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Automorphisms of Baire Measures on Generalized Cubes. II

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1. Introduction

Let $S=\prod I_\alpha$, $\alpha\in A$, where each I_α is the closed unit interval $[0,1]$ and A is an arbitrary index set, possibly uncountable. We call such a set a generalized cube. Let μ be a probability measure on the product σ-algebra of S, which is its Baire σ-algebra in the product topology. If ϕ is an automorphism (not necessarily measure preserving) of the measure algebra of (S,μ), the question arises whether ϕ is induced by a (1-1) Baire (product) measurable point mapping of S onto itself. A classical result of von Neumann [5] states that this is always the case if A is at most countable. $[(S,\mu)$ is then point isomorphic to a probability measure on $I=[0,1]$.$]$ This was generalized to the direct product of uncountably many normalized measures on $[0,1]$ by Maharam [4] and, using many of the ideas of [4], to a wide class of Baire probability measures on an uncountable generalized cube by the author [2]. Using many of the results of this last work, plus others, we prove it for an arbitrary Baire probability measure on an uncountable generalized cube, and thus settle the problem mentioned above. As in the case of all the earlier papers cited above, the same proofs work when S is the uncountable product of Polish spaces, μ a probability measure on the product σ-algebra.

We note, however, that in view of the remarks in the Introduction to [2] and an example of Panzone and Segovia ([6], Sec. 5, Example (c)) there are finite Borel measures μ on the cube and automorphisms of the measure algebra of μ for which no Borel measurable invertible point mapping inducing the automorphism can exist. Since a Borel measure μ and its Baire contraction have the same measure algebra our result shows that there are Baire point automorphisms (in the terminology of [2] or [4]) of a generalized uncountable cube for which no equivalent Borel point automorphism exists. (By equivalent we mean one inducing the same measure algebra automorphism.) The example of Panzone and Segovia shows that there is a compact space X, a finite Borel measure μ and a measure algebra automorphism ϕ for which there exists neither a Borel nor a Baire measurable point automorphism inducing ϕ. For a trivial example in which there exists a Borel but not a Baire measurable point automorphism inducing a given ϕ, let X be a compact space containing a point p which is a $\mathscr{G}_\delta$ and a point q which is not, let μ consist of point masses 1/2 at each of p and q and let ϕ simply interchange the two atoms which generate the measure algebra.

Our earlier proof for the restricted class of measures given in [2] used the highly sophisticated decomposition-disintegration theorem. The present proof

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of the general case is much more elementary and avoids this theorem. This illustrates Halmos' principle [3], p. 35, that a theorem proved using the decomposition theorem can usually be proved by other, more elementary means.

Heavy use is made of the ideas, definitions, notation and results of [2] to which this is a sequel. These are not repeated here. Von Neumann's original theorem and its proof are probably most easily accessible in Billingsley [1], § 5.1, pp. 66 to 73.

2. Notation

In addition to the notation and definitions of [2] we use the following notation in Lemmas A, B, and C. I, J both denote the closed unit interval $[0, 1]$, $S = I \times J$ denotes the unit square. This corresponds, in the notation of [2] to $S = S(A)$ where A has just 2 elements. The projection $S \to I$ is denoted by π_I, from S to J by π_J. S^I denotes the σ-algebra in S of cylinders of the form $\pi_I^{-1}(X)$, where X is a measurable subset of I. If μ is a fixed, Lebesgue-Stieltjes probability measure on S, then, via S^I and π_I, μ induces a measure μ_I on I. The measure algebras of S, S^I and I are denoted by E, E^I and $E(I)$, the last two being canonically isomorphic.

The proofs of Lemmas A and C remain unchanged if I and J are only assumed to be Polish spaces, minor verbal changes are however needed in the proof of Lemma B. The known measure theoretic isomorphism of finite Borel measures on Polish spaces and intervals, makes this exercise unnecessary. The Corollary to Lemma C is in fact the restatement of Lemma C for Polish spaces.

If ϕ is a set automorphism of S such that $\phi(E^I) = E^I$, then ϕ induces a set automorphism ϕ' of I. We say that ϕ leaves the σ-algebra of I-based cylinders invariant.

3. Main Lemmas and Theorem

Lemma A. *Let $S = I \times J$ be the unit square, let μ be a Lebesgue-Stieltjes probability measure on S. Let ϕ be a set automorphism of S leaving the σ-algebra of I-based cylinders invariant, and so inducing a set automorphism ϕ' of (I, μ_I). Let T' be a point automorphism of I inducing ϕ'. Then there exists a subset $\Omega \subset S$ such that $\mu(\Omega) = 1$ [or equivalently $\mu(S - \Omega) = 0$] and a point automorphism T_1 of Ω such that T_1 induces ϕ and such that for $(p, q) \in \Omega$,*

$$\pi_I T_1(p, q) = T' \pi_I(p, q).$$

Proof. By the theorem of von Neumann there exists a point automorphism T_1 of S which induces ϕ. Since ϕ leaves the σ-algebra of I-based cylinders invariant and so induces the set automorphism ϕ' of (I, μ_I), for any measurable subset X of I

$$\phi\{X \times J\} = \phi'\{X\} \times \{J\},$$

and so

$$\mu\big(T_1^n(X \times J) \vartriangle (T'^n X \times J)\big) = 0$$

for all integers n (positive or negative). Let $Z_{X,n}$ denote this set; put $Z_X = \bigcup_{-\infty}^{\infty} Z_{X,n}$ then $\mu(Z_X) = 0$ and for all integers n,

$$T_1^n(X \times J) - Z_X = (T'^n X \times J) - Z_X.$$

Let X_k $(k=1,2,3,\dots)$ be a separating sequence of generators of the σ-algebra of measurable subsets of I (say the intervals with rational end-points). Put $Z=\bigcup_{k=1}^{\infty} Z_{X_k}$ and $N=\bigcup_{n=-\infty}^{\infty} T_1^n Z$, then $\mu(N)=0$, $T_1^n N=N$ and

$$T_1^n(X_k\times J)-N=(T'^n X_k\times J)-N,$$

for all n and $k=1,2,3,\dots$. Hence (since X_n is a separating sequence) for all measurable $X\subset I$,

$$T_1^n(X\times J)-N=(T'^n X\times J)-N;$$

and for all $p\in I$,

$$T_1^n(p\times J)-N=(T'^n p\times J)-N.$$

So if $(p,q)\in\Omega=S-N$,

$$\pi_I T_1(p,q)=T'\pi_I(p,q).$$

Thus T_1 and Ω have the required properties, concluding the proof. [Note that T_1 can be chosen to be *any* point automorphism inducing ϕ.]

Note. I am indebted to the referee for suggesting this short and elegant proof (which is actually almost identical to an argument in the proof of Lemma 5 of our earlier paper [2]). My original proof consisted in following von Neumann's original construction with an added condition to ensure that $\pi_I T=T'\pi_I$ on Ω.

Lemma B. *If μ is a Lebesgue-Stieltjes probability measure on $S=I\times J$, there exists a Borel set Z in S with $\mu(Z)=0$ and such that Z meets every vertical line segment $\pi_I^{-1}(p)$ in a set of cardinal c.*

Proof. Let μ_c be the non-atomic part of μ. The lemma is trivial if $\mu_c=0$. There exists a set L such that L is a countable union of horizontal line segments L_n, $n=1,2,3,\dots$ with $\bar{L}=S$ and such that $\mu_c(L)=0$. (Here $\bar{L}$ denotes the closure of L.) For there are uncountably many such sets which are disjoint and each is Borel and so μ-measurable. Since the intersection of all open horizontal strips (i.e. open sets bounded by two horizontal lines) containing L_n is L_n itself, and since the decreasing directed family of all such strips has a countable cofinal subfamily, it follows that given $\varepsilon>0$ there exists an open horizontal strip V_n^ε containing L_n such that $\mu_c(V_n^\varepsilon)<\frac{\varepsilon}{2^n}$. Let $V^\varepsilon=\bigcup_{n=1}^{\infty} V_n^\varepsilon$, then $\mu_c(V^\varepsilon)<\varepsilon$ and $V^\varepsilon\supset L$. Let $Z'=\bigcap_{k=1}^{\infty} V^{1/k}$. Then $Z'\supset L$, $\mu_c(Z')=0$, Z' is a union of horizontal line segments, Z' is an everywhere dense $\mathscr{G}_\delta$ in S, and so $Z'\cap\pi_I^{-1}(p)$ is, for every p, an everywhere dense $\mathscr{G}_\delta$ in the Polish space $\pi_I^{-1}(p)=p\times J$. Thus $Z'\cap\pi_I^{-1}(p)$ is of cardinal c for every p. There are at most countably many atoms of μ contained in Z'. If M is their union then $Z=Z'-M$ has all the required properties namely $\mu(Z)=0$ and $Z\cap\pi_I^{-1}(p)$ has cardinal c for all $p\in I$.

Alternative Proof. First note that if K is any non-trivial sub-interval of J (i.e. one consisting of more than one point) and $a>0$, then there exist two disjoint non-trivial closed sub-intervals K_0 and K_1 of K such that $\mu(I\times(K_0\cup K_1))\leqq a\,\mu(I\times K)$. Fix a such that $0<a<1$.

Using the above remark we construct inductively for any positive integer r and each sequence $i_1, i_2, \ldots, i_r$ with $i_j = 0$ or 1 for $1 \leqq j \leqq r$, a closed non-trivial interval $J_{i_1, \ldots, i_r}$ with the following properties:

(i) J_0 and J_1 are closed non-trivial disjoint sub-intervals of J such that

$$\mu(I \times (J_0 \cup J_1)) \leqq a\,\mu(I \times J) = a.$$

(ii) $J_{i_1, \ldots, i_{r-1}, 0}$ and $J_{i_1, \ldots, i_{r-1}, 1}$ are closed, non-trivial, disjoint sub-intervals of $J_{i_1, \ldots, i_{r-1}}$ such that

$$\mu(I \times (J_{i_1, \ldots, i_{r-1}, 0} \cup J_{i_1, \ldots, i_{r-1}, 1})) \leqq a\,\mu(I \times J_{i_1, \ldots, i_{r-1}}).$$

It follows that for each r,

$$\mu\Big(I \times \bigcup_{i_1, \ldots, i_r} J_{i_1, \ldots, i_r}\Big) \leqq a^r \to 0 \quad \text{as} \quad r \to \infty.$$

For every infinite sequence $i_1, i_2, \ldots, i_r, \ldots, \bigcap_{r=1}^{\infty} J_{i_1, \ldots, i_r} \neq \emptyset$ (being an intersection of a decreasing sequence of non-empty compact sets) and these intersections are disjoint for different infinite sequences. Since the intersections are in (1-1) correspondence with the binary decimals between 0 and 1, it follows that

$$Z' = \bigcup_{i_1, i_2, \ldots, i_r, \ldots} \Big(\bigcap_{r=1}^{\infty} J_{i_1, \ldots, i_r}\Big)$$

has cardinal c. But $Z' \subset \bigcup_{i_1, \ldots, i_r} J_{i_1, \ldots, i_r}$ for each r and so

$$\mu(I \times Z') \leqq \mu\Big(I \times \bigcup_{i_1, \ldots, i_r} J_{i_1, \ldots, i_r}\Big) \leqq a^r \quad \text{for each } r,$$

so $\mu(I \times Z') = 0$. Thus $Z = I \times Z'$ has the required properties. [This argument was suggested by the referee. Note that although the construction of Z' is similar to that of the Cantor ternary set, Z' may contain non-trivial sub-intervals, and so not be a Cantor set.]

Lemma C. *Under the same hypotheses as in Lemma* A *on $S = I \times J$, μ, ϕ, and T', there exists a point automorphism T of S such that T induces ϕ and such that $\pi_I T = T' \pi_I$.*

Proof. Let Ω, T_1 be as in the conclusion of Lemma A, Z as in Lemma B. Let

$$\Omega' = \Omega - \bigcup_{n=-\infty}^{\infty} T_1^n(\Omega \cap Z).$$

Then $\mu(\Omega') = 1$, $T_1 \Omega' = \Omega'$ and so for $(p, q) \in \Omega'$

$$\pi_I T_1(p, q) = T' \pi_I(p, q).$$

Further if $N = S - \Omega'$, then $\mu(N) = 0$ and since $N \supset Z$, $N \cap \pi_I^{-1}(p)$ has cardinal c for every $p \in I$. Let R_p denote any bijection from $N \cap \pi_I^{-1}(p)$ to $N \cap \pi_I^{-1}(T' p)$. For $(p, q) \in \Omega'$ let $T(p, q) = T_1(p, q)$. For $(p, q) \in N = S - \Omega'$ let $T(p, q) = R_p(p, q)$. Then T is a point automorphism of S which induces ϕ and for all $(p, q) \in S$,

$$\pi_I T(p, q) = T' \pi_I(p, q).$$

Corollary. *Lemma* C *remains true if* I, J *are replaced by arbitrary Polish spaces and* μ *is a Borel probability measure on their product; in particular it remains true if they are replaced by countable products of closed unit intervals.*

Proof. There is a Borel point isomorphism of a Polish space and a closed unit interval, such an isomorphism induces a Lebesgue-Stieltjes measure on S. Alternatively only minor verbal changes, are needed in the proof of Lemma B to make the entire argument of all three lemmas valid for arbitrary Polish spaces.

Theorem 1. *Let* A *be an arbitrary set, possibly uncountable, let* $S=\prod I_\alpha$, $\alpha\in A$, *each* $I_\alpha=[0,1]$. *Let* μ *be a probability measure on the product (Baire)* σ*-algebra of* S. *Then every set automorphism* ϕ *of* μ *on* S *is induced by a point automorphism* T *of* S.

Proof. By Lemma C, Corollary applied to countable products of I_α, ϕ has the countable extension property of [2] for every countable set $F\subset A$, invariant under ϕ. By Lemma 8 of [2] ϕ has the countable extension property. The result then follows by Lemma 7 of [2].

Corollary. *The result still holds if* S *is an arbitrary (possibly uncountable) product of Polish spaces.*

Note. The above proof does not use the last two lemmas of [2] which are the only ones which use the disintegration theorem. It is thus more elementary, apart from being valid for a much wider class of measures μ. In fact even Lemmas 4, 5 and 6 of [2] are not really used. Lemma 4 is used in [2] only in the proof of Lemma 9 and Lemma 5 in the proof of Lemma 6. Lemma 6 appears to be used in the proof of Lemma 8 but in fact its use is entirely unnecessary. The vital point is that the point automorphism $\hat{T}$ on $S(F)$ can, by the countable extension property for countable invariant sets, assumed in the hypothesis of Lemma 8, be extended to a point automorphism $\tilde{T}$ of $S(F\cup C)$ *inducing* $\tilde{\phi}$, the set automorphism induced by ϕ on $S(F\cup C)$ [which exists since $F\cup C$ is invariant]. In the argument as given in [2] it is merely stated that $\hat{T}$ can be extended to *some* point automorphism $\hat{P}$ of $S(F\cup C)$ and then Lemma 6 is invoked to get the required point automorphism $\tilde{T}$, which is clearly unnecessary. [A more vital use of Lemma 6 is however made in Lemma 9 of [2] for which we have now no further use.]

Combining Theorem 1 with the theorem of Lamperti (see Theorem 3.1 of [7]) we have the following.

Theorem 2. *Let* A *be an arbitrary set, let* $S=\prod I_\alpha$, $\alpha\in A$, *with each* $I_\alpha=[0,1]$. *Let* μ *be a probability measure on the product or Baire* σ*-algebra* $\mathscr{B}_0$ *of* S. *Let* U *be an invertible isometry of* $L^p(S,\mathscr{B}_0,\mu)$, $1\leqq p<\infty$, $p\neq 2$. *Then there exists a point automorphism* T *of* $(S,\mathscr{B}_0,\mu)$ *such that*

$$(Uf)(x)=f(T^{-1}x)\,h(x)$$

with $|h(x)|^p=\omega_T(x)$, *where* $\omega_T(x)$ *is defined by*

$$\mu(T^{-1}E)=\int_E \omega_T(x)\,\mu(dx).$$

References

1. Billingsley, P.: Ergodic theory and information. New York: John Wiley 1965.
2. Choksi, J.R.: Automorphisms of Baire measures on generalized cubes. Z. Wahrscheinlichkeitstheorie verw. Gebiete **22**, 195–204 (1972).
3. Halmos, P.R.: Lectures on ergodic theory. Publications of the Math. Soc. of Japan **3** (1956).
4. Maharam, D.: Automorphisms of products of measure spaces. Proc. Amer. math. Soc. **9**, 702–707 (1958).
5. Neumann, J. von: Einige Sätze über meßbare Abbildungen. Ann. of Math. **33**, 574–586 (1932).
6. Panzone, R., and Segovia, C.: Measurable transformations on compact spaces and o.n. systems on compact groups. Revista Un. mat. Argentina **22**, 83–102 (1964).
7. Lamperti, J.: On the isometries of certain function spaces. Pacific J. Math. **8**, 459–466 (1958).

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