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The Maximum Term of Uniformly Mixing Stationary Processes

G.L. O'Brien*

Let $\{X_n\}$ be a uniformly (or strongly) mixing stationary process and let $Z_n = \max(X_1, X_2, \ldots, X_n)$. For $\xi > 0$, let $c_n(\xi) = \inf\{x \in R\colon n P(X_1 > x) \leqq \xi\}$. Under a condition which holds for all φ-mixing processes, necessary and sufficient conditions are given for $P(Z_n \leqq c_n(\xi))$ to converge to each possible limit. Some conditions for convergence of $P(Z_n \leqq d_n)$ for any sequence d_n are also obtained.

1. Introduction

Let $\{X_n\}$ be a strictly stationary process. Assume $\{X_n\}$ is *uniformly* (or strongly) *mixing* with mixing function g: that is, $g(k) \to 0$ as $k \to \infty$ and if $A \in \mathfrak{B}(X_1, \ldots, X_m)$ and $B \in \mathfrak{B}(X_{m+k}, X_{m+k+1}, \ldots)$, then

$$|P(AB) - P(A)\,P(B)| \leqq g(k). \tag{1}$$

We will sometimes assume further that $\{X_n\}$ is φ-mixing, that is, (1) holds with the right side replaced by $\varphi(k)\,P(A)$, where $\varphi(k) \to 0$ as $k \to \infty$.

Let $H(x) = P(X_n \leqq x)$. Let $x_0 = \sup\{x \mid H(x) < 1\}$. For each $\xi > 0$ and each integer $n > \xi$, let $c_n(\xi)$ satisfy $P(X_1 > c_n(\xi)) \leqq \xi n^{-1} \leqq P(X_1 \geqq c_n(\xi))$. Then

$$H^n(c_n(\xi)) \to e^{-\xi} \qquad (\text{that is, } H(c_n(\xi)) = 1 - \xi n^{-1} + o(n^{-1})) \tag{2}$$

if and only if $P(X_1 \geqq x)/P(X_1 > x) \to 1$ as $x \uparrow x_0$ and H is continuous at x_0 (see the author [5]). We assume henceforth that (2) holds.

Let $Z_n = \max(X_1, X_2, \ldots, X_n)$. According to Loynes [4], the only possible limit functions of $P(Z_n \leqq c_n(\xi))$ are $\exp(-\alpha\xi)$ for $0 \leqq \alpha \leqq 1$. We obtain, under Condition C, given below, necessary and sufficient conditions for each limit to occur (Theorem 1). Condition C always holds for φ-mixing processes.

Watson [6], Loynes [4] and Galambos [2] have previously investigated when $P(Z_n \leqq c_n(\xi)) \to \exp(-\xi)$ under various conditions which approximate independence. The methods used here are extensions of those used by Loynes. Examples when $P(Z_n \leqq c_n(\xi)) \to \exp(-\alpha\xi)$ for $\alpha < 1$ are given by the author [5].

In Section 4, we study limits of $P(Z_n \leqq d_n)$ for general sequences $\{d_n\}$. Of particular interest are sequences of the form $d_n = a_n x + b_n$. This problem was investigated in the independent case by Gnedenko [3] and de Haan [1]. We also have some results concerning Gnedenko's concepts of relative stability and the law of large numbers.

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The results given here can easily be modified to apply to minima instead of maxima.

2. The Limit Theorem

Under Condition C stated below, we obtain necessary and sufficient conditions for convergence of $P(Z_n \leqq c_n(\xi))$ to each possible limit. C is related to a sufficient condition of Loynes [4] for convergence to the limit $e^{-\xi}$. (His condition was close to that obtained by taking $r_m = m$ and $\Delta(1) = 0$.) We let $Z_{k,l}$ denote

$$\max(X_k, X_{k+1}, \ldots, X_l)$$

(if $k > l$, let $Z_{k,l} = -\infty$).

A sequence $\{c_n\}$ of real numbers with $c_n < x_0$ is said to satisfy *Condition C* if there are sequences $\{r = r_m\}$, $\{s = s_m\}$ and $\{t = t_m = rs\}$ of positive integers such that $r \to \infty$, $s \to \infty$, $(t_m)^{-1} t_{m+1} \to 1$ and $r g(s) \to 0$, and such that

$$\Delta(j) \equiv \limsup_{m \to \infty} P(Z_{2,j} \leqq c_t, Z_{j+1,s} > c_t | X_1 > c_t) \to 0 \tag{3}$$

as $j \to \infty$.

Remarks. Define

$$\Delta'(j) = \limsup_{m \to \infty} P(Z_{j+1,s} > c_t | X_1 > c_t). \tag{4}$$

Note that $\Delta(j)$ and $\Delta'(j)$ are decreasing sequences. The first four of the five limits of C can always be met by choosing $s = m$ and r some sufficiently slowly increasing sequence. If $\{X_n\}$ is φ-mixing and $\liminf H^t(c_t) > 0$ (so that $\limsup t P(X_1 > c_t) < \infty$), we then have

$$\begin{aligned} \Delta(j) &\leqq \Delta'(j) \leqq \limsup P(Z_{j+1,s} > c_t) + \varphi(j) \\ &\leqq \limsup s P(X_1 > c_t) + \varphi(j) = \varphi(j) \to 0 \quad \text{as} \quad j \to \infty. \end{aligned}$$

In particular, if $c_n = c_n(\xi)$ for some ξ, C is always valid for φ-mixing processes. Similarly by choosing $r = m$ and s some slowly increasing sequence, the first three and the last limits of C can be met if $P(X_i > c_t | X_1 > c_t) \to 0$ for all $i >$ some i_0. Then C holds if $g(s) = o(m^{-1})$.

Lemma 1. *Fix $j \in \{2, 3, \ldots\}$ and let $\{t = t_m\}$ be a sequence such that $t_m \to \infty$ and $(t_m)^{-1} t_{m+1} \to 1$. For $\xi > 0$,*

$$\limsup_{m \to \infty} P(Z_{2,j} \leqq c_t(\xi) | X_1 > c_t(\xi)) = \limsup_{x \uparrow x_0} P(Z_{2,j} \leqq x | X_1 > x)$$

and

$$\liminf_{m \to \infty} P(Z_{2,j} \leqq c_t(\xi) | X_1 > c_t(\xi)) = \liminf_{x \uparrow x_0} P(Z_{2,j} \leqq x | X_1 > x).$$

Proof. Let $\varepsilon > 0$. We may assume without loss of generality that t_m is non-decreasing. Now,

$$\begin{aligned} P(c_{t_m}(\xi) < X_1 < c_{t_{m+1}}(\xi)) &= P(X_1 > c_{t_m}(\xi)) - P(X_1 \geqq c_{t_{m+1}}(\xi)) \\ &\leqq \left(\frac{t_{m+1}}{t_m} - 1\right) \frac{\xi}{t_{m+1}} \\ &\leqq \varepsilon P(X_1 \geqq c_{t_{m+1}}(\xi)), \end{aligned}$$

for m sufficiently large. For $x<x_0$, pick m such that $y \equiv c_{t_m}(\xi) \leqq x < c_{t_{m+1}}(\xi)$. For large x (and hence large m), the above gives

$$P(y<X_1 \leqq x) \leqq \varepsilon P(X_1>y).$$

Thus (if ε is close to 0),

$$\begin{aligned}
P(Z_{2,j} \leqq x | X_1 > x) &\leqq \left(\frac{P(X_1>y)}{P(X_1>x)}\right) \left(\frac{P(Z_{2,j} \leqq x, X_1>y)}{P(X_1>y)}\right) \\
&\leqq \frac{1}{1-\varepsilon} [P(Z_{2,j} \leqq y | X_1 > y) + P(y<Z_{2,j} \leqq x)/P(X_1>y)] \\
&\leqq (1+2\varepsilon)[P(Z_{2,j} \leqq y | X_1>y) + (j-1)\varepsilon] \\
&\leqq P(Z_{2,j} \leqq y | X_1 > y) + 2j\varepsilon.
\end{aligned}$$

This proves the first statement, since ε can be chosen arbitrarily small. On the other hand, with x and y as above,

$$\begin{aligned}
P(Z_{2,j} \leqq x | X_1 > x) &\geqq \frac{P(Z_{2,j} \leqq x, X_1 > x)}{P(X_1>y)} \\
&\geqq \frac{P(Z_{2,j} \leqq y, X_1>y) - \varepsilon P(X_1>y)}{P(X_1>y)} \\
&= P(Z_{2,j} \leqq y | X_1 > y) - \varepsilon,
\end{aligned}$$

which proves the second statement.

We are now ready to prove the main theorem. Let

$$\beta(j) = \limsup_{x \uparrow x_0} P(Z_{2,j} \leqq x | X_1 > x)$$

and let $\gamma(j) = \liminf_{x \uparrow x_0} P(Z_{2,j} \leqq x | X_1 > x)$. Let $\beta = \lim_{j \to \infty} \beta(j)$ and $\gamma = \lim_{j \to \infty} \gamma(j)$.

Theorem 1. *Let $\xi > 0$ and suppose $\{c_n(\xi)\}$ satisfies Condition C. Then*

$$\limsup_{n \to \infty} P(Z_n \leqq c_n(\xi)) = e^{-\gamma \xi}$$

and

$$\liminf_{n \to \infty} P(Z_n \leqq c_n(\xi)) = e^{-\beta \xi}.$$

Thus

$$\lim_{n \to \infty} P(Z_n \leqq c_n(\xi)) = e^{-\alpha \xi}$$

if and only if $\alpha = \beta = \gamma$.

Proof. Write c_n for $c_n(\xi)$. Since $r g(s) \to 0$, it can be shown that there is a sequence $\{q = q_m\}$ of positive integers such that $r g(q) \to 0$ and $q s^{-1} \to 0$. Define $p = p_m = s - q$. By the proof of Lemma 1 of Loynes [4], it is enough to show the results hold with $P(Z_n \leqq c_n)$ replaced by $P(Z_p \leqq c_t)^r$. Fix j. By the definition of $\Delta(j)$,

$$\begin{aligned}
& p P(X_1 > c_t, Z_{2,j} \leqq c_t) - p P(X_1 > c_t, Z_{2,p} \leqq c_t) \\
&\quad = p P(X_1 > c_t, Z_{2,j} \leqq c_t, Z_{j+1,p} > c_t) \qquad (5) \\
&\quad \leqq \xi r^{-1} \Delta(j) + o(r^{-1}).
\end{aligned}$$

Also,

$$P(Z_p > c_t) = \sum_{i=1}^{p} P[X_i > c_t,\ Z_{i+1,p} \leqq c_t] \\ \geqq p\,P[X_1 > c_t,\ Z_{2,p} \leqq c_t]. \tag{6}$$

By (5) and (6),

$$\begin{aligned} &p\,P(X_1 > c_t,\ Z_{2,j} \leqq c_t) - \xi r^{-1} \Delta(j) - o(r^{-1}) \\ &\leqq p\,P(X_1 > c_t,\ Z_{2,p} \leqq c_t) \\ &\leqq P(Z_p > c_t) \\ &\leqq (p-j)\,P(X_1 > c_t,\ Z_{2,j} \leqq c_t) + j\,P(X_1 > c_t) \\ &\leqq p\,P(X_1 > c_t,\ Z_{2,j} \leqq c_t) + o(r^{-1}). \end{aligned}$$

By the above and by (2),

$$\begin{aligned} &\left(1 - \xi r^{-1} P(Z_{2,j} \leqq c_t | X_1 > c_t) + \xi r^{-1} \Delta(j) + o(r^{-1})\right)^r \\ &\geqq P(Z_p \leqq c_t)^r \geqq \left(1 - \xi r^{-1} P(Z_{2,j} \leqq c_t | X_1 > c_t) + o(r^{-1})\right)^r, \end{aligned}$$

which implies that

$$\begin{aligned} &\exp[-\xi P(Z_{2,j} \leqq c_t | X_1 > c_t) + \xi \Delta(j)] + o(1) \\ &\geqq P(Z_p \leqq c_t)^r \geqq \exp[-\xi P(Z_{2,j} \leqq c_t | X_1 > c_t)] + o(1). \end{aligned}$$

Since j is arbitrary, the results follow, using Lemma 1.

Suppose C holds with $\Delta(j) = 0$ for some j. For $i \geqq j$, $\beta(i) = \beta$ and $\gamma(i) = \gamma$. Thus the calculation of β and γ depends only on the j-dimensional distributions of $\{X_n\}$. This happens in particular if $\{X_n\}$ is φ-mixing and $P(X_i > x | X_1 > x) \to 0$ as $x \uparrow x_0$ for all $i > j$, since in this case we have for $k > j$ that

$$\Delta(j) \leqq \Delta'(j) \leqq \sum_{i=j+1}^{k} \limsup P(X_i > c_t | X_1 > c_t) \\ + \limsup P(Z_{k+1,s} > c_t | X_1 > c_t) \leqq \varphi(k). \tag{7}$$

Because k is arbitrary, $\Delta(j) = 0$. The case $j = 1$ generalizes the final paragraph of Loynes [4].

3. Bounds on α

Suppose $P(Z_t \leqq c_t(\xi)) \to e^{-\alpha\xi}$. By Loynes, or by Theorem 1 if C holds, $0 \leqq \alpha \leqq 1$. We did not use C for the lower bound on $\lim P(Z_n \leqq c_n)$ in the proof of Theorem 1. Consequently $\alpha \leqq \beta$ for all uniformly mixing processes. If $P(X_i > x | X_1 > x) \to \delta > 0$ for some $i > 1$, then $\alpha \leqq \beta(i) \leqq 1 - \delta < 1$.

The next lemma leads to a positive lower bound for α in many cases. We call a sequence of events stationary if the sequence of their indicator functions is strictly stationary. Let A^* denote the complement of A.

Lemma 2. *Let $\{A_n\}$ be a stationary sequence of events. For $j > 1$,*

$$j\,P(A_2 A_3 \ldots A_j | A_1^*) \geqq P(A_{j+1} A_{j+2} \ldots A_{2j-1} | A_1^*).$$

Proof. We have

$$\begin{aligned} P(A_1^* A_{j+1} \dots A_{2j-1}) &= \sum_{i=1}^{j} P(A_1^* A_i^* A_{i+1} \dots A_{2j-1}) \\ &\leqq \sum_{i=1}^{j} P(A_i^* A_{i+1} \dots A_{j+i-1}) \\ &= j P(A_1^* A_2 \dots A_j), \end{aligned}$$

which gives the required result.

Theorem 2. *For any uniformly mixing stationary process* $\{X_n\}$,

$$\limsup_{n\to\infty} P(Z_n \leqq c_n(\xi)) \leqq \exp(-\gamma_0 \xi) \quad \textit{where} \quad \gamma_0 = \sup_j [j^{-1}(1-\Delta'(j)) - \Delta(j)].$$

(*Recall* (4).)

Proof. By the proof of Theorem 1 and by Lemma 2 with $A_i = \{X_i \leqq c_t(\xi)\}$ for large t,

$$\begin{aligned} \gamma &\geqq \liminf_{m\to\infty} P(Z_{2,j} \leqq c_t | X_1 > c_t) - \Delta(j) \\ &\geqq j^{-1} \liminf P(Z_{j+1,2j-1} \leqq c_t | X_1 > c_t) - \Delta(j) \\ &\geqq j^{-1}(1-\Delta'(j)) - \Delta(j). \end{aligned}$$

If $\Delta'(j)=0$ for some j, then by Theorem 2, $\limsup P(Z_n \leqq c_n(\xi)) \leqq \exp(-\xi/j)$. This is the case if the conditions above (7) hold. In particular, $\Delta'(j)=0$ if $\{X_n\}$ is j-dependent (see Watson [6]). One of our examples [5] shows how to obtain any limit $e^{-\alpha\xi}$ for $j^{-1} \leqq \alpha \leqq 1$ in the j-dependent case.

In the case of φ-mixing we can obtain the result $\alpha > 0$ without any further assumptions. We first need the following lemma.

Lemma 3. *If* $\{Y_n\}$ *is* φ*-mixing (not necessarily stationary) and if* $A_i \in \mathfrak{B}(Y_{ik})$ *for* $i=1,2,\dots$, *then*

$$|P(A_1 A_2 \dots A_l) - P(A_1) P(A_2) \dots P(A_l)| < \varphi(k).$$

Proof. First note that for any $A \in \mathfrak{B}(Y_1, \dots, Y_m)$ and $B \in \mathfrak{B}(Y_{m+k}, Y_{m+k+1}, \dots)$, $|P(AB) - P(A) P(B)| = |P(A^* B^*) - P(A^*) P(B^*)| \leqq P(A^*) \varphi(k)$. Thus,

$$\begin{aligned} &|P(A_1 A_2 \dots A_l) - P(A_1) P(A_2) \dots P(A_l)| \\ &\quad \leqq |P(A_1 A_2 \dots A_l) - P(A_1) P(A_2 A_3 \dots A_l)| \\ &\qquad + P(A_1) |P(A_2 A_3 \dots A_l) - P(A_2) P(A_3 \dots A_l)| \\ &\qquad + \dots + P(A_1) \dots P(A_{l-2}) |P(A_{l-1} A_l) - P(A_{l-1}) P(A_l)| \\ &\quad \leqq P(A_1^*) \varphi(k) + P(A_1) P(A_2^*) \varphi(k) + \dots + P(A_1) \dots P(A_{l-2}) P(A_{l-1}^*) \varphi(k) \\ &\quad = \varphi(k) [1 - P(A_1) P(A_2) \dots P(A_{l-1})], \end{aligned}$$

which gives the result.

Theorem 3. *If* $\{X_n\}$ *is a* φ*-mixing stationary process, then* $\limsup P(Z_n \leqq c_n(\xi)) < 1$.

Proof. Let k be such that $\varphi(k)<(2e)^{-1}$. Let η be sufficiently large that $e^{-\eta/k}<(2e)^{-1}$. By Lemma 3, $P(Z_n\leqq c_n(\eta))\leqq H^{n/k}(c_n(\eta))+\varphi(k)<e^{-1}$ for large n. Thus, $\limsup P(Z_n\leqq c_n(\eta))\leqq e^{-1}=e^{-\eta/\eta}$. By Theorem 1, $\limsup P(Z_n\leqq c_n(\xi))\leqq e^{-\xi/\eta}$.

4. Limits for General Sequences

The preceeding results may often be used to find $\lim_n P(Z_n\leqq d_n)$ for any sequence $\{d_n\}$.

Theorem 4. (a) *Suppose* $P(Z_n\leqq c_n(\xi))\to\exp(-\alpha\xi)$ *for all* $\xi>0$. *If* $\alpha>0$, *then* $H^n(d_n)\to l$ *if and only if* $P(Z_n\leqq d_n)\to l^\alpha$. *If* $\alpha=0$, $H^n(d_n)\to l>0$ *implies* $P(Z_n\leqq d_n)\to 1$.

(b) *Conversely, under Condition C,* $H^n(d_n)\to l\in(0,1)$ *and* $P(Z_n\leqq d_n)\to l^\alpha$ *together* imply $P(Z_n\leqq c_n(\xi))\to\exp(-\alpha\xi)$.

Proof. Assume $H^n(d_n)\to l>0$ and $\alpha>0$. Let ξ be such that $l>e^{-\xi}$. Since $H^n(c_n(\xi))\to e^{-\xi}$, $c_n(\xi)<d_n$ for large n. Therefore $P(Z_n\leqq d_n)\geqq P(Z_n\leqq c_n(\xi))\to e^{-\alpha\xi}$. Therefore $\liminf P(Z_n\leqq d_n)\geqq l^\alpha$. The rest of the proof is similar.

We give an example with $\varphi(2)=g(2)=0$ for which $P(Z_n\leqq d_n)$ converges but $H^n(d_n)$ and $P(Z_n\leqq c_n(\xi))$ do not converge. Let $X_1, X_3, X_5, \ldots$ be independent random variables, each uniform on (0, 1). Let $X_{2n}=2-3(2^{-k-1})-X_{2n-1}$ when $1-2^{-k}\leqq X_{2n-1}<1-2^{-k-1}$. X_{2n} is also uniform on (0, 1). Let

$$F(x)=P(X_1\leqq x,\, X_2\leqq x).$$

Then

$$F(x)=1-2^{-k} \quad \text{if } 1-2^{-k}\leqq x<1-3(2^{-k-2}) \quad \text{and} \quad F(x)=2x-1+2^{-k-1}$$

if $1-3(2^{-k-2})\leqq x<1-2^{-k-1}$. Thus F is a continuous function with alternating sections of slope 0 and slope 2. Let $d_n=\inf\left\{x\colon F(x)>1-\frac{1}{n}\right\}$. Then $P(Z_n\leqq d_n)=(F(d_n))^{n/2}+o(1)\to e^{-\frac{1}{2}}$. On the other hand, $H(d_n)=d_n$. If $n=2^k$ for some k, $d_n=1-3(2^{-k-2})$ so that $H(d_{2^k})^{2^k}\to e^{-\frac{3}{4}}$. If $n=2^k-1$,

$$d_n=1-(2^{k+1}-1)/(2^{k+1}(2^k-1))=1-2^{-k}+o(2^{-k}),$$

so that $H(d_{2^k-1})^{2^k-1}\to e^{-1}$. Thus $H^n(d_n)$ does not converge. By Theorem 4a, $P(Z_n\leqq c_n(\xi))$ cannot converge. If we now let $d'_n=c_n(1)$, we see that $H^n(d'_n)\to e^{-1}$ while $P(Z_n\leqq d'_n)$ does not converge. In order to achieve stationarity, we modify the process as follows: let J be a random variable which is independent of $\{X_n\}$ and takes values 0 and 1 with equal probability. If $J=0$, make no change, but if $J=1$, replace X_n by X_{n+1} for each n.

The above example can be modified in such a way that F has alternating sections of slope $\varepsilon>0$ and $2-\varepsilon$. Then let H be any strictly increasing distribution such that $F(H(a_n x+b_n))^n\to\Phi(x)$ for some extreme value distribution Φ (see Gnedenko [3]). Let $Y_n=H^{-1}(X_n)$. Then $P(\max(Y_1,\ldots,Y_n)\leqq a_n x+b_n)\to\Phi^{\frac{1}{2}}(x)$ but $H^n(a_n x+b_n)$ does not converge.

The process $\{X_n\}$ is said to satisfy the law of large numbers if there is a sequence of constants $\{A_n\}$ such that $P[|Z_n-A_n|>\varepsilon]\to 0$ for all $\varepsilon>0$. It is said to be relatively stable if there is a sequence of positive constants $\{B_n\}$ such that

$$P[|Z_n-B_n|>\varepsilon B_n]\to 0 \qquad \text{for all } \varepsilon>0.$$

These concepts were introduced by Gnedenko [3]. The next theorem shows in particular that these properties hold for φ-mixing processes if and only if they hold for the i.i.d. process with the same marginal distribution.

Theorem 5. *Assume $\gamma>0$ and $\{c_n(\xi)\}$ satisfies C for all $\xi>0$. For any sequence $\{d_n\}$, $P(Z_n \leqq d_n) \to 1$ (or 0) if and only if $H^n(d_n) \to 1$ (or 0, respectively).*

Proof. Assume $P(Z_n \leqq d_n) \to 1$. Now, $\limsup P(Z_n \leqq c_n(\xi)) = e^{-\gamma\xi} < 1$ for all $\xi > 0$. Thus $d_n > c_n(\xi)$ for large n. Therefore $\liminf H^n(d_n) \geqq \liminf H^n(c_n(\xi)) = e^{-\xi}$; thus, $H^n(d_n) \to 1$. The rest of the proof is similar.

Remarks. The assumption that H is continuous at x_0 only serves to avoid trivialities. The assumption that $P(X_1 \geqq x)/P(X_1 > x) \to 1$ may be dropped by making some minor modifications. Condition C must be changed to C' by replacing (3) by $\Delta'(j) \to 0$.

The conclusions of Theorem 1 take the form (for example)

$$\limsup_{n\to\infty} P(Z_n \leqq c_n(\xi)) - H^{n\gamma}(c_n(\xi)) = 0.$$

This is proved by the methods of the author [5]. All φ-mixing processes are still included. Theorem 5 remains valid under these circumstances.

The stronger Condition C′ mentioned above has the following property: if $\{c_n\}$ is a sequence such that $\limsup H^n(c_n) < 1$ and $\liminf H^n(c_n) > 0$ and $\{c_n\}$ satisfies C', then any sequence $\{d_n\}$ such that $\limsup H^n(d_n) < 1$ and $\liminf H^n(d_n) > 0$ also satisfies C'.

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G.L. O'Brien
York University
Faculty of Arts
Department of Mathematics
Downsview, Ontario M3J 1P3
Canada

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