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A Method for Studying the Integral Functionals of Stochastic Processes with Applications

II. Sojourn Time Distributions for Markov Chains

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1. Introduction

This paper is in continuation to [4], where the work done by several authors in the past on the integral functionals of stochastic processes was briefly surveyed. More importantly in [4] a method was introduced for the study of the distribution of the integral

$$Y(t)=\int_0^t f(X(\tau),\tau)\,d\tau, \tag{1}$$

where $X(t)$, $t\geqq 0$ is a continuous time parameter stochastic process appropriately defined on a probability space $(\Omega,\mathscr{A},\mathscr{P})$, with $\mathscr{X}$ as its state-space; f is a nonnegative (measurable) function defined on $\mathscr{X}\times[0,\infty)$. Here it is assumed that the integral $Y(t)$ exists and is finite almost surely for every $t>0$. The method is based on the introduction of a "quantal response process" $Z(t)$ defined for a hypothetical animal as: $Z(t)$ equals one if the animal is alive at time t and is equal to zero otherwise. In particular, it is assumed that

$$P\big(Z(t+\Delta t)=0\,|\,Z(t)=1,\ X(t)=x\big)=\delta f(x,t)\,\Delta t+o(\Delta t), \tag{2}$$

with $Z(0)=1$ and δ a nonnegative constant. Here the state "zero" is an absorption state for the process $Z(t)$. With this, it is evident that

$$P\big(Z(t)=1\big)=E\left\{\exp\left[-\delta\int_0^t f(X(\tau),\tau)\,d\tau\right]\right\}, \tag{3}$$

which in turn gives the Laplace Transform (L.T.) of the integral $Y(t)$. Thus the study of the integral $Y(t)$ can equivalently be carried out by studying the process $Z(t)$. It is to be noted that the quantal response process $Z(t)$ does not influence the process $X(t)$ in any way, rather as it is clear from (2), is influenced itself by the growth process $X(t)$. Again as was pointed out in [4], f is assumed to be nonnegative here without loss of any generality.

In [4], the above approach was applied to time homogeneous Markov Chains (M.C.) $X(t)$ with $\{1,2,3,\ldots\}$ as the state-space. In this paper, we shall explore further certain aspects of the integral $Y(t)$ of such processes. In particular, for a M.C. $X(t)$ we shall find the joint distribution of times spent by the chain in each

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state of a given finite set J of states, before it hits a taboo set H. Constructively, we define the M.C. $X(t)$ as follows: If $X(t_1)=i$ at some epoch t_1, the value of $X(t)$ will remain constant for an interval $t_1 \leqq t < t_1+\tau$, whose random duration τ is exponentially distributed with density function $c_i \exp(-c_i x)$, $c_i \geqq 0$; the probability that $X(t_1+\tau)=j$ is p_{ij}, where the matrix $\mathbf{p}=(p_{ij})$ is a stochastic transition matrix. We assume that the quantities c_i and p_{ij} are independent of time and that $c_j<\infty$ for all j so that the process $X(t)$ is time homogeneous, stable and strong Markov. Also, we assume that the sample paths of the process $X(t)$ are right continuous. Again since the process is defined constructively, it is separable. We assume hereonwards that the function f depends only on $X(t)$ and not explicitly on t. Also, in order to specify the function f, we are given a sequence of numbers $f(i)=f_i$ with $0 \leqq f_i < \infty$, $i=1,2,\ldots$. It is evident from the construction that the process $\{X(t), Z(t)\}$ is a M.C. with state space $\tilde{\mathscr{X}}=\{(i,r);\, i=1,2,\ldots;\, r=0,1\}$. Also it is not difficult to see that the exponential parameters for various states of $\tilde{\mathscr{X}}$ are given by

$$c_{i1}=(c_i+\delta f_i); \quad c_{i0}=c_i. \tag{4}$$

Furthermore, the transition matrix of the imbedded discrete time M.C. of $\{X(t), Z(t)\}$ is given for $i,j=1,2,\ldots$, by

$$p_{i1,j1}=c_i p_{ij}(c_i+\delta f_i)^{-1}; \quad p_{i1,i0}=\delta f_i (c_i+\delta f_i)^{-1}; \quad p_{i0,j0}=p_{ij}; \quad p_{i0,j1}=0. \tag{5}$$

Let $N(t)$ denote the number of jumps (changes) occurring in the M.C. $\{X(t), Z(t)\}$ during $(0,t]$. We now introduce the following notation.

$$\begin{aligned}
&P_{ij}(t)=P(X(t)=j \mid X(0)=i); \quad \tilde{P}_{ij}(t)=P(X(t)=j, Z(t)=1 \mid X(0)=i, Z(0)=1);\\
&P_{ir,js}(t,n)=P(X(t)=j, Z(t)=s, N(t)=n \mid X(0)=i, Z(0)=r);\\
&P_{ir,js}(t)=P(X(t)=j, Z(t)=s \mid X(0)=i, Z(0)=r);\\
&\boldsymbol{\pi}(\alpha)=(\pi_{ij}(\alpha)); \quad \tilde{\boldsymbol{\pi}}(\alpha)=(\tilde{\pi}_{ij}(\alpha)); \quad \mathbf{C}=(\delta_{ij} c_i); \quad \mathbf{I}=(\delta_{ij});\\
&\mathbf{1}=(1,1,1,\ldots)'; \quad \mathbf{f}=(\delta_{ij} f_i); \quad \mathbf{p}=(p_{ij}),
\end{aligned} \tag{6}$$

where $i,j=1,2,3,\ldots;\, r,s=0,1;\, n=0,1,2,\ldots;\, \alpha>0$; δ_{ij} is the Kronecker delta and $\pi_{ij}(\alpha)$ and $\tilde{\pi}_{ij}(\alpha)$ are L.T. of $P_{ij}(t)$ and $\tilde{P}_{ij}(t)$ respectively. Similarly let $\pi_{ir,js}(\alpha)$ and $\pi_{ir,js}(\alpha,n)$ denote the L.T. of $P_{ir,js}(t)$ and $P_{ir,js}(t,n)$ respectively. Clearly the symbols $\tilde{P}_{ij}(t)$ and $P_{i1,j1}(t)$ represent the same probability. We shall find it convenient to use one notation or the other as the case may be.

Following a standard argument, the L.T. $\pi_{ir,js}(\alpha)$ satisfy the following backward system of equations with $i,k=1,2,\ldots$

$$(\alpha+c_i+\delta f_i)\,\pi_{i1,k1}(\alpha)=\delta_{ik}+c_i \sum_{j=1}^{\infty} p_{ij}\,\pi_{j1,k1}(\alpha), \tag{7a}$$

$$(\alpha+c_i+\delta f_i)\,\pi_{i1,k0}(\alpha)=\delta f_i\,\pi_{i0,k0}(\alpha)+c_i \sum_{j=1}^{\infty} p_{ij}\,\pi_{j1,k0}(\alpha), \tag{7b}$$

$$(\alpha+c_i)\,\pi_{i0,k0}(\alpha)=\delta_{ik}+c_i \sum_{j=1}^{\infty} p_{ij}\,\pi_{j0,k0}(\alpha). \tag{7c}$$

It is known from Feller's construction (see [2]) that there always exists a solution $\boldsymbol{\pi}^{(\infty)}$ of the system (7) which is minimal among all its solutions. For simplicity sake, we now make the following basic assumption (B) throughout this paper.

(B) *The matrices* **p**, **f** *and* **C** *are such that the minimal solution of* (7) *is completely stochastic, or equivalently for* $\alpha>0$, *and* $i=1,2,\ldots$,

$$\alpha\sum_{k=1}^{\infty}\pi_{i1,k1}(\alpha)+\alpha\sum_{k=1}^{\infty}\pi_{i1,k0}(\alpha)=1 \tag{8}$$

and

$$\alpha\sum_{k=1}^{\infty}\pi_{i0,k0}(\alpha)=1. \tag{9}$$

Necessary and sufficient conditions are available for condition (B) to hold. We describe one in the following (see also [3]). Let $X(0)=i$, $Z(0)=r$. Also let τ_0 be the moment the process leaves the state (i,r) for the first time; τ_1 the moment it leaves the next state $(i_1,r_1)=(X(\tau_0),Z(\tau_0))$ for the first time, etc.; τ_n the moment of leaving the state $(i_n,r_n)=(X(\tau_{n-1}),Z(\tau_{n-1}))$ for the first time. In order that M.C. $(X(t),Z(t))$ satisfies the condition (B), it is necessary and sufficient that

$$\sum_{n=1}^{\infty}\frac{1}{c_{X(\tau_n),Z(\tau_n)}}=\infty, \tag{10}$$

with probability one, where c's are as given in (4).

We state without proof the following lemma to be used later.

Lemma 1. *Let the condition* (B) *hold. Then the solution of the backward system* (7) *satisfying Chapman-Kolmogorov equations is unique. The same solution also uniquely satisfies the forward system* (11) *given below. Furthermore, these uniqueness properties are maintained even when the original M.C. is modified by lowering the values of any of the original* c_{ir}'s *(or equivalently of* c_i's *and* f_i's*) or even when for some states these are replaced by zeros thereby making them absorption states for the modified M.C.*

The last statement of the lemma follows from (10). The forward system analogue of (7) is given by

$$(\alpha+c_k+\delta f_k)\,\pi_{i1,k1}(\alpha)=\delta_{ik}+\sum_{j=1}^{\infty}c_j\,p_{jk}\,\pi_{i1,j1}(\alpha), \tag{11a}$$

$$(\alpha+c_k)\,\pi_{i1,k0}(\alpha)=\delta f_k\,\pi_{i1,k1}(\alpha)+\sum_{j=1}^{\infty}c_j\,p_{jk}\,\pi_{i1,j0}(\alpha), \tag{11b}$$

$$(\alpha+c_k)\,\pi_{i0,k0}(\alpha)=\delta_{ik}+\sum_{j=1}^{\infty}c_j\,p_{jk}\,\pi_{i0,j0}(\alpha). \tag{11c}$$

Because of the condition (B), it easily follows that $N(t)$ is almost surely finite for every $t>0$, so that

$$\pi_{ir,js}(\alpha)=\sum_{n=0}^{\infty}\pi_{ir,js}(\alpha,n);\qquad i,j=1,2,\ldots;\ r,s=0,1. \tag{12}$$

Also, note that

$$\pi_{ij}=\pi_{i0,k0}=\pi_{i1,k1}+\pi_{i1,k0};\qquad i,k=1,2,\ldots. \tag{13}$$

We now introduce the following fundamental lemma.

Lemma 2. *Let the condition* (B) *hold. Then* $\pi(\alpha)$ *and* $\tilde{\pi}(\alpha)$ *are related through the identity.*

$$\pi_{ik}(\alpha)-\tilde{\pi}_{ik}(\alpha)=\delta\sum_{j=1}^{\infty} f_j\,\tilde{\pi}_{ij}(\alpha)\,\pi_{jk}(\alpha), \tag{14}$$

or equivalently in matrix form

$$\pi(\alpha)-\tilde{\pi}(\alpha)=\delta\,\tilde{\pi}(\alpha)\,f\,\pi(\alpha). \tag{15}$$

Proof. (14) follows by adding the system of equations

$$\pi_{i1,k0}(\alpha, n+1)=\delta\sum_{j=1}^{\infty} f_j \sum_{m=0}^{n} \pi_{i1,j1}(\alpha, m)\,\pi_{j0,k0}(\alpha, n-m), \tag{16}$$

over $n=0, 1, 2, \ldots$, and using (12) and the fact that $\pi_{i1,k0}(\alpha, 0)\equiv 0$. On the other hand (16) follows from the following backward type system of equations for $\pi_{ir,ks}(\alpha, n)$ and an induction argument on n.

$$(\alpha+c_i+\delta f_i)\,\pi_{i1,k0}(\alpha, n+1)=\delta f_i\,\pi_{i0,k0}(\alpha, n)+\sum_{j=1}^{\infty} c_i\,p_{ij}\,\pi_{j1,k0}(\alpha, n), \tag{17a}$$

$$(\alpha+c_i+\delta f_i)\,\pi_{i1,k1}(\alpha, n+1)=\sum_{j=1}^{\infty} c_i\,p_{ij}\,\pi_{j1,k1}(\alpha, n), \tag{17b}$$

$$(\alpha+c_i)\,\pi_{i0,k0}(\alpha, n+1)=\sum_{j=1}^{\infty} c_i\,p_{ij}\,\pi_{j0,k0}(\alpha, n). \tag{17c}$$

Here $\pi_{i0,k0}(\alpha, 0)=\pi_{ik}(\alpha+c_i)^{-1}$ and $\pi_{i1,j1}(\alpha, 0)=\pi_{ij}(\alpha+c_i+\delta f_i)^{-1}$. The details of the proof are omitted.

2. Main Results

One of the problems treated in this paper (section 2.2) is to find for a M.C. $X(t)$ the joint distribution of the times spent by the chain in each state of a given finite set J of states, before it hits a taboo set H. For this we need to introduce the following notation. Let $X(0)=i$ and H denote an arbitrary taboo set of states which may possible be empty but is such that $i\notin H$. Following Chung [1] we define

$$\rho_i=\inf\{t:\ t>0;\ X(t)\neq i\,|\,X(0)=i\}. \tag{18}$$

$$\alpha_{ij}=\inf\{t:\ t>\rho_i;\ X(t)=j\,|\,X(0)=i\}. \tag{19}$$

Here ρ_i is the first exit time from i and

$$P(\rho_i\geqq t\,|\,X(0)=i)=\exp(-c_i\,t);\qquad t\geqq 0. \tag{20}$$

Also note that $\alpha_{ii}>\rho_i$ almost everywhere. Let

$$_H\alpha_{ij}=\begin{cases}\alpha_{ij} & \text{if } \alpha_{ij}\leqq \inf\limits_{k\in H}\alpha_{ik}\\ \infty & \text{otherwise.}\end{cases} \tag{21}$$

Thus ${}_H\alpha_{ij} \equiv \alpha_{ij}$ if H is empty. Again let

$$\alpha_{iH} = \inf\{t:\ t > \rho_i;\ X(t) \in H | X(0) = i\}. \tag{22}$$

If $i \neq j$, ${}_H\alpha_{ij}$ is the first entrance time from i to j under the taboo H. On the other hand since $i \notin H$, α_{iH} is the first entrance time from i to the set H. We define for $t > 0$ and $i \notin H$,

$$ {}_HP_{ij}(t) = P(\alpha_{iH} \geqq t;\ X(t) = j | X(0) = i), \tag{23}$$

and

$$ {}_H\tilde{P}_{ij}(t) = P(\alpha_{iH} \geqq t;\ X(t) = j,\ Z(t) = 1 | X(0) = i,\ Z(0) = 1). \tag{24}$$

The probability ${}_HP_{ij}(t)$ is the transition probability from i to j in time t under the taboo H. Note that if $i \neq j$ and $j \in H$, then ${}_HP_{ij}(t) \equiv 0$. A similar interpretation is to be taken for ${}_H\tilde{P}_{ij}(t)$. Also it is evident that $({}_HP_{ij}(t))$ and $({}_H\tilde{P}_{ij}(t))$ are in general both substochastic matrices with

$$ {}_HP_{ij}(0) = {}_HP_{ij}(0+) = {}_H\tilde{P}_{ij}(0) = {}_H\tilde{P}_{ij}(0+) = \delta_{ij}. \tag{25}$$

We say k is accessible from j and write $j \rightsquigarrow k$ iff for some $t > 0$, $P_{jk}(t) > 0$. If H is nonempty, we write $j \rightsquigarrow H$ iff there is a state $k \in H$ such that $j \rightsquigarrow k$. If $j \rightsquigarrow H$ it can be shown that $\int_0^\infty {}_HP_{ij}(t)\,dt < \infty$ (see Chung [1], p. 192). Since ${}_H\tilde{P}_{ij}(t) \leqq {}_HP_{ij}(t)$, it follows that $\int_0^\infty {}_H\tilde{P}_{ij}(t)\,dt < \infty$.

To begin with, in the next section, our aim will be to find the joint distribution of the integral $Y(\alpha_{iH})$ and $X(\alpha_{iH})$. To this end, let

$$ {}_HF_{ij}(t) = P(\alpha_{ij} \leqq \min(\inf_{k \in H} \alpha_{ik}, t) | X(0) = i). \tag{26}$$

In particular, let

$$F_{ij}(t) = P(\alpha_{ij} \leqq t | X(0) = i); \quad F_{iH}(t) = P(\alpha_{iH} \leqq t | X(0) = i). \tag{27}$$

Similarly let

$$ {}_H\tilde{F}_{ij}(t) = P(\alpha_{ij} \leqq \min(\inf_{k \in H} \alpha_{ik}, t),\ Z(\alpha_{ij}) = 1 | X(0) = t,\ Z(0) = 1) \tag{28}$$

and

$$\tilde{F}_{ij}(t) = P(\alpha_{ij} \leqq t,\ Z(\alpha_{ij}) = 1 | X(0) = i,\ Z(0) = 1). \tag{29}$$

2.1. Joint Distribution of $Y(\alpha_{iH})$ and $X(\alpha_{iH})$

Let H and the set

$$D = \{j:\ j \notin H,\ j \rightsquigarrow H\} \tag{30}$$

be both nonempty. We then have $\int_0^\infty {}_HP_{ij}(t)\,dt < \infty$, for all $j \in D$. Let $i \notin H$. Also let for $k \in H$,

$$\tilde{G}_{ik}(t|H) = P(\alpha_{iH} \leqq t,\ Z(\alpha_{iH}) = 1,\ X(\alpha_{iH}) = k | X(0) = i,\ Z(0) = 1). \tag{31}$$

It is clear that if $i \notin D$, $\alpha_{iH} = \infty$, a.s., so that $\tilde{G}_{ik}(t|H) \equiv 0$. Thus we assume that $i \in D$, and have the following theorem.

Theorem 1. *Let the condition* (B) *hold. Then for* $i \in D$,

$$E\left(\exp\left[-\delta \int_0^{\alpha_{iH}} f(X(\tau))\,d\tau\right] I_k(\alpha_{iH}) | X(0)=i, \alpha_{iH}<\infty\right) = \left[\sum_{j\in D} c_j p_{jk}\, {}_H\tilde{\pi}_{ij}(0)\right] / F_{iH}(\infty), \tag{32}$$

and

$$E\left(\exp\left[-\delta \int_0^{\alpha_{iH}} f(X(\tau))\,d\tau\right] \Big| X(0)=i\right) = \sum_{j\in D} c_j p_{jH}\, {}_H\tilde{\pi}_{ij}(0) + \lim_{\alpha\downarrow 0} \alpha \sum_{j\notin H} {}_H\tilde{\pi}_{ij}(\alpha), \tag{33}$$

where

$$F_{iH}(\infty) = \sum_{j\in D} c_j p_{jH}\, {}_H\pi_{ij}(0); \qquad p_{jH} = \sum_{k\in H} p_{jk}; \tag{34}$$

${}_H\pi_{ij}(\alpha)$ *and* ${}_H\tilde{\pi}_{ij}(\alpha)$ *are the L.T. of* ${}_HP_{ij}(t)$ *and* ${}_H\tilde{P}_{ij}(t)$ *respectively, with* $\alpha>0$, *and* $I_k(t)$ *is the indicator function of the set* $[X(t)=k]$.

Proof. Observe that the history of the process until the time point α_{iH} does not depend upon c_k and f_k for $k\in H$. Consequently, since we shall be concerned with the process $\{X(t), Z(t)\}$ only until the moment α_{iH} starting from state i, without loss of anything we modify the M.C. by taking $c_k=f_k=0$ for all $k\in H$. With this modification, the states of H become absorption states for the M.C. $\{X(t), Z(t)\}$. Let for this modified M.C.

$$\tilde{Q}_{ij}(t) = P(X(t)=j, Z(t)=1 | X(0)=i, Z(0)=1). \tag{35}$$

Clearly, for $j\notin H$, and $k\in H$,

$$\tilde{Q}_{ij}(t) = {}_H\tilde{P}_{ij}(t); \qquad \tilde{Q}_{ik}(t) = \tilde{G}_{ik}(t|H). \tag{36}$$

In view of condition (B) and Lemma 1, the modified Eqs. (7) and (11) for the modified M.C. also have a unique solution which is completely stochastic. Thus considering the last jump to set H during $(0,t)$ for the modified M.C., we have for $k\in H$, $i\in D$,

$$\tilde{Q}_{ik}(t) = \sum_{j\in D} \int_0^t \tilde{Q}_{ij}(\tau)\, c_j\, p_{jk}\, d\tau. \tag{37}$$

From (36) and (37), it then follows that

$$\tilde{G}_{ik}(t|H) = \sum_{j\in D} \int_0^t {}_H\tilde{P}_{ij}(\tau)\, c_j\, p_{jk}\, d\tau. \tag{38}$$

Using this, we have for $k\in H$, $i\in D$,

$$\begin{aligned} \text{L.H.S. of (32)} &= P(Z(\alpha_{iH})=1, X(\alpha_{iH})=k | X(0)=i, \alpha_{iH}<\infty) \\ &= \tilde{G}_{ik}(\infty|H)/F_{iH}(\infty) = \text{R.H.S. of (32)}, \end{aligned} \tag{39}$$

where $F_{iH}(\infty)$ is positive, since $i \rightsquigarrow H$ by assumption. From the above, it easily follows that for $i\in D$,

$$\text{L.H.S. of (33)} = \sum_{j\in D} c_j p_{jH}\, {}_H\tilde{\pi}_{ij}(0) + \lim_{t\to\infty} P(\alpha_{iH}\geqq t, Z(t)=1 | X(0)=i, Z(0)=1), \tag{40}$$

where it is easily seen that the last limit exists and that, by using a Taubarian argument, is equal to the limit on the right side of (33). This completes the proof.

Subject to the condition (B), it is not difficult to see that for $i, j \in D$, ${}_H\tilde{\pi}_{ij}$ satisfy the following analogues of Eqs. (7a) and (11a)

$$(\alpha + c_i + \delta f_i)\, {}_H\tilde{\pi}_{ij}(\alpha) = \delta_{ij} + \sum_{l \in D} c_i\, p_{il}\, {}_H\tilde{\pi}_{lj}(\alpha), \tag{41}$$

$$(\alpha + c_j + \delta f_j)\, {}_H\tilde{\pi}_{ij}(\alpha) = \delta_{ij} + \sum_{l \in D} {}_H\tilde{\pi}_{il}(\alpha)\, c_l\, p_{lj}. \tag{42}$$

For $j \notin (DUH)$, the Eqs. (41) and (42) are still valid if $l \in D$ is replaced with $l \notin H$ under the summation signs on their right sides. These equations can be solved to yield $({}_H\tilde{\pi}_{ij})$, which in turn using (32) gives for every $i \rightsquigarrow H$ the desired conditional joint distribution of $Y(\alpha_{iH})$ and $X(\alpha_{iH})$ given $\alpha_{iH} < \infty$. Also by taking $f_j = 1$, for $j \notin H$, (33) gives the L.T. of the first passage time α_{iH} to set H.

Consider, in particular, the case where the set D is finite. For this let $\mathbf{C}_R, \mathbf{f}_R, \mathbf{p}_R$ be the finite matrices obtained from $\mathbf{C}$, $\mathbf{f}$ and $\mathbf{p}$ respectively by restricting them only to the states of D. Let

$$\mathbf{h} = (h_{ij}) = \big({}_H\tilde{\pi}_{ij}(0)\big), \qquad i, j \notin D. \tag{43}$$

Since $j \rightsquigarrow H$, we have $h_{ij} < \infty$ for all $i, j \in D$. Also for fixed $k \in H$, let

$$\mathbf{p}_{R,k} = (p_{ik}); \qquad \tilde{\mathbf{G}}_{R,k} = \big(\tilde{G}_{ik}(\infty \mid H)\big); \qquad i \in D, \tag{44}$$

be two finite *column* vectors. We then have the following corollary.

Corollary. *Let D be finite, and $\mathbf{C}_R$, $\mathbf{f}_R$ and $\mathbf{p}_R$ be such that the matrix $(\mathbf{C}_R + \delta \mathbf{f}_R - \mathbf{C}_R\, \mathbf{p}_R)$ is nonsingular. Then under the conditions of Theorem 1,*

$$\tilde{\mathbf{G}}_{R,k} = (\mathbf{C}_R + \delta \mathbf{f}_R - \mathbf{C}_R\, \mathbf{p}_R)^{-1} \cdot \mathbf{C}_R\, \mathbf{p}_{R,k}; \qquad k \in H, \tag{45}$$

which, for the case with $F_{iH}(\infty) = 1$ for $i \in D$, yields the exact expression for (32).

Proof. For fixed $k \in H$, and varying $i \in D$, we have from (38) on letting $t \to \infty$,

$$\tilde{\mathbf{G}}_{R,k} = \mathbf{h} \cdot \mathbf{C}_R \cdot \mathbf{p}_{R,k}; \qquad k \in H. \tag{46}$$

On the other hand letting $\alpha \to 0$ we have from (41) and (42)

$$(\mathbf{C}_R + \delta \mathbf{f}_R - \mathbf{C}_R\, \mathbf{p}_R) \cdot \mathbf{h} = \mathbf{I}; \qquad \mathbf{h} \cdot (\mathbf{C}_R + \delta \mathbf{f}_R - \mathbf{C}_R\, \mathbf{p}_R) = \mathbf{I}, \tag{47}$$

so that

$$\mathbf{h} = (\mathbf{C}_R + \delta \mathbf{f}_R - \mathbf{C}_R\, \mathbf{p}_R)^{-1}, \tag{48}$$

which yields (45). This completes the proof.

2.2. Joint Distribution of Sojourn Times for Different States

Let the taboo set H be nonempty. Let $J = \{l_1, l_2, \ldots, l_M\}$ be another set of states with $l_r \notin H$, but $l_r \rightsquigarrow H$ for $r = 1, 2, \ldots, M$. Given that the process starts with $X(0) = i$, $i \notin H$, $i \rightsquigarrow H$, our aim in this section is to find the joint distribution of the amounts of times ${}_H C_{il_r}$, $r = 1, 2, \ldots, M$, spent by the process $X(t)$ in states $l_1, l_2, \ldots, l_M$ respectively, prior to the moment it hits the taboo set H. To this end let $\beta_r > 0$, $\boldsymbol{\beta} = (\beta_r\, \delta_{rm})$ and $\mathbf{I} = (\delta_{rm})$; $r, m = 1, 2, \ldots, M$. Also, let $\boldsymbol{\eta}(J, J) = (\eta_{l_r l_m}(\alpha))$, and $\boldsymbol{\eta}(j, J) = (\eta_{j l_1}(\alpha), \ldots, \eta_{j l_M}(\alpha))'$, where $\eta_{jk}(\alpha)$ is the L.T. of ${}_H P_{jk}(t)$, for $j, k \notin H$.

In particular, if $k\notin H$ and $k \rightsquigarrow H$, the integral

$$_Hq_{jk}(\infty)\equiv \int_0^\infty {}_HP_{jk}(t)\,dt=\eta_{jk}(0), \tag{49}$$

is finite. Finally let

$$\boldsymbol{\eta}_0(J,J)=\left(\eta_{l_r l_m}(0)\right); \qquad \boldsymbol{\eta}_0(j,J)=\left(\eta_{jl_1}(0),\ldots,\eta_{jl_M}(0)\right)'. \tag{50}$$

Then we have the desired theorem.

Theorem 2. *Let the condition* (B) *hold, and the* $M\times M$ *matrix* $(\mathbf{I}+\boldsymbol{\eta}_0(J,J)\,\boldsymbol{\beta})$ *be nonsingular. Then, if* $X(0)=i$; $i\notin H$ *and* $i\rightsquigarrow H$, *we have*

$$E\left(\exp\left[-\sum_{r=1}^{M}\beta_r\,{}_HC_{il_r}\right]\right)=1-(\beta_1,\ldots,\beta_M)\left[\mathbf{I}+\boldsymbol{\eta}_0(J,J)\,\boldsymbol{\beta}\right]^{-1}\cdot\boldsymbol{\eta}_0(i,J). \tag{51}$$

Proof. It is evident that the distribution of $({}_HC_{il_1},\ldots,{}_HC_{il_M})$ is independent of c_j for $j\in H$. As such, like in section **2.1**, we consider a modified M.C. which is same as the original one except that now $c_j=0$ for $j\in H$, so that every state of H is now an absorption state. Clearly as in (36), the probability $P(X(t)=j|X(0)=i)$ for the modified chain with $i,j\notin H$, is same as ${}_HP_{ij}(t)$. Again, we assume that $\delta f_{l_r}=\beta_r$, $r=1,2,\ldots,M$, and otherwise $f_j=0$ for $j\notin J$. The following relations are now easy to establish.

$$\begin{aligned}
\text{L.H.S. of (51)}&=E\left(\exp\left[-\delta\int_0^{\alpha_{iH}}f(X(\tau))\,d\tau\right]\Big|X(0)=i\right)\\
&=E\left(\exp\left[-\delta\int_0^{\infty}f(X(\tau))\,d\tau\right]\Big|X(0)=i\right)\\
&=\lim_{t\to\infty}E\left(\exp\left[-\delta\int_0^{t}f(X(\tau))\,d\tau\right]\Big|X(0)=i\right)\\
&=\lim_{\alpha\to 0}\alpha\int_0^\infty\exp(-\alpha t)\,E\left(\exp\left[-\delta\int_0^{t}f(X(\tau))\,d\tau\right]\Big|X(0)=i\right)dt\\
&=\lim_{\alpha\to 0}\alpha\sum_{k=1}^{\infty}\tilde{\eta}_{ik}(\alpha),
\end{aligned} \tag{52}$$

where $\tilde{\eta}_{ik}(\alpha)$ is L.T. of $P(X(t)=k, Z(t)=1|X(0)=i, Z(0)=1)$, defined for the modified M.C. the last but one equality of (52) follows from a Taubarian argument, whereas the last one follows from the condition (B) which incidently remains valid for the modified M.C., by virtue of Lemma 1.

We now proceed to evaluate (52). It is easy to see that subject to condition (B), $\boldsymbol{\eta}$ and $\tilde{\boldsymbol{\eta}}$ satisfy the identity (14) which in the present case takes the form

$$\eta_{ik}(\alpha)-\tilde{\eta}_{ik}(\alpha)=\sum_{r=1}^{M}\beta_r\tilde{\eta}_{il_r}(\alpha)\,\eta_{l_rk}(\alpha); \qquad k=1,2,\ldots. \tag{53}$$

The Eq. (53), for $k=1,\ldots,M$, can be written down in the matrix form as

$$\left[\mathbf{I}+\boldsymbol{\eta}(J,J)\,\boldsymbol{\beta}\right]\tilde{\boldsymbol{\eta}}(i,J)=\boldsymbol{\eta}(i,J), \tag{54}$$

where $\tilde{\boldsymbol{\eta}}(i,J)=(\tilde{\eta}_{il_1},\ldots,\tilde{\eta}_{il_M})'$. Since $(\mathbf{I}+\boldsymbol{\eta}_0(J,J)\,\boldsymbol{\beta})$ is nonsingular, $(\mathbf{I}+\boldsymbol{\eta}(J,J)\,\boldsymbol{\beta})$ will be nonsingular for small enough α, so that (53) for $k\in J$ can be uniquely solved for $\tilde{\eta}_{il_r}$, $r=1,2,\ldots,M$, in terms of η's. The remaining $\tilde{\eta}_{ik}$'s with $k\notin J$, can then be obtained explicitly as (see also [4])

$$\tilde{\eta}_{ik}(\alpha)=\eta_{ik}(\alpha)-\sum_{r=1}^{M}\beta_r\,\tilde{\eta}_{il_r}(\alpha)\,\eta_{l_r k}(\alpha). \tag{55}$$

Again, let

$$\eta_{j.}(\alpha)=\sum_{k=1}^{\infty}\eta_{jk}(\alpha). \tag{56}$$

Since (55) is valid for all k, on using it we have from (52)

$$\begin{aligned}\text{L.H.S. of (51)}&=\lim_{\alpha\to 0}\alpha\left[\eta_{i.}(\alpha)-\sum_{r=1}^{M}\beta_r\,\tilde{\eta}_{il_r}(\alpha)\,\tilde{\eta}_{l_r.}(\alpha)\right]\\&=\lim_{\alpha\to 0}\left[1-\sum_{r=1}^{M}\beta_r\,\tilde{\eta}_{il_r}(\alpha)\right]\\&=1-\sum_{r=1}^{M}\beta_r\,\tilde{\eta}_{il_r}(0).\end{aligned} \tag{57}$$

Here we have used the fact that $\alpha\,\eta_{j.}(\alpha)\equiv 1$, for all j and $\alpha>0$. This is a simple consequence of the condition (B). Finally, since the matrix $[\mathbf{I}+\boldsymbol{\eta}_0(J,J)\,\boldsymbol{\beta}]$ has an inverse, (51) follows from (57), thereby completing the proof.

In general the L.T. given by (51) is a rational function of $\beta_1,\beta_2,\ldots,\beta_M$ and can be easily inverted to yield the desired joint distribution of the sojourn times.

Special Cases. As an illustration let us consider the case when $M=1$, so that we are interested in the distribution of ${}_H C_{il}$ for a given state $l\notin H$. From (53) we have

$$\eta_{ik}(\alpha)-\tilde{\eta}_{ik}(\alpha)=\beta\,\tilde{\eta}_i(\alpha)\,\eta_{lk}(\alpha);\qquad k=1,2,\ldots. \tag{58}$$

For $k=l$, this yields

$$\tilde{\eta}_{il}(\alpha)=\eta_{il}(\alpha)\cdot\left(1+\beta\,\eta_{ll}(\alpha)\right)^{-1}. \tag{59}$$

Using this in (51) we finally have

$$\begin{aligned}E(\exp[-\beta\,{}_H C_{il}])&=1-\beta\,\eta_{il}(0)\left(1+\beta\,\eta_{ll}(0)\right)^{-1}\\&=1-\beta\,{}_H q_{il}(\infty)\left(1+\beta\,{}_H q_{ll}(\infty)\right)^{-1},\end{aligned} \tag{60}$$

where ${}_H q_{il}(\infty)$ is given by (49).

Again, we have (see Chung [1], p. 213)

$${}_H q_{il}(\infty)={}_H q_{ll}(\infty)\,{}_H F_{il}(\infty),\qquad i\neq l, \tag{61}$$

so that using these in (60) we obtain

$$E(\exp[-\beta\,{}_H C_{il}])=\begin{cases}\left(1+\beta\,{}_H q_{ll}(\infty)\right)^{-1}; & i=l\\ \left[1+\beta\,{}_H q_{ll}(\infty)\left(1-{}_H F_{il}(\infty)\right)\right]\left(1+\beta\,{}_H q_{ll}(\infty)\right)^{-1}; & i\neq l.\end{cases} \tag{62}$$

Inversion of this transform yields

$$P({}_HC_{il}\leqq t\,|\,X(0)=i)=\begin{cases}1-\exp[-t/{}_Hq_{ll}(\infty)]; & i=l,\\ 1-{}_HF_{il}(\infty)\exp[-t/{}_Hq_{ll}(\infty)]; & i\neq l.\end{cases} \tag{63}$$

This coincides with the known result due to Chung ([1], p. 229), where he gives three different methods of deriving it. Here then we have given a fourth method based on the ideas presented here. Our method however, works for the general case involving any finite number M of states as exhibited above.

Continuing further, let us now consider the case with $M=2$. We use the notation l and s for l_1 and l_2 respectively. With $M=2$, it is easy to show that the matrix $(I+\boldsymbol{\eta}_0(J,J)\,\boldsymbol{\beta})$ has an inverse, so that the Eq. (53) with $k=l$ and s can be uniquely solved for $\boldsymbol{\eta}_{il}$ and $\boldsymbol{\eta}_{is}$. This along with (61), on using in (51) yield after a little simplification

$$\begin{aligned}E(\exp[-\beta_1\,{}_HC_{il}-\beta_2\,{}_HC_{is}])&=[1+\beta_1\beta_2a_1+\beta_1a_2+\beta_2a_3]\\&\quad\cdot[1+\beta_1\beta_2a_4+\beta_1a_5+\beta_2a_6]^{-1}\end{aligned} \tag{64}$$

where

$$\begin{aligned}a_1&=(1-\delta_{il})(1-\delta_{is})\,a_5a_6\big[\big(1+{}_HF_{sl}(\infty)\,{}_HF_{is}(\infty)+{}_HF_{is}(\infty)\,{}_HF_{il}(\infty)\big)\\&\qquad-\big({}_HF_{ls}(\infty)\,{}_HF_{sl}(\infty)+{}_HF_{il}(\infty)+{}_HF_{is}(\infty)\big)\big],\\ a_2&=(1-\delta_{il})\,a_5\big(1-{}_HF_{il}(\infty)\big);\qquad a_3=(1-\delta_{is})\,a_6\big(1-{}_HF_{is}(\infty)\big),\\ a_4&=a_5a_6\big(1-{}_HF_{ls}(\infty)\,{}_HF_{sl}(\infty)\big);\qquad a_5={}_Hq_{ll}(\infty);\qquad a_6={}_Hq_{ss}(\infty).\end{aligned} \tag{65}$$

Here δ's are the Kronecker deltas. The L.T. (64) can easily be inverted to yield for $t_l,\,t_s\geqq 0$,

$$\begin{aligned}P({}_HC_{il}\leqq t_l,\,{}_HC_{is}\leqq t_s)&=1-\left(1-\frac{a_2}{a_5}\right)\exp(-t_s/a_5)-\exp(-a_5\,t_s/a_4)\\&\quad\cdot\left\{1-\frac{a_3}{a_6}-\left(1-\frac{a_2}{a_5}\right)\exp(-t_s/a_5)\right.\\&\quad\left.+\left(\frac{a_3}{a_6}-\frac{a_1}{a_4}\right)\exp(-a_6\,t_s/a_4)\right\}-\exp(-a_5\,t_s/a_4)\\&\quad\cdot\left[\left(1-\frac{a_1}{a_4}\right)\sum_{k=1}^{\infty}\frac{x^k}{k!\,(k-1)!}\int_0^{t_l}\tau^{k-1}\exp(-a_6\,\tau/a_4)\,d\tau\right.\\&\quad-\left(1-\frac{a_2}{a_5}\right)\exp(-t_l/a_5)\sum_{k=1}^{\infty}\frac{x^k}{k!\,(k-1)!}\\&\quad\cdot\int_0^{t_l}\tau^{k-1}\exp\left\{-\left(\frac{a_6}{a_4}-\frac{1}{a_5}\right)\tau\right\}d\tau\\&\quad\left.-\left(\frac{a_3}{a_4}-\frac{a_1a_6}{a_4^2}\right)\sum_{k=1}^{\infty}\frac{x^k}{k!\,k!}\int_0^{t_l}\tau^k\exp\left(-\frac{a_6}{a_4}\tau\right)d\tau\right],\end{aligned} \tag{66}$$

where

$$x=\left(\frac{a_5\,a_6}{a_4^2}-\frac{1}{a_4}\right)t_s. \tag{67}$$

Finally, we present in the following the first two moments of ${}_HC_{il}$ and ${}_HC_{is}$ as obtained by using (64).

Case (i). $s=i$.

$$\begin{aligned}
E({}_HC_{il})&={}_Hq_{ll}(\infty)\cdot{}_HF_{il}(\infty);\\
E({}_HC_{ii})&={}_Hq_{ii}(\infty);\\
\operatorname{Var}({}_HC_{il})&=[{}_Hq_{ll}(\infty)]^2\,{}_HF_{il}(\infty)(2-{}_HF_{il}(\infty));\\
\operatorname{Var}({}_HC_{ii})&=[{}_Hq_{ii}(\infty)]^2;\\
\operatorname{Cov}({}_HC_{il},{}_HC_{ii})&={}_Hq_{ll}(\infty)\cdot{}_Hq_{ii}(\infty)\,{}_HF_{il}(\infty)\,{}_HF_{li}(\infty);\\
\rho({}_HC_{il},{}_HC_{ii})&={}_HF_{il}(\infty)\,{}_HF_{li}(\infty)\cdot[{}_HF_{il}(\infty)(2-{}_HF_{il}(\infty))]^{-\frac{1}{2}}.
\end{aligned} \tag{68}$$

Case (ii). $l\neq i$, $s\neq i$.

$$\begin{aligned}
E({}_HC_{il})&={}_Hq_{ll}(\infty)\cdot{}_HF_{il}(\infty);\\
E({}_HC_{is})&={}_Hq_{ss}(\infty)\,{}_HF_{is}(\infty);\\
\operatorname{Var}({}_HC_{il})&=[{}_Hq_{ll}(\infty)]^2\,{}_HF_{il}(\infty)(2-{}_HF_{il}(\infty));\\
\operatorname{Var}({}_HC_{is})&=[{}_Hq_{ss}(\infty)]^2\cdot{}_HF_{is}(\infty)(2-{}_HF_{is}(\infty));\\
\operatorname{Cov}({}_HC_{il},{}_HC_{is})&={}_Hq_{ll}(\infty)\,{}_Hq_{ss}(\infty)\\
&\quad\cdot[{}_HF_{sl}(\infty)\,{}_HF_{is}(\infty)+{}_HF_{ls}(\infty)\,{}_HF_{il}(\infty)-{}_HF_{il}(\infty)\,{}_HF_{is}(\infty)];\\
\rho({}_HC_{il},{}_HC_{is})&=({}_HF_{sl}(\infty)\,{}_HF_{is}(\infty)+{}_HF_{ls}(\infty)\,{}_HF_{il}(\infty)-{}_HF_{il}(\infty)\,{}_HF_{is}(\infty))\\
&\quad\cdot[{}_HF_{il}(\infty)\,{}_HF_{is}(\infty)(2-{}_HF_{il}(\infty))(2-{}_HF_{is}(\infty))]^{-\frac{1}{2}}.
\end{aligned} \tag{69}$$

Here ρ represents the correlation coefficient which, of course, is defined when ${}_HF_{il}(\infty)$ and ${}_HF_{is}(\infty)$ are both positive. It is clear from (68) that when one of the two states l and s coincides with the initial state i, the two sojourn times ${}_HC_{il}$ and ${}_HC_{is}$ are nonnegatively correlated. However, the author has not been able to conclude one way or the other about the possible sign of ρ for the case (ii) or equivalently of the expression

$${}_HF_{sl}(\infty)\,{}_HF_{is}(\infty)+{}_HF_{ls}(\infty)\,{}_HF_{il}(\infty)-{}_HF_{il}(\infty)\,{}_HF_{is}(\infty), \tag{70}$$

particularly when all the F's involved are positive. The results of [4] and of the present paper have been applied to birth and death processes arising in various practical situations. These along with other results have been reported elsewhere (see [5]). The reader may also refer to a result analogous to our (51) and due to Professor J.F.C. Kingman (Z. Wahrscheinlichkeitstheorie verw. Geb. **11**, 9–17 (1968)). The author is grateful to Professor Kingman for drawing his attention to this result.

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