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On Roots of Transformations

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1. Introduction

Let $(X, \mathfrak{B}, m)$ denote the measure space consisting of the unit interval, Lebesgue measurable sets, and Lebesgue measure respectively. Let $\mathfrak{T}$ denote the class of invertible transformations τ of X onto X which are measurable and non-singular with respect to $(\mathfrak{B}, m)$. This class arises naturally in the problem of invariant measure ([2], p. 81). A topology is defined on $\mathfrak{T}$ by the metric $d(\tau, \sigma) = m(\tau \neq \sigma)$. Our main purpose is to show that on the one hand the class of transformations with roots of every order are dense in $\mathfrak{T}$ while on the other hand the class of ergodic transformations which do not have roots of any order are dense in the antiperiodic transformations. This possibility is suggested by the situation in the invariant measure problem where it is known that on the one hand the class of transformations with a finite invariant measure equivalent to m are dense in $\mathfrak{T}$ while on the other hand the class $\mathfrak{N}$ of transformations with no σ-finite invariant measure absolutely continuous with respect to m are dense in the antiperiodic transformations. The former result may be obtained easily from Theorem 2 below and the latter result is proven in [1].

These results imply in particular that an antiperiodic transformation may be modified on a set of arbitrarily small measure so that the nature of the modified transformation may vary greatly with respect to existence of roots or an invariant measure. However, with respect to the topology defined by d, the class of antiperiodic transformations are not numerous as we show in Theorem 5.

Theorem 1 below is a decomposition result which we utilize in order to modify an antiperiodic transformation. Although Theorem 1 of [1] yields Theorem 1 of this note, we see below that a simple direct proof is possible based on a construction in [1]. Corollary 1 yields a short proof of Theorem 2 which is an unpublished result due to Linderholm that is stated in [5]. The construction in the proof of Lemma 6 was suggested by the technique in [4].

In what follows all set equations as well as conditions involving transformations will be understood to hold modulo null sets. All transformations will be in $\mathfrak{T}$ either by assumption or construction.

2. Preliminaries

A transformation τ is (i) measurable and (ii) nonsingular with respect to $(\mathfrak{B}, m)$ if (i) $B \in \mathfrak{B}$ implies $\tau(B) \in \mathfrak{B}$ and $\tau^{-1}(B) \in \mathfrak{B}$ and (ii) $m(B) = 0$ implies $m(\tau(B)) = m(\tau^{-1}(B)) = 0$. A transformation τ is said to be periodic with strict period n if

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$\tau^i(x) \neq x$, $1 \leqq i \leqq n-1$, $\tau^n(x) = x$, $x \in X$. We say τ is periodic with period n if $\tau^n(x) = x$, $x \in X$, and n is the smallest such positive integer. A transformation τ is antiperiodic if $\tau^n(x) \neq x$ for every positive integer n, $x \in X$. Let $\mathfrak{P}$ denote the class of periodic transformations, $\mathfrak{S}$ the class of strictly periodic transformations, and $\mathfrak{A}$ the class of antiperiodic transformations. Given a positive integer n, a transformation σ is said to be an n^{th} root of τ if $\tau = \sigma^n$. Given τ, the measure $m\tau$ is defined as $m\tau(B) = m(\tau(B))$, $B \in \mathfrak{B}$. Since τ is non-singular the Radon-Nikodym theorem implies there exists a function τ' which is positive and unique a.e. such that $m\tau(B) = \int_B \tau'(x)\,dm$.

Definition 1. Given sets A, $B \in \mathfrak{B}$ of positive measure, we say τ maps A linearly onto B if $\tau(A) = B$, τ is one-to-one, and $\tau'(x) = m(B)/m(A)$, $x \in A$.
The existence of such a mapping is well known.

Definition 2. For each positive integer $n \geqq 2$ we let S_n denote the class of transformations τ for which there exists a set $B \in \mathfrak{B}$ such that $\tau^i(B)$, $0 \leqq i \leqq n-1$, are pairwise disjoint and $X = \bigcup_{i=0}^{n-1} \tau^i(B)$.

Intuitively speaking, $\tau \in S_n$ implies τ yields a stacking $\{\tau^i(B), 0 \leqq i \leqq n-1\}$ of X of height n. We note that Definition 2 implies $\tau^n(B) = B$.

Our starting point is the following lemma, a proof of which is contained in the construction in the proof of Lemma 4 [1].

Lemma 1. *Let $\tau \in \mathfrak{J}$ and let k be a positive integer. If $\tau^l(x) \neq x$, $0 < l \leqq 3^{k-1}$, $x \in X$, then there exists a set $E_k \in \mathfrak{B}$ such that* (i) $m(E_k) \leqq 2^{-k}$, (ii) $E_k = \bigcup_{i=2^k}^{3^k} E_{k,i}$ *where $x \in E_{k,i}$ implies i is the smallest positive integer such that $\tau^i(x) \in E_k$,* (iii) *the sets in the collection $\{\tau^j(E_{k,i}), 0 \leqq j < i, 2^k \leqq i \leqq 3^k\}$ are pairwise disjoint,* $X = \bigcup_{i=2^k}^{3^k} \bigcup_{j=0}^{i-1} \tau^j(E_{k,i})$, and $\tau^{-1}(E_k) = \bigcup_{i=2^k}^{3^k} \tau^{i-1}(E_{k,i})$.

We also utilize the following result due to LINDERHOLM (see [5]), a proof of which follows from Lemma 1.

Lemma 2. *If $\tau \in \mathfrak{S}$ with strict period n then $\tau \in S_n$.*

Proof. Let k be the smallest positive integer such that $2^k \leqq n \leqq 3^k$. It suffices to take $B = E_k$ in Lemma 1 in order to satisfy Definition 2.

3. Main Results

Theorem 1. *If $\tau \in \mathfrak{A}$ and $\varepsilon > 0$ then there exists a positive integer N and $\sigma \in S_N$ such that $d(\tau, \sigma) \leqq \varepsilon$ independently of how σ maps $\sigma^{N-1}(B)$ onto B (where B corresponds to σ in Definition 2).*

Proof. Since τ is non-singular there exists $\delta > 0$ such that $m(A) \leqq \delta$ implies $m(\tau^{-1}(A)) \leqq \varepsilon$. Since $\tau \in \mathfrak{A}$ it follows that Lemma 1 holds for each positive integer k. Let k be so large that $2^{-k} \leqq \delta$. Utilizing Lemma 1, let E_{k,i_j} denote the subsets $E_{k,i}$ such that $m(E_{k,i_j}) > 0$, $1 \leqq j \leqq J$, where $J \leqq (3^k - 2^k)$ is the number of such sets. We define $\sigma(x) = \tau(x)$ for $x \notin \tau^{-1}(E_k)$, hence Lemma 1 (i) implies $d(\tau, \sigma) \leqq \varepsilon$ independently of how σ is defined on $\tau^{-1}(E_k)$. Let σ map $\sigma^{i_j-1}(E_{k,i_j})$ linearly onto $E_{k,i_{j+1}}$, $1 \leqq j \leqq J-1$. We complete the definition of

σ by letting σ map $\sigma^{i_J-1}(E_{k,i_J})$ onto E_{k,i_1}. Let $B = E_{k,i_1}$ and $N = \sum_{j=1}^{J} i_j$. It is easily seen that σ has the desired properties with respect to N and B.

Corollary 1. *$\mathfrak{S}$ is dense in $\mathfrak{A}$.*

Proof. Let $\tau \in \mathfrak{A}$ and $\varepsilon > 0$. Let σ be as in theorem 1 and define $\sigma(x) = \sigma^{-N+1}(x)$, $x \in \sigma^{N-1}(B)$. Therefore σ has strict period N and $d(\tau, \sigma) \leqq \varepsilon$.

Theorem 2. *(Linderholms approximation theorem) $\mathfrak{P}$ is dense in $\mathfrak{T}$.*

Proof. The result follows from the well known decomposition of a transformation into antiperiodic and periodic components and Corollary 1.

Lemma 3. *If $\tau \in \mathfrak{S}$ then τ has roots of every order.*

Proof. Let τ have strict period n and let $B^i = \tau^i(B)$, $1 \leqq i \leqq n-1$, where B is the set corresponding to τ in Lemma 2. Let r be a positive integer. We decompose B into pairwise disjoint subsets B_j, $1 \leqq j \leqq r$, each of measure $m(B)/r$. The sets $B_{i+j} = \tau^i(B_j)$, $1 \leqq j \leqq r$, induce a decomposition of B^i, $1 \leqq i \leqq n-1$. Let η map B_j linearly onto B_{j+1}, $1 \leqq j \leqq r-1$. Assume η is defined on B_l such that $\eta(B_l) = B_{l+1}$, $1 \leqq l \leqq k$. We define $\eta(x) = \tau(\eta^{-r+1}(x))$, $x \in B_{k+1}$, hence by induction we define η on B_l, $r \leqq l \leqq rn$. Furthermore the strict periodicity now implies $\eta(x) = \tau(\eta^{-r+1}(x))$, $x \in B_j$, $1 \leqq j \leqq r-1$. Therefore it is easily seen that $\eta^r(x) = \tau(x)$, $x \in X$.

Lemma 4. *If $\tau \in \mathfrak{P}$ then τ has roots of every order.*

Proof. Apply Lemma 3 on each of the strictly periodic components of τ.

Theorem 3. *The class of transformations with roots of every order is dense in $\mathfrak{T}$.*

Proof. By Lemma 4 and Theorem 2.

Lemma 5. *If $\tau \in S_n$ and τ is ergodic then τ has no n^{th} root.*

Proof. Assume $\eta^n = \tau$. Let B be the set in Definition 2 corresponding to τ. Then for some i, $1 \leqq i \leqq n-1$, either $\eta(B) = \tau^i(B)$ or $0 < m(\eta(B) \cap \tau^i(B)) < m(\tau^i(B))$. We note that $\eta\tau = \tau\eta$. In the former case we have $\eta^n(B) = \tau^{ni}(B) = (\tau^n)^i(B) = B$ since $\tau^n(B) = B$. This contradicts the assumption that $\eta^n = \tau$. In the latter case we let $E = \eta(B) \cap \tau^i(B)$ and $F = \bigcup_{j=0}^{n-1} \tau^j(E)$. We then have $0 < m(F) < 1$ and $\tau(F) = F$ since $\tau^n(E) = \eta(\tau^n(B) \cap \tau^i(\tau^n(B))) = E$. This contradicts the assumption that τ is ergodic.

Concerning Lemma 5, see Theorem 4 [3].

Lemma 6. *There exists an ergodic measure preserving transformation τ which does not have roots of any order.*

Proof. We will construct τ so that τ does not have a p^{th} root for each prime $p > 1$. It easily follows that τ cannot have an r^{th} root for each integer $r > 1$. The transformation τ will be defined inductively in stages so that at the n^{th} stage $\tau \in S_{p(n)}$ where $p(n)$ is the n^{th} prime greater than one. The complete definition of τ will imply that τ is ergodic, hence by Lemma 5 τ will have the desired property. Let $I_0 = (0,1]$. We decompose I_0 into subintervals $I_{0,1} = (0,1/2]$ and $I_{0,2} = (1/2,1]$ and let τ map $I_{0,1}$ linearly (ordinary meaning) onto $I_{0,2}$. We will define τ inductively so that $\tau(I_{0,2}) = I_{0,1}$, hence $\tau \in S_2$. Let $N_0 = 1$ and $N_1 = 2$. At the nth stage we have a positive integer N_n such that $N_n = N_{n-1}p(n)$ and intervals

$(0, b_n] = I_n = I_{n-1,1}$, $(1 - b_n, 1] = J_n = \tau^{N_n - 1}(I_n)$, and $X = \bigcup_{i=0}^{N_n-1} \tau^i(I_n)$. We will define τ on more than one-half of J_n which is the set on which τ is not yet defined. Automatically τ^{-1} will be defined on more than one-half of I_n. In the subsequent stages we will obtain $\tau(J_n) = I_n$, hence letting $B = \bigcup_{i=0}^{N_{n-1}-1} \tau^{ip(n)}(I_n)$ we see that $\tau \in S_{p(n)}$. Let $b_{n+1} = b_n/p(n+1)$, $I_{n,j} = ((j-1)\, b_{n+1}, j b_{n+1}]$, $J_{n,j} = \tau^{N_n - 1}(I_{n,j})$, $1 \leqq j \leqq p(n+1)$. We note that the subintervals $J_{n,j}$, $1 \leqq j \leqq p(n+1)$, generate a decomposition of J_n, each with length b_{n+1}, and $J_{n,\, p(n+1)} = (1 - b_{n+1}, 1]$. Let τ map $J_{n,j}$ linearly onto $I_{n,j+1}$, $1 \leqq j \leqq p(n+1) - 1$. Let $I_{n+1} = I_{n,1}$, $J_{n+1} = J_{n,\, p(n+1)}$, and $N_{n+1} = N_n\, p(n+1)$.

It follows that τ will be defined inductively everywhere except for 0 and 1. Moreover τ is one-to-one and measure preserving since this is fulfilled at each stage. We have $X = \bigcup_{i=0}^{N_n-1} \tau^i(I_n)$ for each positive integer n and it is easily seen that the collection of subintervals $\{\tau^i(I_n),\ 0 \leqq i \leqq N_n - 1,\ n = 1, 2, \ldots\}$ are dense in $(\mathfrak{B}, m)$. It therefore follows that τ is ergodic.

We remark that it is not difficult to construct a transformation with no roots that is not ergodic but which would consist of a countable number of ergodic components.

Lemma 7. *If $m(B) > 0$ then there exists an ergodic measure preserving transformation ζ on B which does not have roots of any order.*

Proof. This follows from Lemma 6 via an isomorphism.

Theorem 4. *The class of ergodic transformations which do not have roots of any order is dense in $\mathfrak{A}$.*

Proof. Let $\tau \in \mathfrak{A}$ and $\varepsilon > 0$. Let σ be as in Theorem 1 and let ζ be a transformation on B as in Lemma 7. We define $\sigma(x) = \zeta(\sigma^{-N+1}(x))$, $x \in \sigma^{N-1}(B)$. It follows that σ is ergodic, σ cannot have a root of any order, and $d(\tau, \sigma) \leqq \varepsilon$.

Corollary 2. *The ergodic transformations are dense in $\mathfrak{A}$.*

We will now show that $\mathfrak{A}$ is nowhere dense in $\mathfrak{T}$. It will follow as a corollary that the class $\mathfrak{R}$ mentioned in the introduction is also nowhere dense. We will need the following result which is known for the measure preserving case ([*2*], p. 75). The proof consists in first appropriately reducing to the measure preserving case and then utilizing the technique in [*2*].

Lemma 8. *The class of ergodic transformations is nowhere dense in $\mathfrak{T}$.*

Proof. Let S be a sphere in $\mathfrak{T}$. It will suffice to show that there exists a sphere $S^* \subset S$ such that S^* contains no ergodic transformations. By Theorem 2 there exists $\sigma \in \mathfrak{P} \cap S$. Let n be the period of σ. We select $\varepsilon > 0$ such that the sphere $\{\tau \mid d(\tau, \sigma) < \varepsilon\} \subset S$. If $F_r = \{x \mid \sigma'(x) \geqq r\}$ then it is easily seen that $m(F_r) \leqq 1/r$. Let $\mu = \sum_{i=0}^{n-1} m\sigma^i$. Since σ is non-singular it follows that there exists $\delta > 0$ such that $m(B) \leqq \delta$ implies $\mu(B) \leqq \varepsilon/2$. We select r such that $1/r < \delta$. Let $\zeta(x) = x$, $x \in \bigcup_{i=0}^{n-1} \sigma^i(F_r)$, and $\zeta(x) = \sigma(x)$ otherwise. It follows that $d(\sigma, \zeta) \leqq \varepsilon/2$, ζ has period

n, and $\zeta' \leqq r$. The measure $\gamma = 1/n \sum_{i=0}^{n-1} m\zeta^i$ is a finite invariant measure for ζ and equivalent to m. It is obvious that $\gamma(B) \geqq m(B)/n$. Furthermore it is easily seen that $m(\zeta^i(B)) \leqq r^i m(B)$, hence $\sum_{i=0}^{n-1} r^i m(B) \geqq n\gamma(B)$. Let $\alpha = \min\{\varepsilon/2, [n\sum_{i=0}^{n-1} r^i]^{-1}\}$ and let $S^* = \{\tau \mid d(\tau, \zeta) < \alpha\}$, hence $S^* \subset S$. Let $\tau \in S^*$ and let $E = \{\tau \neq \zeta\}$, hence $m(E) < \alpha$. If $E^* = \bigcup_{i=0}^{n-1} \zeta^i(E)$ then $\zeta(E^*) = E^*$ and $m(E^*) \leqq n\gamma(E^*) \leqq n^2\gamma(E) \leqq n\sum_{i=0}^{n-1} r^i m(E) < 1$. If $x \in X - E^*$ then $\tau(x) = \zeta(x)$, hence $\tau(X - E^*) = X - E^*$. Since $m(X - E^*) > 0$ τ can be ergodic only if $m(E^*) = 0$. However this implies $m(E) = 0$ and hence τ cannot be ergodic.

Theorem 5. *$\mathfrak{A}$ is nowhere dense in $\mathfrak{T}$.*

Proof. By Corollary 2 and Lemma 6.

Corollary 3. *$\mathfrak{R}$ is nowhere dense in $\mathfrak{T}$.*

Proof. It is easily seen that $\mathfrak{R} \subset \mathfrak{A}$, hence the result follows from Theorem 5.

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