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Splitting Times for Markov Processes and a Generalised Markov Property for Diffusions

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1. Introduction and Summary

In [7], Williams gives the following result on decomposition of the one-dimensional Brownian motion. Let $\{B_t: t \geqq 0\}$ be a BM^0 (Brownian motion starting at 0). Let τ denote the passage time to 1, σ the last time 0 is hit before τ and ρ the time point in $[0, \sigma]$ where the path attains its maximal value α. Then the following construction yields a process identical in law to $\{B_t: t<\tau\}$: choose α uniformly on $[0, 1]$ and run a BM^0 (independent of α) until it first hits α; continue with $\alpha - R_3$ where R_3 is a three-dimensional Bessel process, independent of α and the BM^0, starting at 0 and run until it hits α for the last time; finish with a new Bessel process, independent of the previous items, starting at 0 and run until it first hits 1.

It is an immediate consequence of this result that if ξ is either of the random times ρ or σ, then conditionally on $(\xi, \{B_t: 0 \leqq t \leqq \xi\})$, the law of the post-$\xi$ process $\{B(t+\xi): t \geqq 0\}$ depends only on $B(\xi)$, i.e., BM^0 starts afresh at the random time ξ. (For other decomposition results and proofs, see [8].)

It is the purpose of this paper to define for time-homogeneous Markov processes a class of random times, *splitting times*, for which one might expect this kind of generalised strong Markov property to hold, to discuss the problems arising when one tries to prove general results to this effect, and to show a splitting times theorem for one-dimensional diffusions.

Stopping times τ may be characterised as splitting times enjoying the property that conditionally on the pre-τ behaviour, the post-τ process is a replica of the given Markov process. Williams' decomposition result shows that for splitting times τ the conditional post-τ process may be a Markov process different from the given process.

In [5], Meyer, Smythe and Walsh and in [6], Pittenger and Shih discuss a Markov property with respect to coterminal times. As will be pointed out in Section 3 below, coterminal times come very close to being a special kind of splitting times.

2. Preliminaries

Throughout the paper we shall assume the basic Markov process to possess smooth sample paths and be given in canonical (i.e. function space) form.

Therefore, assume E, the state-space of the process, to be a Polish space with Borel σ-algebra $\mathscr{B}$, and $C(E)$ the space of bounded, real-valued continuous functions on E. Write $T=[0, \infty[$, $\bar{T}=[0, \infty]$ and let Ω be the relevant subset of E^T, i.e. Ω is either the space of continuous paths from T to E or the space of right-continuous paths possessing left-limits everywhere.

Write X_t for the projection $X_t\colon \Omega \to E$ given by $X_t\omega = \omega t$ ($t\in T, \omega\in\Omega$), let $\mathring{\mathscr{F}}$ be the σ-algebra of subsets of Ω generated by all the X_t and write $\mathring{\mathscr{F}}_t(\mathring{\mathscr{F}}^t)$ for the pre-t (post-t) algebra generated by $\{X_s\}_{s\leqq t}(\{X_s\}_{s\geqq t})$. Finally, let θ_t be the shift $X_s\circ\theta_t = X_{s+t}$ on Ω.

Definition 1. A time-homogeneous, canonically-defined Markov process with state-space E is a family $\{P^x\}_{x\in E}$ of probability measures on $(\Omega, \mathring{\mathscr{F}})$ satisfying:

i) for every bounded, $\mathring{\mathscr{F}}$-measurable $Y\colon \Omega\to \mathbb{R}$, the mapping $x\mapsto P^x\, Y$ from E to $\mathbb{R}$ is Borel-measurable;

ii) for every $x\in E$, $P^x\{X_0 = x\} = 1$;

iii) for every bounded, $\mathring{\mathscr{F}}$-measurable $Y\colon \Omega\to\mathbb{R}$ and for every $t\in T$, $x\in E$,

$$P^x_{\mathring{\mathscr{F}}(t)}\, Y\circ\theta_t = P^{X(t)}\,\hat{Y}.$$

The transition semigroup $\{P_t\}_{t\in T}$ for the process is given by

$$(P_t f)\, x = P^x f(X_t)$$

($t\in T$, $x\in E$, $f\colon E\to\mathbb{R}$ bounded Borel).

The notation used here as everywhere else is the following: if Y is P^x-integrable $P^x\, Y$ denotes the P^x-expectation of Y while $P^x(Y; F)$ is the integral of Y over the set F. If $Y = 1_F$ we write of course $P^x F$ instead of $P^x 1_F$. If $\mathscr{G}$ is a sub σ-algebra of $\mathring{\mathscr{F}}$, $P^x_{\mathscr{G}}$ denotes conditional expectation of P^x given $\mathscr{G}$. In case there exists a regular conditional probability, $P^x_{\mathscr{G}}\, Y$ will always denote (pointwise on Ω) the integral of Y with respect to that conditional probability. Finally, functions like

$$\omega\mapsto \int Y(\omega')\, P^{X_t\omega}(d\omega')$$

will be denoted $P^{X_t}\,\hat{Y}$ where more generally the $\hat{}$ is used to show which parts of the P^{X_t}-integrand depend on the integration variable. For instance one writes $P^{X_t} g(\hat{U}, \hat{V})$ for

$$\omega\mapsto\int g(U\omega', V\omega')\, P^{X_t\omega}(d\omega')$$

and $P^{X_t} g(U, \hat{V})$ for

$$\omega\mapsto\int g(U\omega, V\omega')\, P^{X_t\omega}(d\omega').$$

With the setup we are using here, the σ-algebras $\mathring{\mathscr{F}}_t$, $\mathring{\mathscr{F}}^t$ may be characterised as σ-algebras saturated with respect to a measurable partition (cf. [2]). For $t\in T$, let $\underset{t}{\sim}$, $\overset{t}{\sim}$ be the equivalence relations on Ω defined by

$$\omega\underset{t}{\sim}\omega' \quad\text{iff}\quad \omega s = \omega' s \qquad (s\in[0,t]),$$

$$\omega\overset{t}{\sim}\omega' \quad\text{iff}\quad \omega s = \omega' s \qquad (s\in[t,\infty[).$$

As a special case of Lemma 1.2 of [2], it follows that $F\in\mathring{\mathscr{F}}_t$ ($F\in\mathring{\mathscr{F}}^t$) iff $F\in\mathring{\mathscr{F}}$ and F is a union of $\underset{t}{\sim}$ ($\overset{t}{\sim}$) equivalence classes (atoms). Notice that the atoms themselves belong to $\mathring{\mathscr{F}}$ and thus determine a measurable partition of Ω.

The Markov property iii) of Definition 1 may now be formulated as follows: for every $x\in E$, $t\in T$ there exists a regular conditional probability $P^x_{\mathring{\mathscr{F}}(t)}$ of P^x given $\mathring{\mathscr{F}}_t$, uniquely determined by

$$P^x_{\mathscr{F}(t)}\, F\cap\theta_t^{-1}\, G = 1_F\, P^{X(t)}\, \hat{G} \qquad (F\in\acute{\mathscr{F}}_t,\, G\in\acute{\mathscr{F}}).$$

$P^x_{\mathscr{F}(t)}$ is also proper, i.e. for every $\omega\in\Omega$, the probability $P^x_{\mathscr{F}(t)}(\cdot)\,\omega$ is concentrated on the $\underset{t}{\sim}$ equivalence class containing ω.

Because of this, conditional expectations given $\acute{\mathscr{F}}_t$ may be computed treating anything $\acute{\mathscr{F}}_t$-measurable as constant.

A random time is a $\acute{\mathscr{F}}$-measurable mapping τ: $\Omega\to\overline{T}$. The corresponding shift θ_τ is a measurable mapping from $\{\tau<\infty\}$ to Ω, identical to θ_t on $\{\tau=t\}$. Similarly X_τ: $\{\tau<\infty\}\to E$ is measurable and equal to X_t on $\{\tau=t\}$.

For an arbitrary random time τ, the pre-τ algebra $\acute{\mathscr{F}}_\tau$ is defined as the σ-algebra of events which is saturated with respect to the equivalence relation $\underset{\tau}{\sim}$ given by

$$\omega\underset{\tau}{\sim}\omega' \quad \text{iff} \quad \tau\omega=\tau\omega' \quad \text{and} \quad \omega s=\omega' s \qquad (s\in[0,\tau\omega]\cap T)$$

(cf. [2]).

A (strict) stopping time is a random time τ such that $\{\tau=t\}\in\acute{\mathscr{F}}_t$ $(t\in T)$. The process is Markov with respect to the stopping time τ if

$$P^x_{\mathscr{F}(\tau)}\, Y\circ\theta_\tau = P^{X(\tau)}\, \hat{Y} \quad \text{on } \{\tau<\infty\} \tag{2.1}$$

for every bounded, measurable Y: $\Omega\to\mathbb{R}$ and every $x\in E$.

Formally (2.1) is obtained from the Markov property by identifying conditional expectations given $\acute{\mathscr{F}}_\tau$ with those given $\acute{\mathscr{F}}_t$ on $\{\tau=t\}$. Since τ is a stopping time, $\{\tau=t\}\in\acute{\mathscr{F}}_t\cap\acute{\mathscr{F}}_\tau$ with $\underset{t}{\sim}=\underset{\tau}{\sim}$ on $\{\tau=t\}$. This fact partly justifies the identification but does not of course provide a rigorous proof. For that, extra conditions are needed to ensure that one works with the correct versions of the conditional probabilities given the $\acute{\mathscr{F}}_t$.

If (2.1) holds for all stopping times the process is strong Markov. The strong Markov property will appear as a special case of the corollary to Proposition 1 below. The proposition deals with the identification principle in a more general setting.

For the formulation we need the following concept. If τ is a random time and $\{\acute{\mathscr{A}}_t\}_{t\in T}$ a family of sub σ-algebras of $\acute{\mathscr{F}}$ with each $\acute{\mathscr{A}}_t$ being the saturated σ-algebra determined by a measurable equivalence relation $\mathscr{A}_t$, we say that the pre-τ algebra $\acute{\mathscr{F}}_\tau$ is generated by $\{\acute{\mathscr{A}}_t\}$ provided

i) $\{\tau=t\}\in\acute{\mathscr{A}}_t$ $(t\in T)$,

ii) $\underset{\tau}{\sim}=\mathscr{A}_t$ on $\{\tau=t\}$ $(t\in T)$.

Proposition 1. *Let τ be a random time and let $\{\acute{\mathscr{A}}_t\}_{t\in T}$ be a family of σ-algebras which generate $\acute{\mathscr{F}}_\tau$ such that*

$$F\cap\{\tau<t\}\in\acute{\mathscr{A}}_t \qquad (F\in\acute{\mathscr{F}}_\tau,\, t\in T). \tag{2.2}$$

Suppose that for every $x\in E$ regular conditional probabilities $P^x_{\mathscr{A}_t}$ of P^x given $\acute{\mathscr{A}}_t$ exist for all $t\in T$, and suppose that versions of each of these may be chosen such that the following condition is satisfied: for every $n\in\mathbb{N}$, $t_1<\dots<t_n\in T$, $f_1,\dots,f_n\in C(E)$, $\omega\in\{\tau<\infty\}$ the mapping

$$t\mapsto\left(P^x_{\mathscr{A}(t)}\prod_{j=1}^{n} f_j(X_{t_j})\circ\theta_t\right)\omega \tag{2.3}$$

is right-continuous at $t_0=\tau\omega$.

Then conditionally on $\acute{\mathscr{F}}_\tau$ at ω the post-τ process is identical in law to the post-$\tau\omega$ process conditionally on $\acute{\mathscr{A}}_{\tau\omega}$, i.e.

$$(P^x_{\acute{\mathscr{F}}(\tau)}\, Y\circ\theta_\tau)\,\omega = (P^x_{\acute{\mathscr{A}}(\tau\omega)}\, Y\circ\theta_{\tau\omega})\,\omega \qquad (\omega\in\{\tau<\infty\})$$

for every $Y\colon \Omega\to\mathbb{R}$ bounded and measurable.

Proof. It suffices to show that for every $x\in E$, $n\in\mathbb{N}$, $t_1<\dots<t_n\in T$, $f_1,\dots,f_n\in C(E)$, $F\in\acute{\mathscr{F}}_\tau\cap\{\tau<\infty\}$

$$\int_F (P^x_{\acute{\mathscr{A}}(\tau\omega)}\, Y\circ\theta_{\tau\omega})\,\omega\, P^x(d\omega) = P^x(Y;F)$$

with $Y=\prod f_j(X_{t_j})$ and to check that the integrand on the left is $\acute{\mathscr{F}}_\tau$-measurable.

Because $\{\acute{\mathscr{A}}_t\}$ generates $\acute{\mathscr{F}}_\tau$ the integrand is constant on $\acute{\mathscr{F}}_\tau$-atoms. It is $\acute{\mathscr{F}}$-measurable since by (2.3)

$$(P^x_{\acute{\mathscr{A}}(\tau\omega)}\, Y\circ\theta_{\tau\omega})\,\omega = \lim_{n\to\infty}\sum_{k=0}^{\infty}\left(P^x_{\acute{\mathscr{A}}((k+1)2^{-n})}\, Y\circ\theta\left(\frac{k+1}{2^n}\right)\right)\omega\, 1_{F_{nk}}(\omega),$$

where $F_{nk}=\left\{\frac{k}{2^n}\leqq\tau<\frac{k+1}{2^n}\right\}$.

Using this representation, dominated convergence and the fact that

$$F\cap\{s\leqq\tau<t\}\in\acute{\mathscr{A}}_t \qquad (F\in\acute{\mathscr{F}}_\tau,\ s\leqq t\in T)$$

which follows from (2.2) because $\{s\leqq\tau\}\in\acute{\mathscr{F}}_\tau$, we find that

$$\begin{aligned}
&\int_F (P^x_{\acute{\mathscr{A}}(\tau\omega)}\, Y\circ\theta_{\tau\omega})\,\omega\, P^x(d\omega)\\
&\quad=\lim_{n\to\infty}\sum_{k=0}^{\infty} P^x\left(P^x_{\acute{\mathscr{A}}((k+1)2^{-n})}\, Y\circ\theta\left(\frac{k+1}{2^n}\right); F\cap F_{nk}\right)\\
&\quad=\lim_{n\to\infty}\sum_{k=0}^{\infty} P^x\left(Y\circ\theta\left(\frac{k+1}{2^n}\right); F\cap F_{nk}\right)\\
&\quad=P^x(Y\circ\theta_\tau; F).
\end{aligned}$$

In the special case where the $\acute{\mathscr{A}}_t$ increase with t condition (2.2) is always fulfilled, assuming that $\{\acute{\mathscr{A}}_t\}$ generates $\acute{\mathscr{F}}_\tau$, and equivalent to

$$F\cap\{\tau\leqq t\}\in\acute{\mathscr{A}}_t \qquad (F\in\acute{\mathscr{F}}_\tau,\ t\in T).$$

Thus τ is a strict stopping time with respect to the increasing family $\{\acute{\mathscr{A}}_t\}$.

In some cases condition (2.3) may be simplified.

Corollary. *The conclusion of Proposition 1 holds if $\{\acute{\mathscr{A}}_t\}$ generates $\acute{\mathscr{F}}_\tau$, if (2.2) holds and if for every $x\in E$, $\omega\in\{\tau<\infty\}$, $t\in T$, the post-t process conditionally on $\acute{\mathscr{A}}_t$ at ω under P^x is time-homogeneous Markov with initial state $X_t\omega$ and transition semigroup $\{{}^x_\omega Q_s\}_{s\in T}$ not depending on t satisfying*

$$ {}^x_\omega Q_s\colon\ C(E)\to C(E) \qquad (s\in T). \tag{2.4}$$

It is even sufficient that for any $\omega\in\{\tau<\infty\}$ this condition on the post-t process holds for $t\geqq\tau\omega$ only.

Proof. The first assertion is proved by verifying (2.3) of the proposition. By assumption

$$(P^x_{\mathscr{A}(t)}\, Y\circ\theta_t)\,\omega = {}^x_\omega Q^{\omega t}\, Y \tag{2.5}$$

for every $Y\colon \Omega\to\mathbb{R}$ bounded and measurable. (Here $\{{}^x_\omega Q^y\}_{y\in E}$ are the function space probabilities corresponding to the semigroup $\{{}^x_\omega Q_s\}$.)

That the right hand side of (2.5) is right-continuous in t for all Y of the form $\prod f_j(X_{t_j})$ with $t_1<\dots<t_n$, $f_j\in C(E)$ follows if we show that, writing $Q^y={}^x_\omega Q^y$

$$y\mapsto Q^y\, Y \tag{2.6}$$

is continuous. But for $n=1$ this is equivalent to (2.4). Furthermore, if ${}^x_\omega Q_s=Q_s$, then

$$Q^y\prod_{j=1}^{n+1} f_j(X_{t_j}) = Q^y\prod_{j=1}^{n-1} f_j(X_{t_j})\,[f_n\, Q_{t_{n+1}-t_n} f_{n+1}](X_{t_n})$$

so, using (2.4), (2.6) follows by induction.

As for the proof of the second assertion observe that the proof of the proposition applies if each $P^x_{\mathscr{A}(t)}$ is determined only within $\{\tau\leqq t\}$ and there satisfies (2.3).

We shall need the second part of the corollary in Section 5 below.

The first assertion contains a version of the strong Markov property as a special case: if τ is a strict stopping time, then $\{\acute{\mathscr{F}}_t\}$ generates $\acute{\mathscr{F}}_\tau$ and, since the conditional post-t process is the given Markov process itself starting at X_t, the corollary shows (as is of course well known) that the strong Markov property holds if $P_t\colon C(E)\to C(E)$ $(t\in T)$. (Recall that this condition is sufficient for the process to be strong Markov with respect to any stopping time τ, strict or not, and the enlarged pre-τ algebra $\acute{\mathscr{F}}_{\tau+}=\bigcap_{\varepsilon>0}\acute{\mathscr{F}}_{\tau+\varepsilon}$.)

3. Definition and Basic Properties of Splitting Times

We shall study random times τ with respect to which the process obeys the following generalised Markov property: for every $x\in E$, $Y\colon\Omega\to\mathbb{R}$ bounded and measurable, the conditional expectation

$$P^x_{\mathscr{F}(\tau)}\, Y\circ\theta_\tau \tag{3.1}$$

(defined on $\{\tau<\infty\}$) depends only on (τ, X_τ).

Intuitively one would expect this generalised Markov property to hold for random times τ having the property that knowledge that $\tau=t$ may provide information about the behaviour of the path after time t, but only so that this post-t information does not depend on the behaviour of the path prior to t. This leads to the following.

Definition 2. A random time τ is called a *splitting time* if it has the following cross-over property: for any two paths ω_1, ω_2 with $\tau\omega_1=\tau\omega_2$ $(=t$ say) and $\omega_1 t=\omega_2 t$, it is true that $\tau\omega=t$ where

$$\omega u=\begin{cases}\omega_1 u & (u\leqq t)\\ \omega_2 u & (u\geqq t).\end{cases} \tag{3.2}$$

It is evident that any strict stopping time is a splitting time. Furthermore, the definition is symmetric in past and future so that random times that are stopping times for the time-reversed process (e.g. last exit-times) are also splitting times. One also finds that the random times σ, ρ of the introduction are splitting times.

David Williams suggested the name "splitting times" and himself proved that discrete time processes are Markov with respect to arbitrary splitting times. This result, which has not been published, may be formulated as follows.

Let $X=(\Omega, \acute{\mathcal{M}}, \acute{\mathcal{M}}_t, X_t, \theta_t, P^x)$ be a time-homogeneous Markov process in the sense of [1] with discrete time-parameter set $T_d=\{0,1,\ldots\}$ and state-space E. Write $\acute{\mathcal{F}}$ for the σ-algebra generated by $\{X_t\}_{t\in T_d}$. Call $\tau\colon \Omega\to T_d\cup\{\infty\}$ a splitting time if for every $t\in T_d$ there exists $F_t\in\acute{\mathcal{M}}_t$, $G_t\in\acute{\mathcal{F}}$ such that

$$\{\tau=t\}=F_t\cap\theta_t^{-1}\,G_t. \tag{3.3}$$

Define the σ-algebra $\acute{\mathcal{M}}_\tau$ as follows: $M\in\acute{\mathcal{M}}_\tau$ iff $M\in\acute{\mathcal{M}}$ and for every $t\in T_d$ there exists $M_t\in\acute{\mathcal{M}}_t$ with

$$\{\tau=t\}\cap M=M_t\cap\theta_t^{-1}\,G_t.$$

Theorem 1 (Williams). *For every $x\in E$, $Y\colon\Omega\to\mathbb{R}$ bounded and measurable,*

$$(P^x_{\acute{\mathcal{M}}(\tau)}\,Y\circ\theta_\tau)\,\omega=P^{X_\tau\omega}(Y\,|\,G_{\tau\omega})\qquad(\omega\in\{\tau<\infty\}) \tag{3.4}$$

where the right-hand side may be defined arbitrarily (subject to the measurability constraints) for those ω for which $P^{X_\tau\omega}\,G_{\tau\omega}=0$.

In the proof, one works of course on the sets $\{\tau=t\}$ ($t\in T_d$) separately. The proof rests on the Markov property alone. (See the proof of (3.5) below.) Notice that the representation (3.3) is non-unique but that (3.4) holds no matter how the F_t, G_t are chosen.

One reason why this result cannot be used to establish results for continuous time is that, unlike stopping times, splitting times cannot in general be approximated by monotone sequences of splitting times with countable range. The splitting time ρ of Section 1 is an instance of this.

We return now to the continuous time case and the discussion of Definition 2. In the setup we are using, Galmarino's characterisation of strict stopping times (see [4] p. 86) is valid. Definition 2 is the splitting-times analogue of Galmarino's characterisation. The following proposition gives the splitting-times analogue of the customary definition of stopping times, which also matches Williams' definition.

Proposition 2. *A random time τ is a splitting time if for every $t\in T$ there exists $F_t\in\acute{\mathcal{F}}_t$, $G_t\in\acute{\mathcal{F}}$ such that*

$$\{\tau=t\}=F_t\cap\theta_t^{-1}\,G_t.$$

Proof. To verify the cross-over property, assume $\omega_1,\omega_2\in\{\tau=t\}$ with $\omega_1\,t=\omega_2\,t$. Defining ω as in (3.2), since $\omega\underset{t}{\sim}\omega_1$, $\omega\overset{t}{\sim}\omega_2$ it follows that $\omega\in F_t\cap\theta_t^{-1}\,G_t$.

Notice that if τ is a splitting time and one defines F_t ($\theta_t^{-1}\,G_t$) as the set of paths $\underset{t}{\sim}$-equivalent ($\overset{t}{\sim}$-equivalent) to some path in $\{\tau=t\}$, then

$$\{\tau=t\}=F_t\cap\theta_t^{-1}\,G_t.$$

Since F_t $(\theta_t^{-1} G_t)$ is a union of $\hat{\mathscr{F}}_t$-atoms ($\hat{\mathscr{F}}^t$-atoms) we see that the converse to Proposition 2 holds if F_t and $\theta_t^{-1} G_t$ are $\hat{\mathscr{F}}$-measurable.

Consider now a splitting time τ with $\{\tau=t\}$ as in Proposition 2. Define $1_t = 1_{\theta_t^{-1} G_t}$ and let $\underset{\mathscr{G}_t}{\sim}$ denote the equivalence relation

$$\omega_1 \underset{\mathscr{G}_t}{\sim} \omega_2 \quad \text{iff} \quad \omega_1 \underset{t}{\sim} \omega_2 \quad \text{and} \quad 1_t \omega_1 = 1_t \omega_2$$

with $\mathscr{G}_t$ the σ-algebra determined from $\underset{\mathscr{G}_t}{\sim}$.

It is immediate that the family $\{\mathscr{G}_t\}$ generates $\hat{\mathscr{F}}_\tau$. Furthermore, since $\mathscr{G}_t$ is the σ-algebra generated by $(\{X_s\}_{s \leq t}, 1_t)$ we claim that a regular conditional probability of P^x given $\mathscr{G}_t$ is defined by

$$(P^x_{\mathscr{G}(t)} F \cap \theta_t^{-1} G)\,\omega = 1_F(\omega)\, P^{X_t \omega}(G | \{1_{G(t)} = 1_t \omega\}) \qquad (F \in \hat{\mathscr{F}}_t,\, G \in \hat{\mathscr{F}}) \tag{3.5}$$

where the conditional probability on the right may be defined arbitrarily (subject to the $\mathscr{G}_t$-measurability condition) for those ω for which the $P^{X_t\omega}$-measure of the conditioning event is 0.

The proof of (3.5) proceeds as follows: if for example $F' \in \hat{\mathscr{F}}_t$ and $H = F' \cap \{1_t = 1\}$, one finds

$$\begin{aligned} &P^x\big(P^{X(t)}(\hat{G} | \{\hat{1}_{G(t)} = 1_t\});\, F \cap H\big) \\ &= P^x\big(P^{X(t)}(\hat{G} | \{\hat{1}_{G(t)} = 1\});\, F \cap F' \cap \theta_t^{-1} G_t\big) \\ &= P^x\big(P^{X(t)}(\hat{G} \cap \hat{G}_t);\, F \cap F'\big) \\ &= P^x(F \cap \theta_t^{-1} G \cap H). \end{aligned}$$

This is the argument used by Williams in the proof of Theorem 1.

Because of (3.5), one might expect that Proposition 1 could be used straightaway to establish the Markov property for τ. However, it may be true that

$$P^{X_t \omega}\{1_{G(t)} = 1_t \omega\} = 0 \tag{3.6}$$

for all $\omega \in \{\tau = t\}$, which makes it impossible to verify (2.3) of Proposition 1, the limit as $t \downarrow \tau\omega$ not being defined.

An example of this is provided by the splitting times ρ, σ of the introduction. For instance, one has $\{\sigma = t\} = F_t \cap \theta_t^{-1} G_t$ with

$$F_t = \{B_t = 0\} \cap \bigcap_{s \leq t} \{B_s < 1\} \in \hat{\mathscr{F}}_t,$$

$$G_t = \{B_0 = 0\} \cap \{B_s > 0 \text{ for all } s \in]0, \tau]\} \in \hat{\mathscr{F}},$$

using the notation of Section 1. It is now clear that (3.6) holds because BM^0 will with probability 1 cross its initial level infinitely often in any time-interval $]0, u]$.

The generalised Markov property (3.1) states that the process should start afresh at time τ. An important particular case arises naturally when the conditional post-τ process is itself time-homogeneous Markov with law depending only on τ, X_τ. In [5], τ is then called a birth time for the process and it is proved (Theorem 5.1) that any coterminal time L is a birth time in the following sense: the process $\{X_{L+t}\}_{t>0}$ is strong Markov with respect to the family $\{\hat{\mathscr{F}}_{L+t}\}_{t>0}$ of σ-algebras. Also the transition semigroup for the conditional process is given.

It is pointed out in [5] that the restriction to $t>0$ is essential. This way the problem we discussed in connection with (3.6) is avoided.

As we mentioned in the introduction, coterminal times are nearly splitting times. A coterminal time L satisfies in particular that

$$L\circ\theta_t=(L-t)\vee 0 \qquad (t\in T).$$

Assuming $\omega_1\in\{L\leqq t\}$, $\omega_2 \overset{t}{\sim} \omega_1$ it follows that $(L\circ\theta_t)\,\omega_2=(L\circ\theta_t)\,\omega_1=0$ so that $L\omega_2\leqq t$. Thus $\{L\leqq t\}\in\hat{\mathscr{F}}^t$ showing that L is a (non-strict) stopping time for the time-reversed process.

On the other hand, the ρ of Section 1 is a splitting time but not a coterminal time.

In [6], results are given which show that a Markov property (in the sense of (3.1)) is valid with respect to any coterminal time. There the difficulties around (3.6) are solved by showing that certain limits of ordinary conditional probabilities exist (cf. Definition 5.2 and Theorem 1).

4. A Class of Conditional Diffusions

Before formulating and proving splitting-times theorems for diffusions we shall summarise the facts needed from diffusion theory and prove some preliminary results.

We shall only discuss conservative regular diffusions but it is fairly obvious that the results extend to non-singular diffusions with killing.

Let J be a subinterval of the extended real line, with int J denoting the interior of J.

A canonically-defined Markov process $\{P^x\}_{x\in J}$ with state space J is called a conservative, regular diffusion provided

i) the P^x are probabilities on the space of continuous functions from T to J;

ii) the process is strong Markov;

iii) $P^x\{\tau_y<\infty\}>0$ $(x\in \operatorname{int} J,\, y\in J)$.

Here τ_x is the passage time $\inf\{t\in T\colon X_t=x\}$.

Let a be the lower and b the upper boundary of J. We shall need the following facts about diffusions (cf. [3] or [4]).

Any conservative regular diffusion on J may be characterised by a scale $S\colon J\to\mathbb{R}$, which is strictly increasing and continuous, and a speed measure m on the Borel subsets of J which is locally strictly positive and finite (i.e. $0<m[x,y]<\infty$ for all $x<y\in\operatorname{int}J$). S, m must satisfy certain conditions at the endpoints of J, mentioned below for the boundary a.

If $a\notin J$, then either $Sa=-\infty$ or $\int_{]a,x[}(Sy-Sa)\,m(dy)=\infty$ for all $x\in\operatorname{int}J$.

If $a\in J$, then $Sa>-\infty$ and $\int_{]a,x[}(Sy-Sa)\,m(dy)<\infty$ for all $x\in\operatorname{int}J$. If also $m]a,x[=\infty$ for all $x\in\operatorname{int}J$, a is necessarily absorbing. Otherwise a is absorbing iff $m\{a\}=\infty$, and reflecting iff $m[a,x[<\infty$ for all $x\in\operatorname{int}J$.

S, m are related to the exit probabilities and mean exit times by

$$P^x\{\tau_\beta<\tau_\alpha\}=\frac{Sx-S\alpha}{S\beta-S\alpha},$$

$$P^x\,\tau_{\alpha\beta}=\int_{]\alpha,\beta[} G_{\alpha\beta}(x,y)\,m(dy)$$

for all $\alpha<\beta\in J$, $x\in[\alpha,\beta]$. Here $\tau_{\alpha\beta}=\tau_\alpha\wedge\tau_\beta$ and $G_{\alpha\beta}$ is the Green function

$$G_{\alpha\beta}(x,y)=G_{\alpha\beta}(y,x)=\frac{(Sx-S\alpha)(S\beta-Sy)}{S\beta-S\alpha}\qquad (x\leqq y\in[\alpha,\beta]).$$

More generally, if $f\colon[\alpha,\beta]\to\mathbb{R}$ is bounded and measurable, then

$$P^x\int_0^{\tau(\alpha\beta)} f(X_t)\,dt=\int_{]\alpha,\beta[} G_{\alpha\beta}(x,y)\,f(y)\,m(dy).$$

From a special case of this, one finds

$$\begin{aligned}P^x(\tau_{\alpha\beta};\{\tau_\beta<\tau_\alpha\})&=\int_{]\alpha,\beta[} G_{\alpha\beta}(x,y)\frac{Sy-S\alpha}{S\beta-S\alpha}\,m(dy),\\ P^x(\tau_{\alpha\beta};\{\tau_\alpha<\tau_\beta\})&=\int_{]\alpha,\beta[} G_{\alpha\beta}(x,y)\frac{S\beta-Sy}{S\beta-S\alpha}\,m(dy).\end{aligned}\tag{4.1}$$

The proof of (4.1) is as follows:

$$\begin{aligned}\int_{]\alpha,\beta[} G_{\alpha\beta}(x,y)\frac{Sy-S\alpha}{S\beta-S\alpha}\,m(dy)&=P^x\int_0^{\tau(\alpha\beta)}\frac{S(X_t)-S\alpha}{S\beta-S\alpha}\,dt\\ &=\int_0^\infty P^x(P^{X(t)}\{\hat\tau_\beta<\hat\tau_\alpha\};\{\tau_{\alpha\beta}>t\})\,dt\\ &=\int_0^\infty P^x\{\tau_\beta\circ\theta_t<\tau_\alpha\circ\theta_t,\ \tau_{\alpha\beta}>t\}\,dt\\ &=\int_0^\infty P^x\{\tau_\beta<\tau_\alpha,\ \tau_{\alpha\beta}>t\}\,dt\\ &=P^x(\tau_{\alpha\beta};\{\tau_\beta<\tau_\alpha\})\end{aligned}$$

where we have used the Markov property once and Fubini's theorem twice.

It is well known that the transition operators for any conservative regular diffusion on J are operators on $C(J)$.

Suppose $\{P^x\}_{x\in J}$ is conservative and regular on J with $a\notin J$. Then a is an entrance non-exit boundary for $\{P^x\}$ provided

$$Sa=-\infty,\qquad \int_{]a,x[}(Sx-Sy)\,m(dy)<\infty\quad (x\in\operatorname{int}J).\tag{4.2}$$

We shall need the following result about entrance non-exit boundaries.

Proposition 3. *Suppose $\{P^x\}_{x\in J}$ is a regular diffusion on J with a an entrance non-exit boundary. Then there exists a unique probability P^a on the space of contin-*

uous functions from T *to* $J\cup\{a\}$ *satisfying* $P^a\{\tau_x<\infty\}=1$ $(x\in J)$ *and such that* $\{P^x\}_{x\in J\cup\{a\}}$ *defines a canonical strong Markov process with continuous paths and state space* $J\cup\{a\}$. *The transition operators for this process are operators on* $C(J\cup\{a\})$.

For the proof see [4].

Starting from a, the new process immediately moves into J itself never to return to a.

To arrive at the conditional diffusions needed in Section 5, we begin with the following quite general result.

Let $\{P^x\}_{x\in E}$ be a canonical Markov process with state-space E. Let $A\in\hat{\mathscr{F}}$ be an event satisfying this condition: for every $t\in T$ there exists $A_t\in\hat{\mathscr{F}}_t$ such that $A=A_t\cap\theta_t^{-1}A$. Finally, let $E_A=\{x\in E\colon P^x A>0\}$.

Lemma 1. *For every* $x\in E_A$, $t\in T$

$$P^x(X_t\in E_A|A)=1. \tag{4.3}$$

Furthermore, if for $x\in E_A$, $t\in T$, $f\colon E_A\to\mathbb{R}$ *bounded and measurable, one defines*

$$(P_{A,t}f)\,x=P^x\big(f^*(X_t)|A\big), \tag{4.4}$$

where f^* *is an arbitrary bounded and measurable extension of* f *from* E_A *to* E, *then the family* $\{P_{A,t}\}_{t\in T}$ *defines a one-parameter semigroup of stochastic transition operators on* E_A.

Proof. Using the Markov property and the definition of E_A, one finds

$$\begin{aligned}P^x(A\cap\{X_t\in E_A\})&=P^x(P^{X(t)}\hat{A};\,A_t\cap\{X_t\in E_A\})\\&=P^x(P^{X(t)}\hat{A};\,A_t)=P^x A,\end{aligned}$$

proving (4.3). Eq. (4.3) shows that the Definition (4.4) is unambiguous. For the proof of the last assertion of the lemma, only the semigroup property needs verification. But

$$\begin{aligned}&\big(P_{A,t}(P_{A,s}f)\big)\,x\\&\quad=P^x\big((P_{A,s}f)(X_t);\,\{X_t\in E_A\}|A\big)\\&\quad=(P^xA)^{-1}P^x\big[(P^{X(t)}\hat{A})^{-1}P^{X(t)}\big(f(\hat{X}_s);\,\{\hat{X}_s\in E_A\}\cap\hat{A}\big);\,\{X_t\in E_A\}\cap A\big]\\&\quad=(P^xA)^{-1}P^x\big[P^{X(t)}\big(f(\hat{X}_s);\,\{\hat{X}_s\in E_A\}\cap\hat{A}\big);\,\{X_t\in E_A\}\cap A_t\big]\\&\quad=(P^xA)^{-1}P^x\big(f(X_{s+t});\,\{X_t\in E_A\}\cap A_t\cap\theta_t^{-1}A\cap\{X_{s+t}\in E_A\}\big)\\&\quad=(P_{A,t+s}f)\,x.\end{aligned}$$

If it is also known that all paths in A take values in E_A only, the proof of the semigroup property is easily converted into a proof that the conditional process $\{P^x(\cdot|A)\}_{x\in E_A}$ is Markov with state-space E_A.

Again let $\{P^x\}_{x\in J}$ be a regular conservative diffusion on J with scale S and speed measure m and let $\alpha\in J$ with $\alpha<b$. Write

$$A_\alpha=\bigcap_{t>0}\{X_t>\alpha\},\qquad J_\alpha=\{x\in J\colon x>\alpha\}.$$

We have the following dichotomy: either $P^x A_\alpha > 0$ for all $x > \alpha$ or $P^x A_\alpha = 0$ for all $x > \alpha$ with the first possibility occurring iff $Sb < \infty$ and b is not a reflecting boundary (i.e. b is in J and absorbing or not in J with $Sb < \infty$ and $\int (Sb - Sy)\, m(dy) = \infty$).

To see this, observe that for $x > \alpha$

$$P^x A_\alpha = P^x\{\tau_\alpha = \infty\} = \lim_{\beta \uparrow b} P^x\{\tau_\beta < \tau_\alpha\} = \frac{Sx - S\alpha}{Sb - S\alpha} \tag{4.5}$$

provided b is not reflecting. If b is reflecting, $P^b\{\tau_\alpha < \infty\} = 1$ and consequently

$$P^x A_\alpha = P^x\{\tau_b < \tau_\alpha,\ \tau_\alpha \circ \theta(\tau_b) = \infty\} = P^x\{\tau_b < \tau_\alpha\}\, P^b\{\tau_\alpha = \infty\} = 0.$$

This motivates the following.

Definition 3. A conservative regular diffusion $\{P^x\}$ on J is said to be *positively-inclined* if $P^x A_\alpha > 0$ for all $\alpha \in J \setminus \{b\}$, $x \in J_\alpha$.

The next result is basic for the sequel.

Proposition 4. *Let $\{P^x\}_{x \in J}$ be a positively-inclined conservative and regular diffusion on J. For every $\alpha \in J \setminus \{b\}$, the equations*

$$Q^x_\alpha = P^x(\cdot \mid A_\alpha) \qquad (x \in J_\alpha)$$

define a family $\{Q^x_\alpha\}$ of probability measures on the space of continuous functions from T to J_α which determine a conservative regular diffusion on J_α. This diffusion has scale

$$S_\alpha = -(S - S\alpha)^{-1},$$

speed measure

$$m_\alpha(dx) = (Sx - S\alpha)^2\, m(dx)$$

and α as entrance non-exit boundary.

Proof. Although formally defined as a probability on the space of continuous paths from T to J, Q^x_α for $x > \alpha$ may obviously be considered a probability on the space of continuous paths with values in J_α. Furthermore A_α satisfies the condition imposed on the A of Lemma 1 with $E_{A(\alpha)} = J_\alpha$. By that lemma therefore,

$$(Q_{\alpha,t} f)x = Q^x_\alpha(f(X_t))$$

defines a stochastic semigroup $\{Q_{\alpha,t}\}$ of transition operators. According to the remark following Lemma 1 one finds that $\{Q^x_\alpha\}$ is a canonical Markov process on J_α in the sense of Definition 1. To prove that it is a regular conservative diffusion, it thus remains to show that $Q^x_\alpha\{\tau_y < \infty\} > 0$ for all $x \in \operatorname{int} J_\alpha$, $y \in J_\alpha$ (which is trivial) and to verify the strong Markov property.

We shall achieve this by showing that each $Q_{\alpha,t}$ maps $C(J_\alpha)$ into itself. As (4.5) shows $x \mapsto P^x A_\alpha$ is continuous, so this is equivalent to showing that

$$x \mapsto P^x(f(X_t);\ A_\alpha) = P^x(f(X_t);\ \{\tau_\alpha = \infty\}) \tag{4.6}$$

is continuous on J_α for every $f \in C(J_\alpha)$.

But if $x < y \in J_\alpha$, then

$$\begin{aligned} &P^x(f(X(t+\tau_y));\ \{\tau_y < \tau_\alpha,\ \tau_\alpha \circ \theta(\tau_y) = \infty\}) \\ &\quad = \frac{Sx - S\alpha}{Sy - S\alpha} P^y(f(X_t);\ \{\tau_\alpha = \infty\}). \end{aligned}$$

When $y \downarrow x$ the left hand side converges to $P^x(f(X_t); \{\tau_\alpha = \infty\})$ by dominated convergence. It follows that (4.6) is right-continuous.

A similar argument may be used to establish the left-continuity except at b when b is absorbing. But in that case we have for any $x < y \in J_\alpha$ that

$$\begin{aligned}\frac{Sx - S\alpha}{Sy - S\alpha} & P^y(f(X_t); \{\tau_\alpha = \infty\}) \\ &= P^x(f(X(t+\tau_y)); \{\tau_\alpha \circ \theta(\tau_y) = \infty, \tau_y < \tau_\alpha\}) \\ &= P^x(f(X(t+\tau_y)); \{\tau_b < \tau_\alpha = \infty\})\end{aligned}$$

and as $y \uparrow b$ the last term tends to

$$P^x(f(b); \{\tau_b < \tau_\alpha\}) = \frac{Sx - S\alpha}{Sb - S\alpha} f(b).$$

Thus

$$\lim_{y \uparrow b} P^y(f(X_t); \{\tau_\alpha = \infty\}) = f(b) = P^b(f(X_t); \{\tau_\alpha = \infty\})$$

proving the left-continuity at b.

The scale and speed for $\{Q_\alpha^x\}$ may be computed directly. If $\gamma < \delta \in J_\alpha$, $x \in [\gamma, \delta]$, Eq. (4.5) shows that

$$\begin{aligned}Q_\alpha^x\{\tau_\delta < \tau_\gamma\} &= \frac{Sb - S\alpha}{Sx - S\alpha} P^x\{\tau_\delta < \tau_\gamma, \tau_\alpha = \infty\} \\ &= \frac{Sb - S\alpha}{Sx - S\alpha} P^x\{\tau_\delta < \tau_\gamma, \tau_\alpha \circ \theta(\tau_{\gamma\delta}) = \infty\} \\ &= \frac{Sb - S\alpha}{Sx - S\alpha} \frac{Sx - S\gamma}{S\delta - S\gamma} \frac{S\delta - S\alpha}{Sb - S\alpha} \\ &= \frac{S_\alpha x - S_\alpha \gamma}{S_\alpha \delta - S_\alpha \gamma},\end{aligned}$$

proving that S_α is the scale for $\{Q_\alpha^x\}$. Also

$$\begin{aligned}Q_\alpha^x \tau_{\gamma\delta} &= \frac{S_\alpha x}{S_\alpha b} P^x(\tau_{\gamma\delta}; \{\tau_\alpha = \infty\}) \\ &= \frac{S_\alpha x}{S_\alpha b} \left(P^x(\tau_{\gamma\delta}; \{\tau_\delta < \tau_\gamma\}) \frac{S_\alpha b}{S_\alpha \delta} + P^x(\tau_{\gamma\delta}; \{\tau_\gamma < \tau_\delta\}) \frac{S_\alpha b}{S_\alpha \gamma}\right)\end{aligned}$$

and using (4.1) this reduces to

$$\int_{]\gamma, \delta[} G_{\alpha, \gamma\delta}(x, y) S_\alpha^{-2}(y) m(dy)$$

with G_α being the Green function for S_α. Thus m_α is the speed measure for $\{Q_\alpha^x\}$.

Finally it is immediate that (4.2) holds for $a = \alpha$, $S = S_\alpha$, $m = m_\alpha$ so α is entrance non-exit.

This proposition in conjunction with Proposition 3 shows that for every $\alpha \in J$ a probability Q_α^α on the set of continuous paths from T to $J_\alpha \cup \{\alpha\}$ may be adjoined to the family $\{Q_\alpha^x\}_{x\in J_\alpha}$ such that the enlarged family defines a strong Markov process with continuous paths, state space $J_\alpha \cup \{\alpha\}$, and transition operators mapping $C(J_\alpha \cup \{\alpha\})$ into itself.

5. Future-Minimum Times and a Splitting-Times Theorem for Diffusions

We shall be concerned with positively-inclined conservative and regular diffusions on a state-interval J. Ω will denote the space of continuous paths from T to J.

For $t\in T$ let G_t denote the event $\bigcap_{s>t}\{X_s > X_t\}$ and write 1_t for $1_{G(t)}$.

Definition 4. A random time τ on Ω is called a *future-minimum time* if

i) for any $t\in T$ and any $\omega_1\in\{\tau=t\}$ the conditions $\omega_2 \underset{t}{\sim} \omega_1$ and $1_s\omega_2 = 1_s\omega_1$ $(s\leqq t)$ imply $\tau\omega_2 = t$;

ii) $1_\tau = 1$ on $\{\tau<\infty\}$.

Condition i) states that τ is a stopping time with respect to $\{(X_t, 1_t)\}_{t\in T}$.

Suppose that $1_t\omega = 1$ and let $s\leqq t$. Then $1_s\omega = 1$ iff $\omega u > \omega s$ for all $u\in]s,t]$. This being a pre-t condition on ω it follows that i) and ii) are equivalent to

a.i) for any $t\in T$ and any $\omega_1\in\{\tau=t\}$ the conditions $\omega_2 \underset{t}{\sim} \omega_1$ and $1_t\omega_2 = 1$ imply $\tau\omega_2 = t$;

a.ii) $1_\tau = 1$ on $\{\tau<\infty\}$.

Definition 4 as it now stands, is due to J.W. Pitman and supersedes an earlier definition where the random times were described in an implicit manner only.

The two most important examples of future-minimum times are the last time the path is below a given level α

$$\begin{aligned}\tau &= \sup\{t\in T\colon X_t \leqq \alpha\} \qquad &(\text{with } \sup\emptyset = \infty)\\ &= \inf\{t\in T\colon 1_t = 1,\ X_t = \alpha\} \qquad &(\text{with } \inf\emptyset = \infty),\end{aligned}$$

and the last time the path attains its minimum value $\gamma = \inf\{X_t\colon t\in T\}$

$$\begin{aligned}\tau &= \sup\{t\in T\colon X_t = \gamma\}\\ &= \inf\{t\in T\colon 1_t = 1\}.\end{aligned}$$

Proposition 5. *Any future-minimum time τ is a splitting time. Furthermore, if for $t\in T$ $\acute{\mathscr{A}}_t$ is the σ-algebra saturated with respect to the equivalence relation $\underset{\mathscr{A}_t}{\sim}$ given by*

$$\omega_1 \underset{\mathscr{A}_t}{\sim} \omega_2 \quad \textit{iff} \quad \omega_1 \underset{t}{\sim} \omega_2 \quad \textit{and} \quad 1_{\{\tau\leqq t\}}\omega_1 = 1_{\{\tau\leqq t\}}\omega_2,$$

then the $\acute{\mathscr{A}}_t$ increase with t and generate $\acute{\mathscr{F}}_\tau$. In particular (2.2) *holds and τ is a stopping time with respect to $\{\acute{\mathscr{A}}_t\}$.*

Proof. Using condition a.i) it is immediately verified that any future-minimum time has the cross-over property.

For the remainder of the proof observe first that

$$\omega_1 \underset{\mathscr{A}_t}{\sim} \omega_2 \quad \textit{with} \quad \omega_1, \omega_2 \in \{\tau\leqq t\} \quad \textit{implies} \quad \tau\omega_1 = \tau\omega_2, \tag{5.1}$$

since for example the inequality $t_1 = \tau\omega_1 \leqq \tau\omega_2 = t_2$ shows that $1_{t_1}\omega_1 = 1_{t_1}\omega_2$ and hence by a.i) that $\tau\omega_1 = \tau\omega_2$.

Let $t<u$ and $\omega_1 \underset{\mathscr{A}_t}{\sim} \omega_2$. If $\omega_1, \omega_2 \in \{\tau > u\}$ evidently $\omega_1 \underset{\mathscr{A}_t}{\sim} \omega_2$. If $\omega_1, \omega_2 \in \{\tau \leqq u\}$ the equivalence $\omega_1 \underset{\mathscr{A}_t}{\sim} \omega_2$ follows from (5.1). Thus $\acute{\mathscr{A}}_t \subseteq \acute{\mathscr{A}}_u$.

The implication (5.1) also shows that $\{\tau = t\} \in \acute{\mathscr{A}}_t$ $(t\in T)$, and that $\omega_1 \in \{\tau = t\}$, $\omega_1 \underset{\mathscr{A}_t}{\sim} \omega_2$ implies $\omega_1 \underset{\tau}{\sim} \omega_2$. Since the converse implication is obvious, we see that $\{\acute{\mathscr{A}}_t\}$ generates $\acute{\mathscr{F}}_\tau$.

By the remarks following Proposition 1, condition (2.2) is satisfied and τ is a stopping time relative to $\{\acute{\mathscr{A}}_t\}$.

For the proof of Theorem 2 below we need the fact that for any path $\omega \in \{\tau < t\}$ it is possible to pick out the level $X_\tau\omega$ by looking at ω on the interval $[0, t]$ only. This is formalised in Lemma 2 which shows how the information $\omega \in \{\tau < t\}$ decomposes into information about the pre-t behaviour of ω and, relative to that, information concerning the post-t part of ω.

To formulate the lemma introduce the stopping operators $s_t\colon \Omega \to \Omega$ given by $X_u \circ s_t = X(u \wedge t)$ $(u, t \in T)$.

Lemma 2. *Let τ be a future-minimum time and define*

$$M_t = X\bigl(t \wedge (\tau \circ s_t)\bigr) \qquad (t \in T).$$

Then M_t is $\acute{\mathscr{F}}_t$-measurable, $M_t = X_\tau$ on $\{\tau < t\}$ and

$$\{\tau < t\} = \bigcap_{u \geqq t} \{X_u > M_t\} \qquad (t \in T).$$

Proof. The $\acute{\mathscr{F}}_t$-measurability of M_t is evident since $\omega_1 \underset{t}{\sim} \omega_2$ implies $s_t\omega_1 = s_t\omega_2$.

Assume $\omega \in \{\tau < t\}$ so that in particular $\omega \in \bigcap_{u \geqq t} \{X_u > X_\tau\}$. Because τ is a future-minimum time and $s_t\omega \underset{t}{\sim} \omega$ it follows that $(\tau \circ s_t)\omega = \tau\omega$. Hence $M_t\omega = X_\tau\omega$ and $\omega \in \bigcap_{u \geqq t} \{X_u > M_t\}$.

If conversely $\omega \in \bigcap_{u \geqq t} \{X_u > M_t\}$ in particular $\omega t > M_t\omega$ and the definition of M_t then shows that $t_0 = (\tau \circ s_t)\omega < t$. The assumption on ω, the fact that τ is a future-minimum time and the equivalence $\omega \underset{t}{\sim} s_t\omega$ now implies that $\tau\omega = t_0$ so that $\omega \in \{\tau < t\}$.

We come now to the main theorem. For the formulation of this let $\{P^x\}_{x\in J}$ be a positively-inclined conservative and regular diffusion on J, let $\{Q^x_\alpha\}_{x \in J_\alpha \cup \{\alpha\}}$ be the conditional diffusion of Proposition 4 with the entrance non-exit boundary α adjoined and let τ be a future-minimum time with M_t as in Lemma 2.

Theorem 2. *With $\{P^x\}$ and τ as above, conditionally on $\acute{\mathscr{F}}_\tau$ within $\{\tau < \infty\}$, the post-τ process is identical in law to the process $\{Q^x_\alpha\}$ with $\alpha = X_\tau$, starting at its entrance non-exit boundary X_τ. More specifically*

$$P^x_{\acute{\mathscr{F}}(\tau)}\, Y \circ \theta_\tau = Q^{X(\tau)}_{X(\tau)}\, \hat{Y} \quad \text{on } \{\tau < \infty\} \tag{5.2}$$

for every $x \in J$, $Y\colon \Omega \to \mathbb{R}$ bounded and measurable.

Proof. For the proof of the Markov property (5.2), we shall use the second assertion of the corollary to Proposition 1. Since $\{\acute{\mathscr{A}}_t\}$ generates $\acute{\mathscr{F}}_\tau$ and since (2.2) holds (Proposition 5), it suffices to show that

$$P^x_{\mathscr{A}_t}\, Y\circ\theta_t = Q^{X(t)}_{X(\tau)}\,\hat{Y} \quad on\ \{\tau\leqq t\} \tag{5.3}$$

for every $x\in E$, $t\in T$, $Y\colon \Omega\to\mathbb{R}$ bounded and measurable and to check that (2.4) holds.

Given (5.3), this latter fact follows from Propositions 3 and 4 so only (5.3) needs verification.

Because $\acute{\mathscr{A}}_t$ is the σ-algebra generated by $\acute{\mathscr{F}}_t$ and $\{\tau\leqq t\}$ it suffices to show that

$$P^x(Y\circ\theta_t;\, F\cap\{\tau\leqq t\}) = P^x(Q^{X(t)}_{X(\tau)}\,\hat{Y};\, F\cap\{\tau\leqq t\}) \tag{5.4}$$

for any $F\in\acute{\mathscr{F}}_t$.

Because $\{P^x\}$ is a regular diffusion

$$P^x \bigcap_{u>0}\{X_u > X_0\} = 0 \qquad (x\in J)$$

and consequently

$$P^x\{\tau=t\} \leqq P^x \bigcap_{u>t}\{X_u > X_t\} = P^x\, P^{X(t)} \bigcap_{u>0}\{\hat{X}_u > \hat{X}_0\} = 0$$

so in (5.4) we need only integrate over $F\cap\{\tau<t\}$.

Using Lemma 2 and the Markov property it is seen that with A_α as in Section 4

$$\begin{aligned} P^x(Y\circ\theta_t;\, F\cap\{\tau<t\}) &= P^x\big(Y\circ\theta_t;\, F\cap\bigcap_{u\geqq t}\{X_u > M_t\}\big)\\ &= P^x\big(P^{X(t)}(\hat{Y};\, \hat{A}_{M(t)});\, F\big)\\ &= P^x(Q^{X(t)}_{M(t)}\,\hat{Y}\; P^{X(t)}\,\hat{A}_{M(t)};\, F)\\ &= P^x\big(Q^{X(t)}_{M(t)}\,\hat{Y};\, F\cap\bigcap_{u\geqq t}\{X_u > M_t\}\big)\\ &= P^x(Q^{X(t)}_{X(\tau)}\,\hat{Y};\, F\cap\{\tau<t\}) \end{aligned}$$

and the proof is complete. (The fact that there is a proper, regular conditional probability of P^x given $\acute{\mathscr{F}}_t$ justifies the above manipulations with conditional expectations, cf. Section 2.)

The theorem holds in particular for τ the last time the process is below a given level α or τ the last time the process attains its minimum value.

If in particular the diffusion $\{P^x\}$ is a Brownian motion with an upper absorbing boundary b, the associated diffusion $\{Q^x_\alpha\}$ of Proposition 4 becomes $\alpha+R_3$ with b absorbing and R_3 the three-dimensional Bessel process on $[0,\infty[$.

It is now clear why the Bessel processes occur in Williams' decomposition result [7].

That a splitting-times theorem holds at the time where a positively-inclined diffusion attains its minimum is proved in Theorem 2.4 of [8]. The result is first proved for a particular diffusion and then extended to other diffusions by time substitution.

We shall now mention an example of a splitting times theorem slightly different from Theorem 2.

Let $\{B^x\}_{x\in\mathbb{R}}$ be canonical one-dimensional Brownian motion and let $\{B^x_k\}$ be Brownian motion with constant drift $k>0$. Then $B^x_k = B^x\phi^{-1}$ with $\phi\colon \Omega\to\Omega$

given by $X_t \circ \phi = X_t + kt$. Also $\{B_k^x\}$ is positively-inclined and the associated conditional diffusion $\{Q_\alpha^x\}_{x \geqq \alpha}$ has scale and speed

$$S_\alpha x = -(e^{-2k\alpha} - e^{-2kx})^{-1},$$

$$m_\alpha(dx) = (e^{-2k\alpha} - e^{-2kx})^2 \frac{1}{k} e^{2kx}\, dx$$

corresponding to the generator

$$\frac{1}{2} \frac{d^2}{dx^2} + k \coth(kx - k\alpha) \frac{d}{dx}$$

(cf. 2.4 of [8]).

Defining $\tau = \sup\{t \in T\colon X_t \leqq -kt\}$ (with $\sup \emptyset = \infty$) it is readily verified that τ is a splitting time. The map ϕ transforms τ into $\tau^* = \sup\{t \in T\colon X_t \leqq 0\}$. Applying Theorem 2 to τ^* and $\{B_k^x\}$ and transforming back we therefore find

$$(B^x_{\mathscr{F}(\tau)} Y \circ \theta_\tau)\,\omega = Q_0^{X_\tau \omega + k\tau(\omega)}\, Y \circ \psi_{\tau\omega} \qquad (\omega \in \{\tau < \infty\})$$

where $\psi_c\colon \Omega \to \Omega$ is given by

$$X_t \circ \psi_c = X_t - k(t + c).$$

In other words, a Brownian motion starting at $x \in \mathbb{R}$ and conditioned to stay above the line $t \mapsto y - kt$ forever (where $y \leqq x$) is identical in law to the diffusion on $[0, \infty[$ with generator

$$\frac{1}{2} \frac{d^2}{dx^2} + k \coth(kx) \frac{d}{dx}$$

starting at $x - y$ subjected to the transformation ψ_{-y} of Ω. In particular this conditional process is non-homogeneous Markov.

The main result of this paper, Theorem 2 gives a generalised Markov property for a special class of processes and a special class of splitting times. By exploiting the theory of diffusions we were able to describe exactly what the conditional post-τ process should be and to verify the crucial condition 2.4. However, the structure of the problems discussed and solved for diffusions (for instance the fact that certain splitting times are stopping times with respect to families of increasing σ-algebras other than the $\mathscr{F}_t$) suggest that a generalised Markov property with respect to splitting times must hold in great generality.

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