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A Characterization of Regular Solutions of a Linear Stochastic Differential Equation

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§ 1. Introduction

The importance of the Hilbert space structure of wide-sense stationary stochastic processes was pointed out by Cramér [1] and Kolmogorov [4]. The evolution of a stationary process $\{X_t\}$ is described on its space $\mathscr{H}^X$ by a one-parameter group $\{T_t\}$ of unitary operators. The process is regular (completely non-deterministic) if there is a closed subspace D of $\mathscr{H}^X$ such that the span of the T_t-translates is dense in $\mathscr{H}^X$ and their intersection is the zero vector. We call the triple $(\mathscr{H}^X, T_t, D)$ a K-structure. In their investigation of classical wave equations Lax and Phillips [5] discovered that scattering processes can be described by a double K-structure which which we refer to as an LP-structure. In this paper we consider regular stochastic processes which are solutions of a linear stochastic differential equation and show that they are characterized by having an LP-structure. This provides an isomorphism between the Hilbert space structure of such a stochastic process and that of a scattering process. It is this isomorphism which makes possible the mechanical modelling of Brownian motion by arrays of coupled oscillators (see Ford, Kac, and Mazur [3] and earlier papers cited by them). We have used these ideas in showing how to construct a heat bath for a Långevin equation (Lewis and Thomas [6]).

In this paper we are concerned only with the Hilbert space structure of the stochastic processes we refer to. Indeed, in the body of the paper we use the neutral term "process" to denote a family $\{X_t: t\in R\}$ of linear mappings from a Hilbert space $\mathscr{M}$ into a Hilbert space $\mathscr{H}$. If we take $\mathscr{H}$ to be $L^2(\Omega, P)$ of a probability space $(\Omega, \mathscr{B}, P)$ we can interpret the process as a stochastic process over $(\Omega, \mathscr{B}, P)$ taking values in $\mathscr{M}$ and having zero mean and variance given by the inner product in $\mathscr{H}$. On the other hand, if we take $\mathscr{H}$ to be the Hilbert space of Cauchy data for a wave equation with energy norm (see Lax and Phillips [5]) we can interpret the process as a scattering process. The abstract version of a linear stochastic differential equation is given the neutral name "Langevin equation".

In § 2 we review results about K-structures, in § 3 we review results about LP-structures, and in § 4 we state and prove the characterization of the regular solution of a Langevin equation. The central idea is the moving-average representation of a K-structure and the special form which this can be given in the special case of an LP-structure.

Notation. $\mathscr{H}, \mathscr{M}, \mathscr{N}, K, \ldots$ will denote separable Hilbert spaces, either real or complex. The inner product in these spaces is denoted by $(\cdot,\cdot)$ and is always linear in the second argument; when there is risk of confusion we distinguish the inner products in the different spaces by subscripts thus: $(\cdot,\cdot)_{\mathscr{H}}, (\cdot,\cdot)_K, \ldots$. Whenever $[\mathcal{S}_i\colon i\in\mathcal{I}\}$ is a collection of subsets of a Hilbert space we denote by $V\mathcal{S}_i$ the smallest closed subspace which contains every member of the collection, and by $\Lambda\mathcal{S}_i$ the largest closed subspace which is contained in every member of the collection. The Hilbert space of equivalence classes of Borel measurable functions $n(\cdot)$ on R taking values in a Hilbert space $\mathcal{N}$ such that $\int\limits_R \|n(u)\|^2_{\mathcal{N}}du$ is finite is denoted by $L^2(R,\mathcal{N})$. The spaces $L^2((-\infty,0];\mathcal{N})$ and $L^2([0,\infty);\mathcal{N})$ will be identified in the obvious way with closed subspaces of $L^2(R;\mathcal{N})$.

§ 2. *K*-Structures

Let $\mathcal{H}$ be a separable Hilbert space. Let $\{T_t\colon t\in R\}$ be a strongly continuous one-parameter group of unitary operators on $\mathcal{H}$. Let D be a closed subspace of $\mathcal{H}$ and denote by D_t the image of D under T_t. The triple $(\mathcal{H}, T_t, D)$ is said to be a *K-structure* if the family $\{D_t\colon t\in R\}$ satisfies the following three conditions: (i) $D\subseteq D_t$ for all $t\geqq 0$, (ii) $VD_t=\mathcal{H}$, (iii) $\Lambda D_t=\{0\}$. An isometry R from $\mathcal{H}$ onto $L^2(R;\mathcal{N})$ such that

$$(RT_t h)(u)=(Rh)(u-t)$$

is called a *translation-representation of* $\{T_t\}$ *on* $L^2(R;\mathcal{N})$. Lax and Phillips [5] proved the following theorem for K-structures; it is closely relate to a result obtained earlier by Sinai [10].

Theorem 2.1. *Let $(\mathcal{H}, T_t, D)$ be a K-structure. Then there exists a translation-representation $R\colon \mathcal{H}\to L^2(R;\mathcal{N})$ of $\{T_t\}$ in which the image of D is $L^2((-\infty,0];\mathcal{N})$. The dimension of $\mathcal{N}$ is uniquely determined by $(\mathcal{H}, T_t, D)$.* This is an immediate consequence of Mackey's Imprimitivity Theorem [7].

The translation-representation can be pulled back to $\mathcal{H}$ to give a moving-average representation. This is most conveniently done using the notion of quasi-isometric operator-valued measures introduced by Masani [8]. Let $\mathcal{I}$ be the pre-ring of intervals $(a,b]$ of R, let $|\Delta|$ denote the Lebesgue measure of $\Delta\in\mathcal{I}$. A function $\xi(\cdot)$ on $\mathcal{I}$ such that $\xi(\Delta)$ is a linear mapping from $\mathcal{N}$ into $\mathcal{H}$ satisfying

$$(\xi(\Delta_1)n_1,\xi(\Delta_2)n_2)_{\mathcal{H}}=|\Delta_1\cap\Delta_2|\,(n_1,n_2)_{\mathcal{N}} \tag{$*$}$$

for all Δ_1, Δ_2 in $\mathcal{I}$ and all n_1, n_2 in $\mathcal{N}$ is called a *quasi-isometric measure over* $(R,\mathcal{I},|\cdot|)$. The integral $\int\limits_R \xi(du)n(u)$ of a function $n(\cdot)\in L^2(R;\mathcal{N})$ with respect to a quasi-isometric measure $\xi(\cdot)$ is defined as follows: first consider the case of a simple function

$$n(u)=\sum_{i=1}^{m}\chi_{\Delta_i}(u)n_i$$

and define $\int\limits_R \xi(du)n(u)$ to be $\sum\limits_{i=1}^{m}\xi(\Delta_i)n_i$. It follows from $(*)$ that the map from the simple functions into $\mathcal{H}$ defined in this way is an isometry and hence has a unique isometric extension ξ to all of $L^2(R;\mathcal{N})$ since the simple functions are dense and $\mathcal{H}$ is complete. We have

Theorem 2.2. *Let $\xi(\cdot)\colon \mathcal{N}\to\mathcal{H}$ be a quasi-isometric measure over $(R,\mathcal{I},|\cdot|)$. Then there exists a unique isometry ξ of $L^2(R;\mathcal{N})$ into $\mathcal{H}$ denoted by*

$$n(\cdot)\mapsto\int\limits_R \xi(du)\,n(u)$$

such that for all $\Delta \in \mathscr{I}$ and $n \in \mathscr{N}$

$$\int_R \xi(du)\chi_\Delta(u)\,n = \xi(\Delta)\,n.$$

The image $\mathscr{H}^\xi$ *of* $L^2(R;\mathscr{N})$ *under* ξ *is given by*

$$\mathscr{H}^\xi = V\{\xi(\Delta)\,n\colon \Delta \in \mathscr{I},\, n \in \mathscr{N}\}.$$

Combining this result with the translation-representation theorem we get the moving-average representation:

Theorem 2.3. *Let* $(\mathscr{H}, T_t, D)$ *be a* K*-structure. Let* $R\colon \mathscr{H} \to L^2(R;\mathscr{N})$ *be a translation-representation which maps* D *onto* $L^2((-\infty,0];\mathscr{N})$*. Then there exists a quasi-isometric measure* $\xi(\cdot)\colon \mathscr{N} \to \mathscr{H}$ *over* $(R, \mathscr{I}, |\cdot|)$ *such that the unique isometry* $\xi\colon L^2(R;\mathscr{N}) \to \mathscr{H}$ *determined by* $\xi(\cdot)$ *is the inverse of* R.

In particular for each $h \in D$

$$T_t h = \int_{-\infty}^{t} \xi(du)(Rh)(u-t).$$

Proof. For each $\Delta \in \mathscr{I}$ define the linear mapping $\xi(\Delta)\colon \mathscr{N} \to \mathscr{H}$ by

$$\xi(\Delta)\,n = R^{-1}(\chi_\Delta(\cdot)\,n).$$

It is easily checked that $\xi(\cdot)$ is a quasi-isometric measure. The isometry ξ which it determines agrees with R^{-1} on the simple functions and hence on the whole of $L^2(R;\mathscr{N})$. Written explicitly

$$\begin{aligned} T_t h &= \int_R \xi(du)(RT_t h)(u) \\ &= \int_R \xi(du)(Rh)(u-t). \end{aligned}$$

For $h \in D$ the support of $(Rh)(\cdot)$ lies in $(-\infty,0]$ and so we may write

$$T_t h = \int_{-\infty}^{t} \xi(du)(Rh)(u-t).$$

§ 3. *LP*-Structures

Let $\mathscr{H}$ be a separable Hilbert space, let $\{T_t\colon t \in R\}$ be a strongly-continuous one-parameter group of unitary operators on $\mathscr{H}$ and let D_+ and D_- be a pair of orthogonal closed subspaces of $\mathscr{H}$. Then $(\mathscr{H}, T_t, D_-, D_+)$ is said to be an *LP-structure* if $(\mathscr{H}, T_t, D_-)$ and $(\mathscr{H}, T_t^*, D_+)$ are both K-structures. The subspace $K = (D_- \oplus D_+)^\perp$ is very important. We say that the LP-structure is *trivial* if $K = \{0\}$, *non-trivial* if $K \neq \{0\}$, and *cyclic* if $VK_t = \mathscr{H}$ where K_t denotes the image of K under T_t. [Each K-structure $(\mathscr{H}, T_t, D)$ determines a trivial LP-structure $(\mathscr{H}, T_t, D, D^\perp)$.] The *direct sum* of two LP-structures $(\mathscr{H}, T_t, D_-, D_+)$ and $(\bar{\mathscr{H}}, \bar{T}_t, \bar{D}_-, \bar{D}_+)$ is the LP-structure $(\mathscr{H} \oplus \bar{\mathscr{H}}, T_t \oplus \bar{T}_t, D_- \oplus \bar{D}_-, D_+ \oplus \bar{D}_+)$.

Lemma. *A non-trivial LP-structure is either cyclic or the direct sum of a cyclic LP-structure and a trivial LP-structure.*

Proof. Let $(\mathscr{H}, T_t, D_-, D_+)$ be non-trivial so that $K=(D_-\oplus D_+)^\perp$ is not the zero-vector. Let $\mathscr{H}^c = VK_t$. If $\mathscr{H}=\mathscr{H}^c$ the LP-structure is cyclic. If $\mathscr{H}\neq\mathscr{H}^c$ then $(\mathscr{H}^c)^\perp \neq \{0\}$, and $\mathscr{H}^c$ is invariant under T_t for all t and $(\mathscr{H}^c)^\perp$ is invariant under T_t^* for all t so that the orthogonal projection P of $\mathscr{H}$ onto $\mathscr{H}^c$ commutes with $\{T_t\colon t\in R\}$. It is easily checked that $(P\mathscr{H}, T_tP, PD_-, PD_+)$ is a cyclic LP-structure and that $(Q\mathscr{H}, T_tQ, QD_-, QD_+)$ is a trivial LP-structure where $Q=1-P$.

Lax and Phillips [5] have shown that the restriction of T_t to K is a semi-group of contractions which tends strongly to zero as $t\to\infty$.

Theorem 3.1. *Let $(\mathscr{H}, T_t, D_-, D_+)$ be a non-trivial LP-structure, let P_+, P_- be the orthogonal projections onto $D_+^\perp$, $D_-^\perp$ respectively and for $t\geqq 0$ let $S_t=P_+\,T_t\,P_-$. Then*

(i) *S_t annihilates D_+ and D_- and maps K into itself.*

(ii) *On K the operators $\{S_t\colon t\geqq 0\}$ form a strongly continuous semi-group of contractions.*

(iii) *$\{S_t\}$ tends strongly to zero as $t\to\infty$; for each $k\in K$*

$$\lim_{t\to\infty} S_t k = 0.$$

We sketch the proof; Lax and Phillips [5] give full details. (i) and (ii) are straightforward consequences of the definitions. To prove (iii) we use the fact that the translates of D_+ are dense in $\mathscr{H}$. Thus for every k in K and every $\varepsilon>0$ there exists h in D_+ and $t_0>0$ such that

$$\|k-T_{-t_0}h\|<\varepsilon.$$

Since $P_+\,T_t$ is a contraction

$$\|P_+\,T_t(k-T_{-t_0}h)\|<\varepsilon$$

so that

$$\|S_t k - P_+\,T_{t-t_0}h\|<\varepsilon.$$

Choose $t>t_0$; then $T_{t-t_0}h\in D_+$ and hence $\|S_t k\|<\varepsilon$.

The following result which is a special case of a theorem of Sz-Nagy and Foias [11] was proved by Lax and Phillips [5]. We sketch the proof.

Theorem 3.2. *Let $\{S_t\colon t\geqq 0\}$ be a strongly continuous semi-group of contractions on a Hilbert space K which tends strongly to zero as $t\to\infty$. Let B be the generator of $\{S_t\}$. Then there exists a Hilbert space $\mathscr{N}$, a linear mapping $A\colon D(B)\to\mathscr{N}$ and an isometry $\mathscr{R}$ of K into $L^2(R;\mathscr{N})$ given on $D(B)$ by*

$$(\mathscr{R}k)(s)=\begin{cases} AS_{-s}k & s\leqq 0\\ 0 & s>0\end{cases}$$

which sends S_t into right-translation by t followed by restriction to $(-\infty, 0]$.

Proof. Let B be the infinitesimal generator of S_t. The domain $D(B)$ of B is dense in K and since S_t, is a contraction the form $[\cdot,\cdot]$ defined on $D(B)$ by

$$[k,k] = -(k, Bk)_K - (Bk, k)_K$$

is non-negative. Let $D(B)_0$ be the set of vectors in $D(B)$ for which $[k,k]$ is zero. The form $[\cdot,\cdot]$ induces an inner product $(\cdot,\cdot)_{\mathscr{N}}$ on $D(B)/D(B)_0$ and we define $\mathscr{N}$

to be the completion of $D(B)/D(B)_0$ in the associated norm. Let A be the quotient map of $D(B)$ into $\mathcal{N}$. For k in $D(B)$ define

$$(Rk)(s)=\begin{cases} AS_{-s}k & s\leqq 0\\ 0 & s>0.\end{cases}$$

Then

$$\begin{aligned}\int_R \|(Rk)(s)\|^2\,ds &= -\int_{-\infty}^{0}\{(BS_{-s}k, S_{-s}k)_K+(S_{-s}k, BS_{-s}k)_K\}\,ds\\ &= \int_{-\infty}^{0}\frac{d}{ds}\|S_{-s}k\|_K^2\,ds=\|k\|_K^2\end{aligned}$$

since $\lim\limits_{s\to\infty}\|S_s k\|_K^2=0$. Thus R is an isometry from $D(B)$ into $L^2(R;\mathcal{N})$ which extends by continuity to all of K. It is clear from the construction that S_t goes into translation by t followed by restriction to $(-\infty, 0]$.

It has been shown by Douglas [2] that the unitary group of translations on $L^2(R;\mathcal{N})$ is a minimal unitary dilation of the semigroup RS_tR^{-1} so that the translates of RK are dense in $L^2(R;\mathcal{N})$. This enables us to get explicit forms for the translation representations of the K-structures associated with a cyclic LP-structure $(\mathcal{H}, T_t, D_-, D_+)$. Let $\{S_t\colon t\geqq 0\}$ be the semigroup of contractions on $K=(D_+\oplus D_-)^\perp$ which it determines and let B be its generator. Applying the construction of Theorem 3.2 to this we get a translation representation $R_+\colon \mathcal{H}\to L^2(R;\mathcal{N}_+)$ where $\mathcal{N}_+$ is the completion of $D(B)/D(B)_0$ with respect to the norm got from the form

$$[k,k]_+=-(k,Bk)_K-(Bk,k)_K.$$

Denoting the quotient map of $D(B)$ into $\mathcal{N}_+$ by A_+ we have

$$(R_+k)(s)=\begin{cases} A_+\exp(-sB)k & s\leqq 0\\ 0 & s>0.\end{cases}$$

This extends to a translation representation of $\{T_t\}$ on $L^2(R;\mathcal{N}_+)$ in which D_+ is mapped onto $L^2([0,\infty);\mathcal{N}_+)$. Starting with $D(B^*)$ in place of $D(B)$ we get a translation representation $R_-\colon \mathcal{H}\to L^2(R;\mathcal{N}_-)$ where $\mathcal{N}_-$ is the completion of $D(B^*)/D(B^*)_0$ with respect to the norm got from the form

$$[k,k]_-=-(k,B^*k)-(B^*k,k)_K.$$

Denoting by A_- the quotient map of $D(B^*)$ into $\mathcal{N}_-$ we have

$$(R_-k)(s)=\begin{cases} 0 & s<0\\ A_-\exp(sB^*) & s\geqq 0.\end{cases}$$

This extends to a translation-representation of $\{T_t\}$ on $L^2(R;\mathcal{N}_-)$ in which D_- is mapped onto $L^2((-\infty,0];\mathcal{N}_-)$. Thus we have

Theorem 3.3. *Let $(\mathcal{H}, T_t, D_-, D_+)$ be a cyclic LP-structure. Let $\{S_t\colon t\geqq 0\}$ be the semi-group of contractions got by restricting $\{T_t\}$ to $K=(D_-\oplus D_+)^\perp$. Then there exist translation-representations $R_\pm\colon \mathcal{H}\to L^2(R;\mathcal{N}_\pm)$ of $\{T_t\}$ such that $R_+D_+=L^2([0,\infty);\mathcal{N}_+)$, $R_-D_-=L^2((-\infty,0];\mathcal{N}_-)$. Let B be the infinitesimal* *generator of $\{S_t\}$. Then there exist linear mappings $A_+ : D(B) \to \mathcal{N}_+, A_- : D(B^*) \to \mathcal{N}_-$ such that $R_\pm$ are given by*

$$(R_+ k)(s) = \begin{cases} A_+ S_{-s} k & s \leqq 0, \\ 0 & s > 0, \end{cases} \quad \text{for } k \in D(B),$$

$$(R_- k)(s) = \begin{cases} 0 & s < 0, \\ A_- S_s^* k & s \geqq 0, \end{cases} \quad \text{for } k \in D(B^*).$$

Combining this result with Theorem 2.3 we have

Theorem 3.4. *Let $(\mathcal{H}, T_t, D_-, D_+)$ be a cyclic LP-structure. Then there exist quasi-isometric measures $\xi_\pm(\cdot) : \mathcal{N}_\pm \to \mathcal{H}$ such that*

$$T_t k = \int_{-\infty}^{t} \xi_+(du) A_+ e^{B(t-u)} k, \qquad k \in D(B)$$

$$= \int_{t}^{\infty} \xi_-(du) A_- e^{B^*(t+u)} k, \qquad k \in D(B^*).$$

§ 4. Langevin Equations

Let $\xi(\cdot) : \mathcal{N} \to \mathcal{H}$ be a quasi-isometric measure over $(R, \mathcal{I}, |\cdot|)$. Define the function $t \mapsto \xi_t$ as follows: for each $t \in R$ let $\xi_t : \mathcal{N} \to \mathcal{H}$ be the linear mapping given by

$$\xi_t = \begin{cases} \xi((0,t]) & t > 0 \\ 0 & t = 0 \\ -\xi((t,0]) & t < 0, \end{cases}$$

so that for all $s, t \in R$ we have

$$\xi((s,t]) = \xi_t - \xi_s. \tag{$*$}$$

It follows that

$$(\xi_t n, \xi_s n')_{\mathcal{H}} = (s \wedge t)(n, n')_{\mathcal{N}}. \tag{$**$}$$

We say that a function $t \to \xi_t$ taking values in the linear mappings from $\mathcal{N}$ into $\mathcal{H}$ is an *operator-valued Wiener process* if it satisfies ($**$). It is easy to see that given such a function we can define a quasi-isometric measure on $(R, \mathcal{I}, |\cdot|)$ by means of ($*$). We are now in a position to derive an "integration-by-parts" formula which generalizes that for stochastic integrals. We adapt the proof given by Nelson [9] to the general situation.

A function $n(\cdot) \in L^2(R; \mathcal{N})$ is said to be a *function of bounded variation with compact support* if for some orthonormal basis $\{e_i\}$ for $\mathcal{N}$ all the components $n^{(i)}(\cdot) = (e_i, n(\cdot))$ of $n(\cdot)$ are functions of bounded variation with compact support. For such a function $n(\cdot)$ we define an integral $\int_R \xi_t \, dn(t)$ of an operator-valued Wiener process as follows: for each i define the function $t \to \xi_t^{(i)} = \xi_t e_i$; then $\|\xi_t^{(i)}\|_{\mathcal{H}} = |t|^{\frac{1}{2}}$ and for each $h \in \mathcal{H}$

$$|(h, \xi_t^{(i)}) - (h, \xi_s^{(i)})| \leqq \|h\| \, |s-t|^{\frac{1}{2}}$$

so that for each $h \in \mathcal{H}$ the Stieltje's integral

$$\int_R (h, \xi_t^{(i)}) \, dn^{(i)}(t)$$

exists, and

$$h \to \int_R (h, \xi_t^{(i)})\, dn^{(i)}(t)$$

is a bounded linear functional on $\mathscr{H}$. Let $\int_R \xi_t^{(i)}\, dn^{(i)}(t)$ be the element of $\mathscr{H}$ which it determines through the Riesz Representation Theorem. In the case where $n^{(i)}(\cdot) = \chi_{(a,b]}(\cdot)$ we have

$$\int_R \xi_t^{(i)}\, dn^{(i)}(t) = \xi_a^{(i)} - \xi_b^{(i)} = -\int_R \xi(dt)\, e_i\, n^{(i)}(t).$$

Taking a sequence $\{f_m\}$ of step-functions such that $f_m \to n^{(i)}$ in $L^2(R)$ and $df_m \to dn^{(i)}$ in the weak*-topology of measures we have

$$\int_R \xi_t^{(i)} dn^{(i)}(t) = -\int_R \xi(dt)\, e_i\, n^{(i)}(t)$$

for an arbitrary function of bounded variation with compact support. It follows that

$$\Big\|\int_R \xi_t^{(i)}\, dn^{(i)}(t)\Big\|^2 = \int_R |n^{(i)}(t)|^2\, dt.$$

Hence $\sum_{i=1}^{\infty} \big\|\int_R \xi_t^{(i)}\, dn^{(i)}(t)\big\|^2 = \|n(\cdot)\|^2 < \infty$ since we assumed that $n(\cdot) \in L^2(R; \mathscr{N})$ and so we may define $\int_R \xi_t\, dn(t)$ as an element of $\mathscr{H}$ by

$$\int_R \xi_t\, dn(t) = \sum_{i=1}^{\infty} \int_R \xi_t^{(i)}\, dn^{(i)}(t).$$

It follows that

$$\int_R \xi_t\, dn(t) = -\sum_{i=1}^{\infty} \int_R \xi(dt)\, e_i (e_i, n(t)) = -\int_R \xi(dt)\, n(t).$$

The right-hand side is independent of the choice of basis $\{e_i\}$ so the value of the left-hand side is the same for any basis $\{e_i\}$ in which each component $n^{(i)}(\cdot)$ is of bounded variation with compact support. Hence $\int_R \xi_t\, dn(t)$ is well-defined and we have the "Integration-by-parts" formula:

Theorem 4.1. *Let $n(\cdot)$ be a function in $L^2(R; \mathscr{N})$ of bounded variation with compact support. Let $\{\xi_t : t \in R\}$ be an operator-valued Wiener process. Then*

$$\int_R \xi(dt)\, n(t) = -\int_R \xi_t\, dn(t).$$

We define a *process* as a family $\{X_t : t \in R\}$ of continuous linear mappings $X_t : \mathscr{M} \to \mathscr{H}$ such that for all $m \in \mathscr{M}$ the function $t \to X_t m$ is continuous. The *space of the process* $\mathscr{H}^X$ is defined by $\mathscr{H}^X = V\{X_t m : t \in R, m \in \mathscr{M}\}$. The *history of the process up to time t* is $\mathscr{H}_t^X$, defined by

$$\mathscr{H}_t^X = V\{X_s m : s \leqq t, m \in \mathscr{M}\}.$$

Clearly we have

$$V \mathscr{H}_t^X = \mathscr{H}^X.$$

If in addition we have

$$\Lambda \mathscr{H}_t^X = \{0\}$$

we say the process is *regular*. We say that the process is *stationary* if $(X_s m, X_{s+t} m')$ is independent of s for all t and all $m, m' \in \mathcal{M}$. In which case there exists a bounded operator $R(t)\colon \mathcal{M} \to \mathcal{M}$ such that

$$(X_s m, X_{s+t} m') = (m, R(t) m')$$

for all $s \in R$ and all $m, m' \in \mathcal{M}$. The family $\{T_t\colon t \in R\}$ of linear operators on $\mathcal{H}^X$ defined by $T_t X_s m = X_{s+t} m$ for all $s \in R$ and all $m \in \mathcal{M}$ is a strongly continuous group of unitary operators. Evidently $\{X_t\colon t \in R\}$ is a regular stationary process if and only if $(\mathcal{H}^X, T_t, \mathcal{H}_0^X)$ is a K-structure.

Let $\{S_t\colon t \geqq 0\}$ be a strongly continuous semi-group of contractions on a Hilbert space $\mathcal{M}$ which tends strongly to zero as $t \to \infty$. Let B be the infinitesimal generator of $\{S_t\}$ with domain $D(B)$. Let $A\colon D(B) \to \mathcal{N}$ be a linear mapping such that

$$\int_R \|A e^{tB} k\|_{\mathcal{N}}^2 \, dt = \|k\|_{\mathcal{M}}^2.$$

Let $\xi_t\colon \mathcal{N} \to \mathcal{H}$ be an operator-valued Wiener process. We say that a process $\{X_t\colon t \in R\}$ where $X_t\colon \mathcal{M} \to \mathcal{H}$ is a solution of the Langevin equation

$$dX_t = X_t B \, dt + \xi(dt) A$$

if for all $m \in D(B)$ we have

$$X_t m - X_s m = \int_s^t X_u B m \, du + \xi_t A m - \xi_s A m$$

for all $s, t \in R$.

Theorem 4.2. *The Langevin equation*

$$dX_t = X_t B \, dt + \xi(dt) A$$

has a unique regular solution given for each $m \in D(B)$ *by*

$$X_t m = \int_{-\infty}^t \xi(ds) A e^{B(t-s)} m.$$

It is necessarily stationary with

$$(X_s m, X_{s+t} m')_{\mathcal{H}} = \begin{cases} (m, e^{Bt} m')_{\mathcal{M}} & t \geqq 0 \\ (m, e^{-B^* t} m')_{\mathcal{M}} & t \leqq 0. \end{cases}$$

Proof. The integral $\int_{-\infty}^t \xi(ds) A e^{B(t-s)} m$ exists by virtue of the condition

$$\int_0^\infty \|A e^{Bt} m\|^2 \, dt = \|m\|^2,$$

since then the function

$$n(s) = \begin{cases} A e^{B(t-s)} m & s \leqq t \\ 0 & s > t \end{cases}$$

has $L^2(R; \mathcal{N})$-norm equal to $\|m\|$.

By polarization

$$\int_0^\infty (A e^{Bt} m, A e^{Bt} m') \, dt = (m, m').$$

Hence

$$(X_s m, X_{s+t} m') = \int_{-\infty}^{\min(s,s+t)} (A e^{B(s-u)} m, A e^{B(s+t-u)} m')\, du$$

$$= \begin{cases} (m, e^{Bt} m')_{\mathcal{M}} & t \geqq 0 \\ (m, e^{-B^* t} m')_{\mathcal{M}} & t \leqq 0. \end{cases}$$

To see that $X_t m$ satisfies the Langevin equation we use Theorem 4.1 to integrate by parts.

We have

$$X_t m = X_0 e^{Bt} m + \int_0^t \xi(du) A e^{B(t-u)} m$$

$$= X_0 e^{Bt} m + \xi_t A m - \xi_0 A e^{Bt} m + \int_0^t \xi_u A e^{B(t-u)} B m\, du.$$

We see that $X_t m - \xi_t A m$ is differentiable with derivative

$$\frac{d}{dt}(X_t m - \xi_t A m) = X_0 e^{Bt} B m - \xi_0 A e^{Bt} B m$$

$$+ \xi_t A B m + \int_0^t \xi_u A e^{B(t-u)} B^2 m\, du = X_t B m.$$

Hence

$$X_t m - X_s m = \int_s^t X_u B m\, du + \xi_t A m - \xi_s A m.$$

Thus $\{X_t \colon t \in R\}$ is a solution of the Langevin equation, and $\mathscr{H}_t^X = \mathscr{H}_t^\xi$ for all t. Hence $\Lambda \mathscr{H}_t^X = \{0\}$, so that $\{X_t\}$ is regular. Suppose $\{Y_t\}$ is another regular process satisfying the Langevin equation; put

$$W_t = X_t - Y_t.$$

Then

$$W_t k - W_s k = \int_s^t W_u B k\, du$$

so that

$$W_t k = W_0 e^{Bt} k$$

for all $k \in D(B)$ and all $t \in R$.

Hence $\mathscr{H}_t^W = V\{W_0 m \colon m \in D(B)\} = W$ say. But for all t

$$\mathscr{H}_t^W \subseteq \mathscr{H}_t^X \vee \mathscr{H}_t^Y$$

so that $W \subseteq \mathscr{H}_t^X \vee \mathscr{H}_t^Y$ for all t.

Since $\{Y_t\}$ satisfies the Langevin equation, $Y_t m - Y_s m - \int_s^t Y_u B m\, du = \xi_{(s,t]} A m$, we see that

$$\xi_{(s,t]} A m \in \mathscr{H}_t^Y \quad \text{which implies} \quad \mathscr{H}_t^X \subseteq \mathscr{H}_t^Y$$

and so

$$W \subseteq \mathscr{H}_t^Y \qquad \text{for all } t;$$

hence

$$W \subseteq \Lambda \mathscr{H}_t^Y.$$

But $\Lambda\mathscr{H}_t^Y=\{0\}$ since $\{Y_t\}$ is regular and so $W_t\equiv 0$.

We can restate Theorem 3.4 as

Theorem 4.3. *Let* $(\mathscr{H}, T_t, D_-, D_+)$ *be a cyclic LP-structure. Let* $\{S_t\colon t\geqq 0\}$ *be the semi-group of contractions got by restricting* $\{T_t\}$ *to* $K=(D_-\oplus D_+)^\perp$, *and let* B *be the infinitesimal generator of* $\{S_t\}$. *Let* j *be the injection of* K *in* $\mathscr{H}$. *Then there exists a Hilbert space* $\mathscr{N}$, *an operator-valued Wiener process* $\{\xi_t\}$, $\xi_t\colon \mathscr{N}\to\mathscr{H}$, *and a linear mapping* $A\colon D(B)\to\mathscr{N}$ *such that the process* $\{X_t\}$ *given by* $X_t=T_t\circ j$ *is the unique regular solution of the Langevin equation* $dX_t=X_tB\,dt+\xi(dt)A$.

Finally, we prove a converse to this:

Theorem 4.4. *Let* $\{X_t\colon t\in R\}$ *be the unique regular solution of the Langevin equation*

$$dX_t=X_tB\,dt+\xi(dt)A.$$

Let $K=V\{X_0m\colon m\in\mathscr{M}\}$, $D_+=(\mathscr{H}_0^X)^\perp$ *and let* $D_-=(D_+\oplus K)^\perp$. *Then* $(\mathscr{H}^X, T_t, D_-, D_+)$ *is a cyclic LP-structure.*

Proof. We have remarked that $(\mathscr{H}^X, T_t, \mathscr{H}_0^X)$ is a K-structure since $\{X_t\}$ is regular, and so $(\mathscr{H}^X, T_t^*, D_+)$ is also a K-structure. It is clear from the definition that K is a cyclic subspace and that $\Lambda T_tD_-\subseteq\Lambda T_t\mathscr{H}_0^X=\Lambda\mathscr{H}_t^X=\{0\}$ so all that remains to be proved is that

$$VT_tD_-=\mathscr{H}^X.$$

Since $\mathscr{H}^X=\mathscr{H}^\xi$ it is enough to show that for an arbitrary interval Δ and $n\in\mathscr{N}$ the vector $\xi(\Delta)n$ can be approximated arbitrarily closely by a T_t-translate of a vector in D_-.

Now there exists t_0 such that for all $t>t_0$

$$T_{-t}\,\xi(\Delta)\,n\in\mathscr{H}_0^\xi=\mathscr{H}_0^X=D_-\oplus K.$$

Hence there exists $k\in K$, $\|k\|=1$ such that

$$T_{-t}\,\xi(\Delta)\,n=d+\lambda k$$

where $d\in D$. Given $\varepsilon>0$ there exists $m\in\mathscr{M}$, such that

$$\left\|k-\int_{-\infty}^{0}\xi(du)\,A\,e^{-Bu}m\right\|<\frac{\varepsilon}{2}.$$

Then $\|T_{-t}\,\xi(\Delta)\,n-d\|=|\lambda|$ and

$$\lambda=\big(T_{-t}\,\xi(\Delta)\,n,\,k\big)$$

so that

$$|\lambda|\leqq\left|\left(T_{-t}\xi(\Delta)\,n,\ \int_{-\infty}^{0}\xi(du)\,A\,e^{-Bu}m\right)\right|+\frac{\varepsilon}{2}\leqq\int_{\Delta-t}|(n,Ae^{-Bu}m)|\,du+\frac{\varepsilon}{2}$$

$$\leqq\|n\|\;|\Delta|\;\|A\,e^{(t-|\Delta|)B}m\|+\frac{\varepsilon}{2}.$$

But $e^{Bt}m\to 0$ so we may choose $t>t_0$ such that $|\lambda|<\varepsilon$ and then

$$\|\xi(\Delta)\,n-T_td\|<\varepsilon.$$

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