\leqq b$ and $(x,y)\in S\times S$, Σ-measurable in (x,y) for each s and t, and for which

$$Q(s,x,t,E)=\int_E q(s,x;t,y)\,\lambda(dy)\qquad a\leqq t\leqq b,\ x\in S,\ E\in\Sigma. \tag{3.1}$$

We define

$$p(s,x;t,y;u,z)=\frac{q(s,x;t,y)\,q(t,y;u,z)}{q(s,x;u,z)},\quad a\leqq s<t<u\leqq b,\ (x,y,z)\in S\times S\times S, \tag{3.2}$$

and

$$\begin{aligned}P(s,x;t,E;u,y)&=\int_E p(s,x;t,z;u,y)\,\lambda(dz),\\ &\qquad a\leqq s<t<u\leqq b,\ (x,y)\in S\times S,\ E\in\Sigma.\end{aligned} \tag{3.3}$$

It is easy to verify that $P(s,x;t,E;u,y)$ is a reciprocal transition probability function; we say that it is *derived from* $q(s,x;t,y)$. We observe that $P(s,Y_s;t,E;u,Y_u)$ is a version of $P(Y_t\in E\mid Y_s,Y_u)$.

Let μ be an arbitrary probability measure on $\Sigma\times\Sigma$. By virtue of theorem 2.1, if S is a σ-compact Hausdorff space with Σ its topological Borel sets there is a unique measure P_μ on the measurable space $(\Omega, \mathscr{F})$ of paths such that the coordinate functions $\{X_t, a \leqq t \leqq b\}$ constitute a reciprocal process for which

(i) $$P_\mu(X_a \in A, X_b \in B) = \mu(A \times B) \qquad A \in \Sigma, B \in \Sigma,$$

and

(ii) $$P_\mu(X_t \in A \mid X_s, X_u) = P(s, X_s; t, A; u, X_u), \qquad A \in \Sigma,\ a \leqq s < t \leqq u \leqq b.$$

We call μ the *(joint) endpoint distribution* of $\{X_t, a \leqq t \leqq b\}$. The measures μ_a and μ_b defined by $\mu_a(E) = \mu(E \times S)$ and $\mu_b(E) = \mu(S \times E)$ are called the *marginal endpoint distributions.* We denote the joint distribution of X_s and X_t by $\mu_{s,t}$ for $a \leqq s < t \leqq b$. Thus $\mu_{a,b} = \mu$. The distribution of X_s is denoted by μ_s, $a \leqq s \leqq b$. If either μ_a or μ_b concentrates all its mass on a single point of S, $\{X_t, a \leqq t \leqq b\}$ is not only reciprocal but Markovian by virtue of Lemma 1.4. In the following theorem we characterize for S metric all endpoint distributions μ for which $\{X_t, a \leqq t \leqq b\}$ is a Markov process.

Theorem 3.1. *Let $Q(s, x; t, E)$, $a \leqq s < t \leqq b$, $x \in S$, $E \in \Sigma$ be a Markov transition probability function. Assume that S is a σ-compact metric space and that Σ is the σ-field of topological Borel sets C. (Then Σ is generated by a countable class of sets.) Suppose there is a σ-finite measure λ on Σ and a function $q(s, x; t, y)$, $a \leqq s < t \leqq b$, $(x, y) \in S \times S$ which is strictly positive, $\Sigma \times \Sigma$-measurable in (x, y), and for which* (3.1) *holds. Let $P(s, x; t, E; u, y)$, $a \leqq s < t \leqq b$, $(x, y) \in S \times S$, $E \in \Sigma$, be the reciprocal probability function derived from $q(s, x; t, y)$, let μ be a probability measure on $\Sigma \times \Sigma$, and let X_t, $\{a \leqq t \leqq b\}$ be the corresponding reciprocal process with endpoint distribution μ. The following are equivalent:*

(a) *$\{X_t, a \leqq t \leqq b\}$ is a Markov process.*

(b) *There are measures ν_a and ν_b on Σ such that*

$$\mu(G) = \int_G q(a, x; b, y)\, d(\nu_a \times \nu_b)(x, y), \qquad G \in \Sigma \times \Sigma.$$

Proof. (b)$\Rightarrow$(a). Suppose (b) holds. Let $a < t_1 < \cdots < t_n < b$, and $E_i \in \Sigma$, $i = 1, \ldots, n$. For each $(z_1, \ldots, z_n) \in S^n$ let

$$\alpha(z_1, \ldots, z_n) = q(t_1, z_1; t_2, z_2) \cdot \cdots \cdot q(t_{n-1}, z_{n-1}; t_n, z_n).$$

Let f be any non-negative Σ-measurable function on S. Referring to (2.2), (3.2), and (3.3), we see, after some cancellations, that

$$\int_{\{X_{t_1} \in E_1, \ldots, X_{t_n} \in E_n\}} f(X_{t_n})\, dP$$
$$= \int_{S \times E_1 \times \cdots \times E_n \times S} q(a, x; t_1, z_1)\, \alpha(z_1, \ldots, z_n)\, q(t_n, z_n; b, y) f(z_n)\, d\gamma(x, z_1, \ldots, z_n, y),$$

where γ is the product measure $\nu_a \times \lambda^n \times \nu_b$, λ^n being the n-fold product of λ with itself. This last expression can be written as

$$\int_{S \times E_1 \times \cdots \times E_n} q(a, x; t_1, z_1)\, \alpha(z_1, \ldots, z_n) f(z_n)$$
$$\cdot \left[\int_S q(t_n, z_n; b, y)\, \nu_b(dy)\right] d\bar{\gamma}(x, z_1, \ldots, z_n),$$

with $\bar{\gamma} = \nu_a \times \lambda^n$. Suppose f is defined by

$$f(w) = \frac{\int\limits_{F \times S} q(t, z; b, y)\, d(\lambda \times \nu_b)(z, y)}{\int\limits_S q(t_n, w; b, y)\, \nu_b(dy)}$$

where $F \in \Sigma$. Substituting in the previous expression, we have

$$\begin{aligned} \int\limits_{\{X_{t_1} \in E_1, \ldots, X_{t_n} \in E_n\}} f(X_{t_n})\, dP_\mu = & \int\limits_{S \times E_1 \times \cdots \times E_n} q(a, x; t_1, z_1)\, \alpha(z_1, \ldots, z_n) \\ & \cdot \Big[\int\limits_{F \times S} q(t, z; b, y)\, d(\lambda \times \nu_b)(z, y) \Big]\, d\bar{\gamma}(x, z_1, \ldots, z_n) \\ & \int\limits_{S \times E_1 \times \cdots \times E_n \times F \times S} q(a, x; t_1, z_1)\, \alpha(z_1, \ldots, z_n) \\ & \cdot q(t, z; b, y)\, d\rho(x, z_1, \ldots, z_n, z, y), \end{aligned}$$

where $\rho = \nu_a \times \lambda^{n+1} \times \nu_b$. Again using (2.2), (3.2), and (3.3), we see that this last expression is equal to $P_\mu(X_{t_1} \in E_1, \ldots, X_{t_n} \in E_n, X_t \in F)$. Since all this is independent of the choice of $E_1, \ldots, E_n$, what we have shown is that

$$P(X_t \in F \mid X_{t_1}, \ldots, X_{t_n}) = f(X_{t_n}),$$

whence $P(X_t \in F \mid X_{t_1}, \ldots, X_{t_n}) = P(X_t \in F \mid X_{t_n})$. Similar calculations lead to the same conclusion if $t_1 = a$ or $t_n = b$ or both. Thus (a) holds.

(a)$\Rightarrow$(b). Suppose (a) holds. Then there is a Markov transition probability function $\bar{Q}(s, x; t, E)$, $a \leqq s < t \leqq b$, $E \in \Sigma$, for the Markov processes $\{X_t, a \leqq t \leqq b\}$, and we may assume that $\bar{Q}$ satisfies the Chapman-Kolmogorov equations in the following sense: for each $E \in \Sigma$ and $a \leqq s < t < u \leqq b$,

$$\bar{Q}(s, x; u, E) = \int \bar{Q}(s, x; t, dy)\, \bar{Q}(t, y; b, E) \tag{3.4}$$

for μ_s-almost all $x \in S$. Then

$$\mu(E \times F) = \int\limits_E \mu_a(dx)\, \bar{Q}(a, x; b, F) \tag{3.5}$$

for E, F in Σ.

(i) *For each* $a \leqq s < t < b$, $\bar{Q}(s, x; t, \cdot)$ *is equivalent to* λ *for* μ_s*-almost all* $x \in S$.

To prove (i) it suffices to verify that

$$\bar{Q}(s, X_s; t, F) = \int \bar{Q}(s, X_s; b, dy) \int\limits_F p(s, X_s; t, z; b, y)\, \lambda(dz) \tag{3.6}$$

for each $F \in \Sigma$ except for a P_μ-null set of $\omega \in \Omega$. For each $\omega \in \Omega$, however, both sides of (3.6) are, as functions of F, probability measures on Σ. Since Σ is generated by a countable subfield, it suffices to show that, for each $F \in \Sigma$, (3.6) holds P_μ-almost surely. It follows from the definition of P_μ that

$$P_\mu(X_s \in E, X_t \in F) = \int\limits_{E \times S} d\mu_{s,b}(x, y) \int\limits_F p(s, x; t, z; b, y)\, \lambda(dz). \tag{3.7}$$

Also

$$\mu_{s,b}(A \times B) = \int\limits_A \mu_s(dx)\, \bar{Q}(s, x; b, B), \tag{3.8}$$

which implies that for any non-negative $\Sigma \times \Sigma$-measurable f,

$$\int_{E\times S} d\mu_{s,b} f(x,y) = \int_E \mu_s(dx) \int_S \bar{Q}(s,x;b,dy) f(x,y).$$

If

$$f(x,y) = \int_F p(s,x;t,z;b,y)\,\lambda(dz),$$

we have, using (3.7),

$$P_\mu(X_s\in E, X_t\in F) = \int_E \mu_s(dx) \int \bar{Q}(s,x;b,dy) \int_F p(s,x;t,z;b,y)\,\lambda(dz).$$

Since this last equation holds for each $E\in\Sigma$, it follows that the right hand side of (3.6) is a version of $P_\mu(X_t\in F\,|\,X_s)$. But so is $\bar{Q}(s,X_s;t,F)$, whence (3.6) follows P_μ-almost surely. This completes the proof of (i).

(ii) *For each $a\leqq s<b$, $\bar{Q}(s,x;b,\cdot)$ is absolutely continuous with respect to μ_b for μ_s-almost all $x\in S$.*

To prove (ii), pick $t\in(s,b)$ and observe that since $\mu_b(E)=\int \mu_s(dx)\,\bar{Q}(s,x;b,E)$ we have

$$\mu_b(E) = \int \mu_s(dx) \int \bar{Q}(s,x;t,dy)\,\bar{Q}(t,y;b,E) \tag{3.9}$$

by virtue of (3.4). Let $S'=\{x\colon \bar{Q}(s,x;t,\cdot)\sim\lambda\}$. The complement of S' is μ_s-null by (i). Let S'' be the set of $x\in S$ such that (3.4) holds for all $E\in\Sigma$. Since Σ is generated by a countable field, the complement of S'' is μ_s-null. Let $S_0=S'\cap S''$. Suppose $x_0\in S_0$, and $\mu_b(E)=0$. Then $\bar{Q}(s,x;b,E)=0$ for μ_s-almost all $x\in S$, hence for some $x_1\in S_0$. Since (3.4) holds for $x=x_1$, $\{y\colon \bar{Q}(t,y;b,E)>0\}$ has $\bar{Q}(s,x_1;t,\cdot)$-measure zero, hence λ-measure zero, hence $\bar{Q}(s,x_0;t,\cdot)$-measure zero. Since (3.4) holds for $x=x_0$, $\bar{Q}(s,x_0;b,E)=0$. Since E is arbitrary, $\bar{Q}(s,x_0;b,\cdot)\ll\lambda$. Since the complement of $S_0=S'\cap S''$ is μ_s-null, this proves (ii).

Fix $t\in(a,b)$. Let

$$\gamma(x,z) = \int \bar{Q}(s,x;b,dy)\,p(s,x;t,z;b,y) \qquad (s,z)\in S\times S.$$

By virtue of the proof of (i) (see (3.6)), $\gamma(x,\cdot)$ is, for μ_s-almost all x, a density of $\bar{Q}(a,x;t,\cdot)$ with respect to λ. Let ν be the probability measure on $\Sigma\times\Sigma$ determined by

$$\nu(E\times F) = \int_E \mu_t(dz)\,\bar{Q}(t,z;b,F), \qquad E,F\in\Sigma.$$

We may infer from (ii) that ν is absolutely continuous with respect to the product measure $\mu_t\times\mu_b$ on $\Sigma\times\Sigma$. By an argument of Doob ([5], Chapt. VII, § 8) the Radon-Nikodym derivative $d\nu/d(\mu_t\times\mu_b)$ has a $\Sigma\times\Sigma$-measurable version $\delta(z,y)$, so that

$$\nu(E\times F) = \int_E \mu_t(dz) \int_F \mu_b(dy)\,\delta(z,y).$$

Comparing the last two expressions for $\nu(E\times F)$, we see that for μ_t-almost all z, $\delta(z,\cdot)$ is a density of $\bar{Q}(t,z;b,\cdot)$ with respect to μ_b. Because $\bar{Q}(s,x;t,E)$ is a transition function for the Markov process $\{X_t, a \leqq t \leqq b\}$, we have

$$P_\mu(X_a \in A, X_t \in B, X_b \in C) = \int_A \mu_a(dx) \int_B \lambda(dz)\, \lambda(x, z) \int_C \delta(z, y)\, \mu_b(dy) \tag{3.10}$$

for each A, B, C in Σ. Using Doob's argument again, we find a $\Sigma \times \Sigma$-measurable function $\rho(x, y)$ such that $\rho(x, \cdot)$ is the Radon-Nikodym derivative of $Q(a, x; b, \cdot)$ with respect to μ_b for μ_a-almost all $x \in S$. By (3.5) and the definition of $\{X_t, a \leqq t \leqq b\}$ as the reciprocal process with endpoint distribution μ and reciprocal transition function given by (3.2), we have

$$\begin{aligned} &P_\mu(X_a \in A, X_t \in B, X_b \in C) \\ &\quad = \int_A \mu_a(dx) \int_C \mu_b(dy)\, \rho(x, y) \int_B p(a, x; t, z; b, y)\, \lambda(dz), \end{aligned} \tag{3.11}$$

for all A, B, C in Σ. But (3.10) and (3.11) imply that for $\mu_a \times \lambda \times \mu_b$-almost all (x, z, y),

$$\rho(x, y)\, p(a, x; t, z; b, y) = \gamma(x, z)\, \delta(z, y).$$

By Fubini's theorem there is a z_0 such that for $\mu_a \times \mu_b$-almost all (x, y),

$$\rho(x, y)\, p(a, x; t, z_0; b, y) = \gamma(x, z_0)\, \delta(z_0, y).$$

Referring to (3.2), we see that for these (x, y),

$$\rho(x, y) = f(x)\, q(a, x; b, y)\, g(y), \tag{3.12}$$

where f and g are defined by

$$f(x) = \frac{\gamma(x, z_0)}{q(a, x; t, z_0)}, \qquad g(y) = \frac{\delta(z_0, y)}{q(t, z_0; b, y)}.$$

From (3.12) and (3.5) we get

$$\begin{aligned} \mu(A \times B) &= \int_A \mu_a(dx) f(x) \int_B q(a, x; b, y)\, g(y)\, \mu_b(dy) \\ &= \int_A \nu_a(dx) \int_B q(a, x; b, y)\, \nu_b(dy), \end{aligned}$$

where $\nu_a = f \cdot d\mu_a$, $\nu_b = g \cdot d\mu_b$. This shows that (a) $\Rightarrow$ (b), which completes the proof of the theorem.

Remark. Condition (b) on μ is simply that there exists a product measure π on $\Sigma \times \Sigma$ such that $d\mu/d\pi = q$, where $q(x, y) = q(a, x; b, y)$.

Consider the following problem. Suppose that (S, Σ, λ) is a σ-finite measure space, and that $q(x, y)$ is an everywhere positive, $\Sigma \times \Sigma$-measurable function on $S \times S$ for which $\int q(x, y)\, \lambda(dy) = 1$ for each $x \in S$. Suppose μ_1 and μ_2 are probability measures on Σ. Is there a probability measure μ on $\Sigma \times \Sigma$ which has μ_1 and μ_2 for marginals and which satisfies condition (b) of the theorem? That is, can we find measures ν_1 and ν_2 on Σ such that, if μ is defined on $\Sigma \times \Sigma$ by

$$\mu(E \times F) = \int_E \nu_1(dx) \int_F q(x, y)\, \nu_2(dy), \qquad E \in \Sigma,\ F \in \Sigma \tag{3.13}$$

then

$$\left.\begin{array}{ll}\mu(E\times S)=\mu_1(E) & E\in\Sigma\\ \mu(S\times F)=\mu_2(F) & F\in\Sigma\end{array}\right. \tag{3.14}$$

both hold? Also, is μ uniquely determined by (3.13) and (3.14)? Since μ determines and is uniquely determined by the pair of measures ν_1 and ν_2, the problem we are posing is that of the existence and uniqueness of solutions ν_1 and ν_2 for the functional equations

$$\mu_1(E)=\int_E \nu_1(dx)\,q(x,y)\,\nu_2(dy) \qquad E\in\Sigma, \tag{3.15}$$

and

$$\mu_2(F)=\int_F \nu_2(dy)\,q(x,y)\,\nu_1(dx) \qquad F\in\Sigma. \tag{3.16}$$

Suppose that $\mu_i \ll \lambda$, $i=1,2$. Let $f=d\mu_1/d\lambda$, $g=d\mu_2/d\lambda$. Since $d\mu/d(\nu_1\times\nu_2)=q$, $d(\nu_1\times\nu_2)/d\mu=1/q$, from which it easily follows that $\nu_i\ll\mu_i$, $i=1,2$. Thus $\nu_i\ll\lambda$, $i=1,2$. Let $\psi=d\nu_1/d\lambda$ and $\phi=d\nu_2/d\lambda$. Then (3.15) and (3.16) are equivalent to

$$f(x)=\psi(x)\int q(x,y)\,\phi(y)\,\lambda(dy) \qquad x\in S, \tag{3.17}$$

$$g(y)=\phi(y)\int q(x,y)\,\psi(x)\,\lambda(dx) \qquad y\in S. \tag{3.18}$$

These equations, with (S,Σ,λ) being the real line with Lebesgue measure, and with

$$q(x,y)=\frac{1}{\sqrt{2\pi\sigma^2}}\,e^{-\frac{(x-y)^2}{2\sigma^2}}, \tag{3.19}$$

were derived by Schrödinger ([11, 12]), who conjectured the existence and uniqueness (up to multiplicative constants) of the functions ϕ and ψ except perhaps when f or g are especially "tückisch". In [1], S. Bernstein stated without proof that the pair of functional equations, with q given by (3.19), has a solution provided f and g are continuous. In [7], Fortet used the method of successive approximations to prove the existence and uniqueness of non-negative solutions (3.17) and (3.18) for a wide class of continuous functions $q(x,y)$ including (3.19), but with (S,Σ,λ) still the real line. In [2], Beurling formulated a problem which (when his $n=2$) is that of the existence and uniqueness of solutions ν_1 and ν_2 to (3.15) and (3.16), except that S is a locally compact Hausdorff space, q is required to be continuous and the requirement that $\int q(x,y)\,\lambda(dy)=1$ for all x is dropped, there being no underlying measure λ. It turns out that if $0<a\leq q<b<\infty$, then (3.15) and (3.16) have uniquely determined solutions ν_1 and ν_2 (if q is a Markovian density relative to λ, this requires that λ be finite, and so excludes the case for which λ is Lebesgue measure on the real line). Relaxing the assumption that q be bounded away from 0, he proves existence and uniqueness of positive but not necessarily finite measures ν_1 and ν_2 for which (3.15) and (3.16) hold if $q>0$ and if in addition

$$\iint \log q(x,y)\,\mu_1(dx)\,\mu_2(dy)$$

is finite. Beurling shows that this last condition can be replaced by a weaker but more complicated one. His uniqueness proof, however, is valid without his condition and we can extend his proof to yield existence as well.

Theorem 3.2. *Suppose S is a σ-compact metric space, that μ_1 and μ_2 are probability measures on its σ-field Σ of Borel sets, and that q is an everywhere continuous, strictly positive function on $S \times S$. Then there is a unique pair μ, π of measures on $\Sigma \times \Sigma$ for which*

(a) *μ is a probability measure and π is a σ-finite product measure.*

(b) $$\mu(E \times S) = \mu_1(E), \quad \mu(S \times E) = \mu_2(E), \quad E \in \Sigma,$$

(c) $$\frac{d\mu}{d\pi} = q.$$

Proof. To say that S is σ-compact means that there is an increasing sequence $A_1, A_2, \ldots$ of compact subsets of S for which $S = \bigcup_{n=1}^{\infty} A_n$. Let $B_n = A_n \times A_n$, and let $\Sigma_n = \Sigma \cap A_n = \{E \cap A_n \colon E \in \Sigma\}$. Then $\Sigma_n \times \Sigma_n$ is the class of Borel subsets of B_n. On B_n, q is bounded above and away from zero below. By theorem *I* of [2] there exists a finite product measure π^n on $\Sigma_n \times \Sigma_n$ and a measure $\mu^{(n)}$ on $\Sigma_n \times \Sigma_n$ such that

(i) $$\left.\begin{aligned} \mu^{(n)}(E \times A_n) &= \mu_1(E) \\ \mu^{(n)}(A_n \times E) &= \mu_2(E) \end{aligned}\right\} E \in \Sigma_n$$

(ii) $$\frac{d\mu^{(n)}}{d\pi^n} = q \quad \text{on } B_n.$$

We extend $\mu^{(n)}$ and π^n to all of Σ by setting them equal to 0 on sets $E \in \Sigma$ disjoint from B_n. π^n remains a product measure and (ii) holds throughout $S \times S$ π^n-almost surely. Let $\mu_i^{(n)}$ be the marginals of $\mu^{(n)}$ as so extended. There is a sequence $\{n_k\}$ such that the restriction of $\mu^{(n_k)}$ to B_m converges weakly for each $m = 1, 2, \ldots$. It is easy to see that this implies the existence of a measure μ on $\Sigma \times \Sigma$ whose restriction to $\Sigma_m \times \Sigma_m$ is for each m the weak limit relative to $C(B_m)$ of the sequence formed by the restrictions of $\mu^{(n_k)}$ to Σ_m. Then $\iint g\, d\mu^{(n_k)} \to \iint g\, d\mu$ for any continuous g on $S \times S$ with support contained in one of the compact sets B_m. I claim that this convergence holds provided only that g is bounded and continuous on $S \times S$. Since $\mu^{(n)}(S) = \mu^{(n)}(B_n) = \mu^{(n)}(A_n \times A_n) = \mu_1(A_n) \leqq 1$ by (i), this certainly holds if we establish that μ is a probability measure. It is clear from (ii), however, that the marginals $\mu_i^{(n)}$ of $\mu^{(n)}$ converge weakly to μ_i, $i = 1, 2$. Since these are probability measures, the sequence $\{\mu^{(n)}\}$ of probability measures is tight and the limit μ of $\mu^{(n_k)}$ is a probability measure ([3], p. 30) with μ_1, μ_2 as marginals, which establishes (b) and half of (a). Now fix m and assume $f \in C(S \times S)$ has support in B_m. The restriction of f/q to B_m belongs to $C(B_m)$, so by (ii)

$$\int f\, d\mu^{n_k} = \int (f/q)\, d\mu^{(n_k)} \to \int (f/q)\, d\mu \tag{3.20}$$

as $n \to \infty$. This shows that the restriction of π^{n_k} to B_m converges weakly to a limit π_m as $k \to \infty$. Again, it is easy to see that there is a measure π on $\Sigma \times \Sigma$ whose restriction to B_m is π_m, $m = 1, 2, \ldots$. It follows from (3.20) that $d\pi/d\mu = 1/q$, whence $d\mu/d\pi = q$. Since $\pi = \pi_m$ is finite on B_m, π is σ-finite. π^n is a product measure for each n, and an easy argument shows that each π_m, hence π, must therefore be a product measure. This shows the existence of measures π and μ as described in the theorem.

To establish that π and μ are unique, assume that π' is a product measure and μ' a probability measure for which (a) and (b) hold. Then

$$\mu_1(E)=\int_{E\times S} q\,d\pi=\int_{E\times S} q\,d\pi' \tag{3.21}$$

and

$$\mu_2(E)=\int_{S\times E} q\,d\pi=\int_{S\times E} q\,d\pi'$$

for each $E\in\Sigma$. Suppose $\pi=\nu_1\times\nu_2$, $\pi'=\nu_1'\times\nu_2'$. Let $h_1(x)=\int q(x,y)\,\nu_2(dy)$, $x\in S$, $h_2(y)=\int q(x,y)\,\nu_1(dx)$, $y\in S$, and let $h(x,y)=h_1(x)\,h_2(y)$, $(x,y)\in S\times S$. Let k_1, k_2, and k be similarly defined but with ν_i' replacing ν_i, $i=1,2$. Let g_1 and g_2 be bounded Σ-measurable functions on S, and let $g(x,y)=g_1(x)\,g_2(y)$, $(x,y)\in S\times S$. By virtue of (3.21), $\int g_i\,d\mu_i=\int g_i h_i\,d\pi_i$ $i=1,2$. Multiplying corresponding sides of these two equations, we have

$$\int g\,d(\mu_1\times\mu_2)=\int g h\,d\pi.$$

Since h is strictly positive, we can rewrite this as

$$\int g h^{-1}\,d(\mu_1\times\mu_2)=\int g\,d\pi. \tag{3.22}$$

(Of course h^{-1} denotes the reciprocal, not the inverse, of h.) Similarly

$$\int g k^{-1}\,d(\mu_1\times\mu_2)=\int g\,d\pi'. \tag{3.23}$$

The definition of $\Sigma\times\Sigma$ as the σ-field generated by the field of finite disjoint unions of rectangles $E\times F$ with $E, F\in\Sigma$ ensures that (5) and (6) hold for all non-negative $\Sigma\times\Sigma$-measurable functions g. Let σ_1 and σ_2 be bounded Σ-measurable functions on S, and let $\sigma(x,y)=\sigma_1(x)+\sigma_2(y)$, $(x,y)\in S\times S$. Then

$$\begin{aligned}\int \sigma\,d(\mu_1\times\mu_2)&=\int\sigma_1\,d\mu_1+\int\sigma_2\,d\mu_2\\&=\iint\sigma_1(x)\,q(x,y)\,\nu_1(dx)\,\nu_2(dy)+\iint\sigma_2(y)\,q(x,y)\,\nu_1(dx)\,\nu_2(dy)\\&=\int\sigma q\,d(\nu_1\times\nu_2)=\int\sigma q h^{-1}\,d(\mu_1\times\mu_2)\end{aligned}$$

by virtue of (3.21) and (3.22). Using ν_i' instead of ν_i, $i=1,2$, we obtain similarly $\int\sigma\,d(\mu_1\times\mu_2)=\int\sigma q k^{-1}\,d(\mu_1\times\mu_2)$. We conclude that

$$\int\sigma q h^{-1}\,d(\mu_1\times\mu_2)=\int\sigma q k^{-1}\,d(\mu_1\times\mu_2). \tag{3.24}$$

Since σ is bounded and since $q\,d\pi$ is a probability measure, the common value of the two sides of (3.24) is finite by virtue of (3.22) and (3.23). Thus (3.24) yields

$$\int\sigma q(h^{-1}-k^{-1})\,d(\mu_1\times\mu_2)=0.$$

In particular, this last equation holds if

$$\sigma(x,y)=\frac{h_1^{-1}(x)}{h_1^{-1}(x)+k_1^{-1}(x)}-\frac{h_2^{-1}(y)}{h_2^{-1}(y)+k_2^{-1}(y)},\qquad (x,y)\in S\times S \tag{3.25}$$

from which we deduce

$$\int q(h^{-1}-k^{-1})^2/r\, d(\mu_1\times\mu_2)=0, \tag{3.26}$$

where $r(x, y)$ is the product of the denominators of the two fractions on the right hand side of (3.25). Since $q>0$, $h=k$ on the support of $\mu_1\times\mu_2$. It now follows from (3.22) and (3.23) that $\pi=\pi'$, and it follows from $d\mu/d\pi=d\mu'/d\pi'$ that $\mu=\mu'$. This completes the proof of the theorem. (The very elegant proof of uniqueness is due to Beurling; we have changed his notation to conform with ours, and rearranged his proof to exhibit its independence from his condition (8.1) on p. 198 of [2].)

§4

If a reciprocal transition function $P(s,x;t,\cdot;u,z)$ is absolutely continuous relative to a σ-finite measure λ on Σ, then there is a function $p(s,x;t,y;u,z)$ for which

$$P(s,x;t,E;u,z)=\int_E p(s,x;t,y;u,z)\,\lambda(dy),\qquad a\leqq s<t<u\leqq b\quad x,z\in S,\ E\in\Sigma. \tag{4.1}$$

If $P(s,x;t,\cdot,u,z)$ is derived from a Markov transition density $q(s,x;t,y)$ we have in fact

$$p(s,x;t,y;u,z)=\frac{q(s,x;t,y)\,q(t,y;u,z)}{q(s,x;u,z)},\qquad a\leqq s<t<u\leqq b,\ \ x,y,z\in S. \tag{4.2}$$

Is any reciprocal transition density derived from a Markov transition density? More precisely, given that a function $p(s,x;t,y;u,z)$ satisfies (4.1), does there exist a Markov transition density $q(s,x;t,y)$ such that (4.2) holds? In this section we give partial answers to this question. First, to motivate our definition of reciprocal transition density, we list those properties of $p(s,x;t,y;u,z)$ which follow by virtue of (4.1) and properties (A1), (A2) and (A3) of $P(s,x;t,\cdot;u,z)$. As usual, $a\leqq s<t<u\leqq b$, x,y,z are in S, and $E\in\Sigma$.

(a1) $y\to p(s,x;t,y;u,v)$ is λ-measurable, with

$$p(s,x;t,y;u,v)\geqq 0\qquad \lambda\text{-almost all } y$$

and

$$\int p(s,x;t,y;u,v)\,\lambda(dy)=1.$$

(a2) $(x,y)\to\int_E p(s,x;t,y;u,z)\,\lambda(dy)$ is $\Sigma\times\Sigma$ measurable.

(a3) For each $a\leqq s<t<u<v\leqq b$, and each x, w in S,

$$p(s,x;u,z;v,w)\,p(s,x;t,y;u,z)=p(s,x;t,y;v,w)\,p(t,y;u,z;v,w)$$

for $\lambda\times\lambda$-almost all $(y,z)\in S\times S$.

This last property is an almost immediate consequence of (A3), which in turn is analogous to the Chapman-Komogorov equation satisfied by Markov transition functions. However, its consequence (a3) for densities is not an integral equation as in the Markov case but a pointwise, nonintegrated equality which right away yields our first result. We require of our definition sharper versions of (a1)–(a3).

As in Section 2, $[a, b]$ is a non-degenerate closed interval, and (S, Σ) is a measurable space. We use $\mathscr{E}$ to denote the set of all ordered sextuples (s, x, t, z, u, y) for which x, y, and z are in S and $a \leqq s < t < u \leqq b$. Let λ be a σ-finite measure on Σ. A function p on $\mathscr{E}$ to the (positive) non-negative reals is called a *(strictly positive) reciprocal transition probability λ-density* if the following conditions are satisfied.

(b 1) For each $a \leqq s < t < u \leqq b$, the map $(s, y, z) \to p(s, x, t, y, u, z)$ is $\Sigma \times \Sigma \times \Sigma$-measurable, and

$$\int p(s, x, t, y, u, z)\, \lambda(dz) = 1 \qquad x, y \text{ in } S.$$

(b 2) For each $a \leqq s < t < u < b \leqq b$ and x, y, z, w in S,

$$p(s, x, u, z, v, w)\, p(s, x, t, y, u, z) = p(s, x, t, y, v, w)\, p(t, y, u, z, v, w).$$

If (b 1) and (b 2) are satisfied then the function P on $\mathscr{D}$ (see Section 2) defined by

$$P(s, x, t, E, u, z) = \int_E p(s, x, t, z, u, y)\, \lambda(dz) \tag{4.3}$$

is a reciprocal transition probability function. We write $p(s, x; t, y; u, z)$ for $p(s, x, t, y, u, z)$. We pose but otherwise ignore the question of whether a density $p(s, x; t, y; u, z)$ satisfying (a 1), (a 2), and (a 3) has a version satisfying (b 1) and (b 2).

Theorem 4.1. *Let $p(s, x; t, y; u, v)$ be a strictly positive reciprocal transition λ-density on $[a, b]$. Then for each $b' \in (a, b)$ there is a Markov transition λ-density for which*

$$p(s, x; t, y; u, v) = \frac{q(s, x; t, y)\, q(t, y; u, z)}{q(s, x; u, z)}, \qquad a \leqq s < t < u \leqq b',\ x, y, z \text{ in } S.$$

Proof. In property (b 2) set $v = b'$, fix $w \in S$ and let $q(s, x; t, y) = p(s, x; t, y; w, b')$.

There are processes defined on $(-\infty, \infty)$ which are reciprocal on an interval $[a, b]$ but on no strictly larger super-interval (for example, the process discussed by Slepian in [12]). Thus we wish to replace $b' < b$ by b itself. One would think it possible to concoct some simple limiting argument and let $b' \to b$. We are able to obtain the result only under some restrictions on $p(s, x; t, y; u, z)$ and by rather involved reasoning. We first give an example to show that not all discrete-parameter reciprocal processes are derived from Markov transition functions. Given $X_1, \ldots, X_n$ reciprocal, then $X_1, \ldots, X_{n-1}$ is derived from a Markov transition function, but there may be an "endpoint effect" ensuring that $X_1, \ldots, X_n$ is not so derived. Any process X_1, X_2, X_3 is reciprocal for the same trivial sort of reason that any process X_1, X_2 is Markovian. Take $S = \{0, 1\}$, and let $p(x|y|z) = P(X_2 = y | X_1 = x, X_2 = z)$, where x, y, and z range over $\{0, 1\}$. For our example of a reciprocal process not derived from a Markov process we choose $p(x|y|z)$ so that there is no system of Markov transition functions $q(i, x; j, y)$, $1 \leqq i < j \leqq 3$, x, y in S for which

$$p(x|y|z) = \frac{q(1, x; 2, y)\, q(2, y; 3, z)}{q(1, x; 3, z)} \qquad x, y, z = 0, 1. \tag{4.4}$$

We determine $p(x|y|z)$ by the condition that

$$p(x|0|z)=\begin{cases}\frac{1}{3} & x=z\\ \frac{2}{3} & x\neq z\end{cases}\qquad x,z=0,1. \tag{4.5}$$

Define $F(x)$ to be the quotient of $p(x|0|0)/p(x|0|1)$ by $p(x|1|0)/p(x|1|1)$. Suppose (4.4) holds. Then $p(x|y|z)=f(x,y)\,g(y,z)\,h(x,z)$ for some functions f, g, h, and

$$F(x)=\frac{f(x,0)\,g(0,0)\,h(x,0)}{f(x,0)\,g(0,1)\,h(x,1)}\cdot\frac{f(x,1)\,g(1,1)\,h(x,1)}{f(x,1)\,g(1,0)\,h(x,0)}$$

which is independent of x. We see from (4.5), however, that $F(0)=1/4$, $F(1)=1/3$. Contradiction: (4.4) cannot hold, and $p(x|y|z)$ is not derived from a Markov transition function.

We need the following lemma, which is of interest in its own right as a partial converse to the results of Section 3.

Lemma 4.2. *Let (S,d) be a σ-compact metric space with Σ the σ-field generated by the open sets of S, and let λ be a σ-finite measure on Σ. Let $p(s,x;t,y;u,z)$ be a reciprocal transition λ-density on $[a,b]$. Let μ be a probability measure on $\Sigma\times\Sigma$ both of whose marginals are absolutely continuous with respect to λ and have strictly positive densities. Let $\{X_t, a\leqq t\leqq b\}$ be the reciprocal process with transition function given by (4.3) and endpoint distribution μ. If $\{X_t, a\leqq t\leqq b\}$ is Markov, then $\mu\ll\lambda\times\lambda$, $d\mu/d(\lambda\times\lambda)$ has a strictly positive version, and there is a Markov transition density $q(s,x;t,y)$ such that (4.2) holds for each $a\leqq s<t<u\leqq b$ and $\lambda\times\lambda\times\lambda$-almost all (x,y,z).*

Proof. Assume the hypotheses of the theorem. There are everywhere positive measurable functions f and g on S for which

$$\mu(E\times S)=\int_E f\,d\lambda,\qquad \mu(S\times E)=\int_E g\,d\lambda,\qquad E\in\Sigma.$$

Let $x\to\mu(\cdot,x)$ and $x\to\mu(x,\cdot)$ be conditional distributions of μ given the sub-σ-fields $\{E\times S\colon E\in\Sigma\}$ and $\{S\times E\colon E\in\Sigma\}$ respectively. For each x, y in S, let

$$\begin{aligned} r(s,x,t,y)&=\int p(a,x';s,x;b,z)\,p(s,x;t,y;b,z)\,d\mu(x',z), && a<s<t<b,\\ r(a,x,t,y)&=f(x)\int p(a,x;t,y;b,z)\,\mu(x,dz), && a<t<b, \\ r(t,x,b,y)&=g(y)\int p(a,x';t,x;b,y)\,\mu(dx',y), && a<t<b. \end{aligned}\tag{4.6}$$

For each choice of (s,t) with $a\leqq s<t\leqq b$ other than $(s,t)=(a,b)$, $r(s,x;b,y)$ is the value at (x,y) of the joint density of (X_s, X_t) with respect to $\lambda\times\lambda$. This can be checked using (2.2). For example,

$$\begin{aligned}\int_{E\times F} r(a,x,t,y)\,d(\lambda\times\lambda)(x,y)&=\int_E\int_F\int p(a,x;t,y;b,z)\,\lambda(dy)\,\mu(x,dz)\,f(x)\,\lambda(dx)\\ &=\int_{E\times S} P(a,x;t,F;b,z)\,d\mu(x,z)\\ &=P_\mu(X_a\in E, X_t\in F),\end{aligned}$$

and the others are similar. Let

$$\rho(t,x)=\int r(a,x',t,x)\,\lambda(dx') \qquad a<t<b;$$

then $\rho(t,x)$ is the value at x of the conditional density of X_t. For each x, y in S define

$$q(a,x;t,y)=\frac{r(a,x,t,y)}{f(x)} \qquad a<t<b,$$

$$q(t,x;b,y)=\frac{r(t,x,b,y)}{\rho(t,x)} \qquad a<t<b,$$

$$q(s,x;t,y)=\frac{r(s,x,t,y)}{\rho(s,x)} \qquad a<s<t<b.$$

Then, for each (s,t) with $a\leqq s<t\leqq b$, except $(s,t)=(a,b)$, $q(s,x;t,y)$ is the value at y of the conditional density of X_t given $X_s=x$. Were μ assumed absolutely continuous with respect to $\lambda\times\lambda$, then $d\mu/d(\lambda\times\lambda)$ would be the joint density of X_a and X_b, and we could write down the corresponding conditional density. However, we are assuming only that μ has λ-absolutely continuous marginals, and this by itself does not imply that $\mu\ll\lambda\times\lambda$. This last is indeed true, because $\{X_t, a\leqq t\leqq b\}$ is Markovian. Fixing $t\in(a,b)$, we have already established the existence of joint densities for (X_a,X_t) and for (X_t,X_b). It follows from the Chapman-Kolmogorov equation that X_a and X_b have a joint density, in other words, that $\mu<\lambda\times\lambda$. The argument of Doob used in the proof of Theorem 3.1 shows that there is a version $r(x,y)$ of this joint density which is $\Sigma\times\Sigma$-measurable in (x,y), and then $q(a,x;b,y)=r(x,y)/f(x)$ is the value at y of the conditional density of X_b given $X_a=x$. The Chapman-Kolmogorov equation also shows that we may choose $r(x,y)$, hence $q(a,x;b,y)$, strictly positive. The conditional densities $q(s,x;t,y)$ are now defined for all $a\leqq s<t\leqq b$ and x,y in S. On the one hand, the value at y of the conditional density of X_t given $X_s=x$ and $X_n=z$ is given by $p(s,x;t,y;u,z)$. On the other hand, the fact that $\{X_t, a\leqq t\leqq b\}$ is a Markov process with transition density q enables us to write this conditional density as the quotient of $p(s,x)\,q(s,x;t,y)\,q(t,y;b,z)$ by $\rho(s,x)\,q(s,x,u,y)$. This establishes (4.2) and proves the lemma.

Theorem 4.3. *Assume the conditions on S, Σ, and λ given in the previous lemma. Suppose that $p(s,x;t,y;u,z)$ is a strictly positive transition λ-density on $[a,b]$ satisfying the following conditions:*

(i) *For each $a\leqq s<t<u\leqq b$, the map $(x,y,z)\to p(s,x;t,y;u,z)$ is continuous on $S\times S\times S$ and for each $a<s_0<t_0<u_0<v_0<b$ is bounded uniformly in $s\in[a,s_0]$, $u\in[v_0,b]$, and x,y,z in S.*

(ii) *For each $t\in(a,b)$ and x,y,z in S,*

$$\lim_{a'\downarrow a,\, b'\uparrow b} p(a',x;t,y;b',z)=p(a,x;t,y;b,z)$$

the limit approach being uniform for (s,y,z) in any compact subset of $S\times S\times S$.

(iii) *There is a* $\eta>0$ *such that for each* $\delta>0$ *and* x_0, z_0 *in* S, *and compact* K

$$\lim_{u\to 0}\frac{1}{u}\max_1\int_{R(y,\delta)} p(s,x;t,y;b,z_0)\,\lambda(dy)=0,$$

$$\lim_{u\to 0}\frac{1}{u}\max_2\int_{R(y,\delta)} p(a,x_0;s,y;t,x)\,\lambda(dy)=0,$$

where $\max_1$ *is taken over* $a\leqq s\leqq a+\eta$, $s<t<s+u$ *and* $x\in K$, $\max_2$ *is taken over* $b-\eta\leqq s\leqq b$, $t-u<s<t$, *and* $x\in K$, *while* $R(y,\delta)$ *is the complement of the sphere of radius* δ *centered at* y.

Then there is a Markov transition λ-*density* $q(s,x;t,y)$, $a\leqq s<t\leqq b$, x,y *in* S, *such that* (4.2) *holds for* $\lambda\times\lambda\times\lambda$-*almost all* (x,y,z) *in* $S\times S\times S$, *and all* (x,y,z) *if* $s>a$ *and* $u<b$.

Proof. Assume the hypotheses of the theorem. Let f and g be strictly positive Σ-measurable functions on S with $\int f\,d\lambda=\int g\,d\lambda=1$. Let $a_n\downarrow a<b_n\uparrow b$. By virtue of the lemma, there is a Markov transition density $q_n(s,x;t,y)$, $a_n<s<t<b_n$ such that (4.2) holds. By (ii), we may assume that $q_n(s,x;t,y;t,z)$ is continuous in (x,y,z) for each $a_n\leqq s<t<u\leqq b_n$. By theorem 3.2, for each n there is a measure μ_n on $\Sigma\times\Sigma$ whose marginals are given by the λ-densities f and g such that the reciprocal process $\{X_t, a_n\leqq t\leqq b_n\}$ with transition density $p(s,x;t,y;b,z)$ and endpoint measure μ_n is Markov. Since the marginals of μ_n do not depend on n, $\{\mu_n\}$ has a weakly convergent subsequence. Let μ be its limit. The marginals of μ are given by the densities f and g. We may assume without loss of generality that $\{\mu_n\}$ itself converges. Consider now the process $\{X_t, a\leqq t\leqq b\}$ determined by the reciprocal transition density $p(s,x;t,y;u,t)$ and the endpoint measure μ.

(a) $\{X_t, a<t<b\}$ *is Markov*.

To prove (a), let $a<s_1<\cdots<s_k<t<b$. Choose n large enough so that $a_n<s_1$, $b_n<t$. Let

$$\pi(x,x_1,\ldots,x_k,y,z)=p(a,x;s_1,x_1;b,z)\,p(s_1,x_1;s_2,x_2;b,z)\cdot\cdots$$
$$\cdots\cdot p(s_k,x_k;t,y;b,z),$$
$$\sigma(x,x_1,\ldots,x_k,z)=p(a,x;s_1,x_1;b,z)\,p(s_1,x_1;s_2,x_2;b,z)\cdot\cdots$$
$$\cdots\cdot p(s_{k-1},x_{k-1};s_k,x_k;b,z)$$

for $x,x_1,\ldots,x_k,y,z$ in S, and let π_n and σ_n be defined in exactly the same way, but with a_n and b_n replacing a and b respectively. $(X_{s_1},\ldots,X_{s_k},X_t)$ has a joint λ-density, and the value at y of the conditional density of X_t given $X_{s_1}=x_1,\ldots,X_{s_k}=x_k$ is easily seen to be equal to

$$\frac{\int\pi(x,x_1,\ldots,x_k,y,z)\,d\mu(x,z)}{\int\sigma(x_1,x_1,\ldots,x_k,z)\,d\mu(x\,z)}. \tag{4.7}$$

It follows from conditions (i) and (ii) that for fixed $x_1,\ldots,x_k$ and y $\{\pi_n(x,x_1,\ldots,x_k,y,z)\}$ is bounded uniformly in x,y and z and converges to $\pi(x,x_1,\ldots,x_k,y,z)$

uniformly for (x, y, z) in compact subsets of $S \times S \times S$. Therefore (4.7) is equal to

$$\frac{\lim_n \int \pi_n(x, x_1, \ldots, x_k, y, z)\, d\mu_n(x, z)}{\lim_n \int \sigma_n(x, x_1, \ldots, x_k, z)\, d\mu_n(x, z)} = \lim_n \frac{\int \pi_n(x, x_1, \ldots, x_k, y, z)\, d\mu_n(x, z)}{\int \sigma_n(x, x_1, \ldots, x_k, z)\, d\mu_n(x, z)}.$$

The expression whose limit is being taken is the value at y of the corresponding conditional density relative to the process $\{X_t, a_n \leqq t \leqq b\}$ with transition density $p(s, x; t, y; u, z), a_n \leqq s < t < u \leqq b_n$, and endpoint measure μ_n, This process is Markov, so the conditional density in question is independent of $x_1, \ldots, x_{k-1}$. The same must be true of the limit (4.7). Since k and $a < s_1 < \cdots < s_k < t < b$ are arbitrary, this proves (a).

(b) *Almost all paths of* $\{X_t, a \leqq t \leqq b\}$ *are continuous on* $[a, a+\eta]$ *and* $[b-\eta, b]$.

First, fix x_0 and z_0 in S, and consider the reciprocal process $\{X_t, a \leqq t \leqq b\}$ with transition density $p(s, x; t, y; b, z)$ and endpoint measure equal to the point mass $\delta_{(x_0, y_0)}$. Then $\{X_t, a \leqq t \leqq b\}$ is a Markov process with transition q-density $p(s, x; t, y; b, z_0)$, as is therefore $\{X_t, a \leqq t \leqq a+\eta\}$. The first condition of (iii) is guarantees that this latter process has continuous paths by virtue of the corollary to Theorem 6.6 of [6]. $\{X_t, a \leqq t \leqq b\}$ with time reversed is also a Markov process, with transition density $\rho(t, x; s, y) = p(a, x_0; s, y; t, x)$ for $a \leqq s < t \leqq b$, $x, y \in S$. The second condition of (iii) guarantees that $\{X_t, b-\eta \leqq t \leqq b\}$ has continuous paths. Thus $\{X_t, a \leqq t \leqq b\}$ has the desired continuity if the endpoint measure is a point mass, and it follows immediately that the same is true for any endpoint measure, in particular for the endpoint measure μ. This proves (b).

(c) $\{X_t, a \leqq t \leqq b\}$ *is a Markov process.*

First we show that $\{X_t, a \leqq t < b\}$ is a Markov process. For each $a \leqq c < d \leqq b$ let $\mathscr{I}_{[c,d]}$ be the σ-field generated by $\{X_t \colon c \leqq t \leqq b\}$. We need to show that $E\{h(X_{t'}) | \mathscr{I}_{[0,t]}\} = E\{h(X_{t'}) | X_t\}$ for any $a \leqq t < t' \leqq b$ and bounded Σ-measurable function h. Since $\{X_t, a \leqq t < b\}$ is a Markov process by (a), we know that $E\{h(X_{t'}) | \mathscr{I}_0\} = E\{h(X_{t'}) | X_t\}$, where $\mathscr{I}_0$ is the smallest σ-field containing all the σ-fields $\mathscr{I}_{[a+\frac{1}{n}, t]}$, $n = 1, 2, \ldots$. Let Z be a bounded random variable measurable with respect to $\mathscr{I}_0$, let k be a bounded continuous function on S and let $Y = k(X_a)\, Z$. Then

$$\int Y h(X_{t'}) = \int k(X_a)\, Z h(X_{t'}) = \int \lim_n k(X_{a+\frac{1}{n}})\, Z h(X_{t'}) = \lim_n \int k(X_{a+\frac{1}{n}})\, Z h(X_{t'}),$$

the last two equalities holding by virtue of the continuity of k, the continuity (by (b)) at a of the paths of $\{X_t, a \leqq t \leqq b\}$ and the bounded convergence theorem. But $k(X_{a+\frac{1}{n}})\, Z$ is measurable with respect to $\mathscr{I}_0$ for any $n = 1, 2, \ldots$, so

$$\int k(X_{a+\frac{1}{n}})\, Z h(X_{t'}) = \int k(X_{a+\frac{1}{n}})\, Z E\{h(X_{t'}) | X_t\}.$$

Letting $n \to \infty$, and using again continuity and the bounded convergence theorem, we have $\int Y h(X_{t'}) = \int Y E\{h(X_{t'}) | X_t\}$. This is true for any Y of the form $k(X_a)\, Z$, with k continuous and Z bounded and $\mathscr{I}_0$-measurable. But bounded $\mathscr{I}_{[a,t]}$-measurable functions can be approximated by linear combinations of such Y's, so the last equality holds for arbitrary bounded $\mathscr{I}_{[a,t]}$ measurable Y. It follows that $E\{h(X_{t'})|\mathscr{I}_{[0,t]}\}=E\{h(X_{t'})|X_t\}$. Reversal of the direction of time and use of the continuity of the paths of $\{X_t, a\leqq t\leqq b\}$ at b show that $\{X_t, a\leqq t\leqq b\}$ is Markovian, establishing (c).

Since $\{X_t, a\leqq t\leqq b\}$ is a Markov process, and since the marginals f and g of the endpoint measure μ are strictly positive, Lemma 4.2 applies to yield a Markov transition function $g(s,x;t,y)$ for which (4.2) holds for $\lambda\times\lambda\times\lambda$-almost all (x,y,z) in $S\times S\times S$. If $s>a$ and $u<b$, the joint density $r(s,x;t,y)$ of (X_s, X_t) is continuous in (x,y) by virtue of assumption (i), (4.6), and the bounded convergence theorem, while the density $\int p(a,w;s,x;b,y)\,d\mu(w,y)$ of X_s is continuous in x again by (ii) and the bounded convergence theorem. Thus $q(s,x;t,y)$, which is the quotient of $r(s,x;t,y)$ and the (everywhere positive) density of x, is itself a continuous function of (x,y), from which it follows that both sides of (4.2) are continuous functions of (x,y,z) if $s>a$ and $u<b$, and (4.2) therefore holds for all (x,y,z) under this restriction. This completes the proof of the theorem.

We note that in keeping with the statement of the theorem, the reciprocal process $\{X_t, 0\leqq t\leqq 1\}$ studied by Slepian in [13] has a reciprocal transition density which satisfies (4.2) with $q(s,x;t,y)$ being the transition density of the standard Brownian motion on [0, 1] and endpoint measure μ the product measure whose factors are the one-dimensional Gaussian measures with zero mean and unit variance.

References

1. Bernstein, S.: Sur les liaisons entre les grandeurs aléatoires, Verh. des intern. Mathematikerkongr. **I** Zürich 1932
2. Beurling, A.: An automorphism of product measures, Ann. of Math. **72**, 189–200 (1960)
3. Billingsley, P.: Convergence of Probability Measures. New York: Wiley 1968
4. Blumenthal, R.M., Getoor, R.K.: Markov Processes and Potential Theory. New York: Academic Press 1968
5. Doob, J.L.: Stochastic Processes. New York: Wiley 1953
6. Dynkin, E.B.: The Theory of Markov Processes. London: Pergamon Press 1960
7. Fortet, R.: Résolution d'un système d'équations de M. Schroedinger. J. Math. Pures Appl. **IX**, 83–105 (1940)
8. Jamison, B.: Reciprocal Processes: The stationary Gaussian case, Ann. Math. Statist. **41**, 1624–1630 (1970)
9. Loève, M.: Probability Theory, (3rd ed.). Princeton: Van Nostrand 1963
10. Loève, M.: Probability Methods in Physics I. Statistical Equilibrium (Seminar Notes) Statistical Laboratory, Department of Mathematics, University of California, Berkeley, 1949
11. Schrödinger, E.: Über die Umkehrung der Naturgesetze, Sitz. Ber. der Preuss. Akad. Wissen., Berlin Phys. Math. **144** (1931)
12. Schrödinger, E.: Theorie relativiste de l'electron et l'interpretation de la méchanique quantique, Ann. Inst. H. Poincaré **2**, 269–310 (1932)
13. Slepian, D.: First passage time for a particular Gaussian process, Ann. Math. Statist. **32**, 610–612 (1961)

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