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A Decomposition of Bessel Bridges

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1. Introduction

For any $d \geqq 0$, the "d-dimensional" Bessel process, BES^d, is the continuous diffusion process, valued in $[0, \infty)$, whose infinitesimal generator coincides on $C^2(0, \infty)$ with:

$$\text{(1.a)} \qquad \tfrac{1}{2} D^2 + \frac{d-1}{2x} D, \quad \text{where } D = \frac{d}{dx}.$$

Recall that, for $d \geqq 2$, 0 is an entrance (entrance, not exit) boundary point for the process; for $0 < d < 2$, 0 is chosen to be an instantaneously reflecting regular (exit and entrance) boundary; for $d = 0$, 0 is a trap. We are, in fact, interested in the *square* of BES^d, which we denote by BESQ^d; this is obviously again a (positive) diffusion process, whose infinitesimal generator, when restricted to $C^2(0, \infty)$, is:

$$\text{(1.a')} \qquad 2x D^2 + d D.$$

We define Q_x^d, the distribution of BESQ^d starting at $x \geqq 0$, on the canonical space $C = C([0, \infty); [0, \infty))$, equipped with the σ-field

$$\mathscr{F} = \sigma\{\omega \to X_s(\omega) \equiv \omega(s);\ s \geqq 0\}.$$

The reason for our interest in these squares is the important observation by Shiga and Watanabe [19] that for any $d, d', x, x' \geqq 0$:

$$\text{(1.b)} \qquad Q_x^d \oplus Q_{x'}^{d'} = Q_{x+x'}^{d+d'},$$

where, for P and Q two probabilities on $(C, \mathscr{F})$, $P \oplus Q$ denotes the distribution of $(X_t + Y_t,\ t \geqq 0)$, with (X_t) and (Y_t) two independent processes, respectively P and Q distributed.

A much deeper result of Shiga and Watanabe [19] is that, up to a trivial scale factor, there is a one-to-one correspondance between families, indexed by

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$d \geqq 0$, of positive diffusions $(Q_x^d; x \geqq 0)_{d \geqq 0}$ which satisfy (1.b), and the set of real numbers, the correspondance being that, for any $\beta \in \mathbb{R}$, the family $({}^\beta Q_x^d; x \geqq 0)_{d \geqq 0}$ admits the following operator, as (restricted) infinitesimal operator, for the "d-dimensional" process $({}^\beta Q_x^d, x \geqq 0)$:

$$(1.c) \qquad {}^\beta A_d = 2xD^2 + (2\beta x + d)D.$$

For integer d, $({}^\beta Q_x^d, x \geqq 0)$ is the diffusion obtained by taking the square of the (Euclidian) norm of the d-dimensional Ornstein-Uhlenbeck process with parameter β. However, Girsanov's theorem and some classical space-time transformations easily reduce the study of the family $({}^\beta Q^d, d \geqq 0)$ to that of (BESQ^d, $d \geqq 0$), that is to the case $\beta = 0$. These reductions are presented at the end of the paper (see Sect. 6).

We begin a systematic exploitation of (1.b) by determining the laws of the "quadratic functionals" $\int d\mu(s)\, \rho_s^2$, where $(\rho_s, s \geqq 0)$ is $\mathrm{BES}^d_{\sqrt{x}}$, and μ is a positive Radon measure on $(0, \infty)$, with compact support (for simplicity). Indeed, one deduces from (1.b) that, if:

$$(1.d) \qquad I_\mu(\omega) \stackrel{\text{def}}{=} \int_{(0,\infty)} d\mu(s)\, X_s(\omega),$$

there exist two strictly positive reals $A(\mu)$ and $B(\mu)$ such that: for any d, $x \geqq 0$,

$$(1.e) \qquad Q_x^d(e^{-I_\mu}) = A(\mu)^x B(\mu)^d.$$

Moreover, $A(\mu)$ and $B(\mu)$ may be expressed explicitly (see Theorem (2.1)) in terms of a solution of the Sturm-Liouville equation $\frac{1}{2}\phi'' = \mu \cdot \phi$, thanks to the celebrated Ray-Knight theorems on Brownian local times. This is a development of work by D. Williams [26], who used the Ray-Knight theorems to obtain the well-known Cameron-Martin formula:

$$W_0\left[\exp -\frac{\alpha^2}{2} \int_0^t X_s^2\, ds\right] = (\mathrm{ch}\, \alpha t)^{-\frac{1}{2}} \qquad (\alpha \in \mathbb{R};\ t \geqq 0),$$

where $((X_t); W_0)$ denotes BM_0, the one-dimensional Brownian motion starting at 0.

The Ray-Knight theorems are now recalled, since they play an important part in the paper: let $(l_t^b; b \in \mathbb{R}, t \geqq 0)$ denote a jointly continuous version of Brownian local times, and let $T_0 = \inf\{t\colon X_t = 0\}$, $\tau_x = \inf\{t\colon l_t^0 = x\}$. Then

(R.K.1) *Under W_a, the distribution of BM starting at $a > 0$, the law of*

$$(l_{T_0}^b, 0 \leqq b \leqq a) \quad \textit{is } Q_0^2.$$

(R.K.2) *Under W_0, the law of $(l_{\tau_x}^b, b \geqq 0)$, for given $x \geqq 0$, is Q_x^0.*

The appendix of Walsh's paper [20] explains how the appearance of the divers Q_x^d's in (R.K) is being forced by the additivity property (1.b).

Before discussing some general facts about Bessel bridges, we present the following particular formula: for any $b, x, y > 0$,

$$Q_x^d\left[\exp\left(-\frac{b^2}{2}\int_0^t X_s\,ds\right)\middle| X_t=y\right] = \left(\frac{bt}{sh\,bt}\right)\exp\left\{\frac{x+y}{2t}(1-bt\coth bt)\right\}\frac{I_\nu\left(\frac{zb}{sh\,bt}\right)}{I_\nu\left(\frac{z}{t}\right)}, \tag{2.m}$$

where $z=\sqrt{xy}$, and $\nu=d/2-1$.

This formula is obtained in Sect. 2, after calculating $A(\mu)$ and $B(\mu)$, for $\mu=\alpha\varepsilon_t+\frac{b^2}{2}1_{[0,t]}(s)\,ds$, for any $\alpha\geqq 0$.

It is the (obvious) remark that the right-hand side of (2.m) splits naturally in a product of 4 terms which led us to suspect the existence of some interesting Bessel bridge decomposition. This is indeed the case, but we need first some definitions, in order to give a precise statement: let $Q_{x\to y}^d$ be the d-dimensional squared Bessel bridge from x to y over (the time interval) $[0,1]$, that is the Q_x^d conditional distribution of $(X_s,\ 0\leqq s\leqq 1)$, given $X_1=y$, viewed as a probability on $C([0,1],[0,\infty))$, and chosen to be weakly continuous for $y\geqq 0$ if $d>0$.

For $d=x=0$, and $y>0$, we are led, since 0 is a trap for BES^0 (see details in Sect. 5, paragraph (5.3)), to define $Q_{0\to y}^0$ as $\hat{Q}_{y\to 0}^0$, that is: the image of $Q_{y\to 0}^0$ under time reversal: $t\to(1-t)$. This being said, we show in Theorem (5.8), that, for all $d,x,y\geqq 0$:

$$Q_{x\to y}^d = Q_{x\to 0}^0 \oplus Q_{0\to y}^0 \oplus Q_{0\to 0}^d \oplus \sum_{n=0}^{\infty} b_{\nu,z}(n) Q_{0\to 0}^{4n}, \tag{1.f}$$

where $\nu=d/2-1$, $z=\sqrt{xy}$, and $b_{\nu,z}(n)=(z/2)^{2n+\nu}/n!\,\Gamma(n+\nu+1)\,I_\nu(z)$ ($(b_{\nu,z}(n), n\in\mathbb{N})$ defines a probability on $\mathbb{N}$).

An immediate consequence of (1.f) is the following generalization of (2.m): if μ is a positive Radon measure on $[0,1]$,

$$Q_{x\to y}^d(e^{-I_\mu}) = A_0(\mu)^x A_0(\hat{\mu})^y B_0(\mu)^2 \frac{I_\nu(zB_0(\mu)^2)}{I_\nu(z)} \tag{1.g}$$

where $\hat{\mu}$ is the image of μ under the map $s\to 1-s$, and the pair $(A_0(\mu), B_0(\mu))$ is defined by:

$$Q_{x\to 0}^d(e^{-I_\mu}) = A_0(\mu)^x B_0(\mu)^d. \tag{1.h}$$

The existence of such a pair is a consequence of the additivity property: for any $d,d',x,x'\geqq 0$,

$$Q_{x\to 0}^d \oplus Q_{x\to 0}^{d'} = Q_{x+x'\to 0}^{d+d'}. \tag{1.b$_0$}$$

It is also proven, at the end of Sect. 5 (see (5.z)), that $A_0(\mu)$ and $B_0(\mu)$ may be expressed quite simply in terms of $A(\mu)$, $A(\hat{\mu})$, $B(\mu)$ and $B(\hat{\mu})$.

As a preliminary to the proof of the decomposition (1.f), we offer in Sect. 4 a simultaneous construction for all $d\geqq 0$, $x\geqq 0$ of processes with laws Q_x^d as sums of excursions in a Poisson point process of the type described by Itô [6]. This yields also a Lévy-Hinčin representation of Q_x^d: there exist two σ-

finite positive measures M and N on $(C, \mathscr{F})$ such that, for any positive Radon measure μ on $[0, \infty)$,

$$Q_x^d(e^{-I_\mu}) = \exp(xM + dN)(e^{-I_\mu} - 1).$$

In Sect. 3, it is explained how the measure M can be viewed as an excursion law for BESQ^0, despite the fact that 0 is a trap for BESQ^0. Using the same notation as for (R.K.2) above, M is the characteristic measure of the C-valued Poisson point process $(l^{\bullet}_{\tau_x} - l^{\bullet}_{\tau_{x-}}, x > 0)$, and there are similar descriptions of N.

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2. Laplace Transforms of Certain Bessel Quadratic Functionals

The main result of this paragraph is the following theorem.

(2.1) **Theorem.** *Let μ be a positive (Radon) measure on $(0, \infty)$ such that, for all n, $\mu(0, n) < \infty$. Then, one has:*

(2.a) $$Q_x^d[\exp - \int X_t \, d\mu(t)] = [\phi_\mu(\infty)]^{d/2} \exp\left[\frac{x}{2} \phi_\mu^+(0)\right]$$

where $\phi_\mu(\infty)$ and $\phi_\mu^+(0)$ are respectively the limit at ∞, and the right derivative at 0 of the unique solution ϕ_μ of:

(2.b) $$\tfrac{1}{2}\phi'' = \mu \cdot \phi \quad \text{on } (0, \infty); \quad \phi(0) = 1, \ 0 \leqq \phi \leqq 1.$$

Notes. 1) Since $\mu(0, a) < \infty$ $(a > 0)$, then $\phi_\mu^+(0)$ exists and belongs to $(-\infty, 0]$.

Also, ϕ_μ is convex, and decreasing, so $\phi_\mu(\infty)$ exists (and belongs to $[0, 1]$), but $\phi_\mu(\infty) = 0$ is a possibility. If so, formula (2.a) still holds with the conventions $0^0 = 1$, $\exp(-\infty) = 0$. As a consequence of Proposition (2.2) below, $\phi^\mu(\infty) > 0$ holds iff $\int^\infty t \, d\mu(t) < \infty$.

2) If the support of μ is contained in $[0, a]$, $\phi_\mu(\infty) = \phi_\mu(a)$ obviously.

3) Replacing μ by $(\alpha\mu)$ gives the Laplace transform of $(\int d\mu(t) X_t)$ under Q_x^d, and formula (2.a) shows up the infinite divisibility of this r.v., from the presence of the multiplicative parameters d and x.

We proceed to the proof of the theorem by several steps.

Step 1. $\phi_\mu(x, d) \overset{\text{def}}{=} Q_x^d[\exp - \int d\mu(t) X_t]$ satisfies, from (1.c):

$$\phi_\mu(x, d) \, \phi_\mu(y, d') = \phi_\mu(x + y, d + d')$$

for all $x, y, d, d' \geqq 0$. In particular:

$$\phi_\mu(x, d) = \phi_\mu(x, 0) \, \phi_\mu(0, d),$$

and the functions: $x \to \phi_\mu(x, 0)$, $d \to \phi_\mu(0, d)$ are multiplicative.

They are equal to 1 for $x = d = 0$, since 0 is a holding point for BESQ^0. Moreover, they are measurable, as a consequence of the measurability of $Q_x^d(A)$

$(x \geqq 0;\, d \geqq 0)$ for any $A \in \mathscr{F}_\infty$. This last measurability property may be seen from the Yamada-Watanabe paper [27]; another proof of this may be obtained from the following Radon-Nikodym formula of [16], which will be used again below: for $t>0$, $x>0$, $d \geqq 2$:

$$Q_x^d|_{\mathscr{F}_t} = \left(\frac{X_t}{x}\right)^{\nu/2} \exp\left(-\frac{\nu^2}{2}\int_0^t \frac{ds}{X_s}\right) \cdot Q_x^2|_{\mathscr{F}_t}, \tag{2.c}$$

where $\nu = \frac{d}{2} - 1$. (For our purpose, the condition: $d \geqq 2$ implies no loss of generality, because of the multiplicative property of ϕ_μ.)

Finally, there exist $A(\mu)$, $B(\mu) \geqq 0$, such that:

$$\phi_\mu(x, d) = A(\mu)^x B(\mu)^d. \tag{2.d}$$

Step 2. We fix $x=0$, and compute $B(\mu)$. From Ray-Knight's theorem (recalled in the introduction), the law of $\int_0^a d\mu(s) X_s$, under Q_0^2, is that of $\int_0^a \mu(ds)\, l_{T_0}^s$, under W^a, the distribution of linear Brownian motion starting at a, where $l_{T_0}^s$ denotes the local time at point s of this Brownian motion until the first time it hits 0.

Therefore, one has:

$$Q_0^2\left[\exp - \int_0^a \mu(ds) X_s\right] = \phi_\mu(a, a),$$

where

$$\phi_\mu(a, b) = W^a\left[\exp - \int_0^b \mu(ds)\, l_{T_0}^s\right].$$

Remark that $\phi_\mu(a, b)$ is decreasing as either a or b increases.

So, by dominated convergence, one gets:

$$Q_0^2[\exp - \int d\mu(s) X_s] = \lim_{a\to\infty} \phi_\mu(a, a) = \lim_{a\to\infty} \phi_\mu(a),$$

where $\phi_\mu(a) \overset{\text{def}}{=} \lim_{b\to\infty} \phi_\mu(a, b) = W^a[\exp - \int d\mu(s)\, l_{T_0}^s]$.

But, it is an easy application of martingale calculus (for instance) that ϕ_μ is characterized by (2.b).

Step 3. We fix $d=0$, and compute $A(\mu)$. Again, from the Ray-Knight theorem, one has, denoting by (τ_x) the right-continuous inverse of $(l_t^0, t \geqq 0)$:

$$Q_x^0[\exp(-\int \mu(ds) X_s)] = W^0[\exp(-\int \mu(ds)\, l_{\tau_x}^s)],$$

and the right-hand side is, from martingale calculus again, well-known to be equal to $\exp(\frac{x}{2}\phi_\mu^+(0))$ (see Itô-McKean [7], Sect. (6.2), or Jeulin-Yor [9]). □

Before applying formula (2.a) to particular measures μ, we show, among other things, that $\phi_\mu(\infty) > 0$ iff $\int^\infty t\, d\mu(t) < \infty$.

(2.2) **Proposition.** *Let $d>0$, and $x\geqq 0$. Then, for any positive Radon measure on* $(0,\infty)$, *one has:*

a) $$Q_0^d\Big[\int_{0+} X_t\,d\mu(t)<\infty\Big]=1 \quad \text{iff} \quad \int_{0+} t\,d\mu(t)<\infty,$$

b) $$Q_x^d\Big[\int^{\infty} X_t\,d\mu(t)<\infty\Big]=1 \quad \text{iff} \quad \int^{\infty} t\,d\mu(t)<\infty.$$

For $d=1$, and μ absolutely continuous, the results of Proposition (2.2) were already obtained by L.A. Shepp ([18], Sect. 19), with a proof which is quite different from the following one.

Remark that the case $(d=0)$ is of no interest since 0 is a holding point for BESQ^0. One of the ingredients of our proof of a) and b) above is that, from Watanabe [21], the law of X_t under Q_x^d is that of $(t^2 X_{1/t})$ under $Q_0^{d,x}$, where $Q_0^{d,x}$ is the distribution of the square of the d-dimensional Bessel process, with drift $\sqrt{x}$, starting at 0 (see either Watanabe [21], or Pitman-Yor [16] for the definitions and notations concerning these generalized Bessel processes).

However, since $Q_0^{d,x}|_{\mathscr{F}_t}$ and $Q_0^d|_{\mathscr{F}_t}$ are mutually equivalent, one gets:

$$Q_x^d\Big[\int^{\infty} X_t\,d\mu(t)<\infty\Big]=Q_0^d\Big[\int_{0+} t^2 X_{1/t}\,d\mu(t)<\infty\Big].$$

Note in particular that the sets $(\int^{\infty} X_t\,d\mu(t)<\infty)$ (and of course, $(\int_{0+} X_t\,d\mu(t)<\infty)$) have either 0 or 1 Q_x^d probability.

Finally, Watanabe's time-inversion result permits the reduction of assertions a) and b) to

b′) $$Q_0^d\Big[\int^{\infty} X_t\,d\mu(t)<\infty\Big]=1 \quad \text{iff} \quad \int^{\infty} t\,d\mu(t)<\infty.$$

Now, since the variables X_t/t, $t>0$, are identically distributed under Q_0^d, with probability law λ such that: $\lambda\{0\}=0$; $\int\lambda(dx)x<\infty$, *b′*) is an immediate consequence of the following (deterministic) version of a very useful lemma due to Jeulin [8].

(2.3) **Lemma.** *If $(R_t)_{t>0}$ is a measurable positive process such that for every t, R_t has the same law λ, such that $\lambda\{0\}=0$ and $\int\lambda(dx)x<\infty$, then for every positive Radon measure μ on $(0,\infty)$, the set $\left\{\int_1^{\infty} R_t\,d\mu(t)<\infty\right\}$ can only have probability 0 or 1. It has probability 1 iff $\int_1^{\infty} d\mu(t)<\infty$.*

Turning to examples of our general formula (2.a), we shall use – for notational convenience – $C(\mu)$ and $B(\mu)$ instead of, respectively, $(-\frac{1}{2})\,\phi_\mu^+(0)$ and $(\phi_\mu(\infty))^{1/2}$. Here are some general remarks about A, B, C:

(i) As a consequence of the Markov property, one has the following iteration formulae:

(2.e) $$C(\mu)=C[\mu_{[0,t]}+C(\mu_t)\varepsilon_t]; \qquad B(\mu)=B[\mu_{[0,t]}+C(\mu_t)\varepsilon_t]\,B(\mu_t)$$

where $\mu_{[0,t]}$ denotes the restriction of μ to $[0,t]$, and

$$\mu_t(\Gamma)=\mu(]t,\infty[\cap(\Gamma+t)) \qquad (\Gamma\in\mathscr{B}(\mathbb{R}_+))$$

(ii) The infinite divisibility of $\int X_t\,d\mu(t)$ under Q_x^0 and Q_0^d implies the existence of two positive σ-finite measures m_μ and n_μ on $[0,\infty)$ such that, for every $\alpha>0$,

$$C(\alpha\mu)=\int(1-e^{-\alpha u})m_\mu(du) \tag{2.f}$$

and, in the case where $\int t\,d\mu(t)<\infty$,

$$B(\alpha\mu)=\exp\int(e^{-\alpha u}-1)n_\mu(du), \tag{2.f'}$$

with $\int(u\wedge 1)(m_\mu+n_\mu)(du)<\infty$.

(iii) *Example* 0. For $\mu=\alpha\varepsilon_t$ ($\alpha>0$, $t>0$), one gets:

$$C(\alpha\varepsilon_t)=\frac{\alpha}{1+2\alpha t}; \qquad B(\alpha\varepsilon_t)=(1+2\alpha t)^{-1/2}, \tag{2.g}$$

and it is easily deduced that:

$$m_{\varepsilon_t}(du)=\frac{1}{4t^2}\exp\left(-\frac{u}{2t}\right)du; \qquad n_{\varepsilon_t}(du)=\frac{1}{2u}\exp\left(-\frac{u}{2t}\right)du \tag{2.h}$$

These results provide a means of obtaining the transition probabilities of BESQ^d:

$$q^d(t,x,y)=\frac{1}{2t}\left(\frac{y}{x}\right)^{\nu/2}\exp\left(-\frac{x+y}{2t}\right)I_\nu\left(\frac{\sqrt{xy}}{t}\right) \tag{2.i}$$

$\left(\nu=\frac{d}{2}-1;\right.$ remark also, from (2.g), that $\left.Q_x^0(X_t=0)=\exp\left(-\frac{x}{2t}\right)\right)$. One then recovers the transition probabilities of BES^d (cf. Molchanov [12]):

$$p^d(t,x,y)=\frac{1}{t}\left(\frac{y}{x}\right)^\nu y\exp\left(-\frac{x^2+y^2}{2t}\right)I_\nu\left(\frac{xy}{t}\right) \tag{2.i'}$$

(iv) Using (i) and (iii) in conjunction, one gets recurrence formulae for $C(\mu)$ and $B(\mu)$, with

$$\mu=\sum_{i=1}^n\lambda_i\varepsilon_{t_i} \qquad (\lambda_i>0;\ 0=t_0<t_1<t_2\ldots<t_n).$$

Indeed, one has:

$$C\left[\sum_{i=1}^n\lambda_i\varepsilon_{t_i}\right]=C[\tilde\lambda_1\varepsilon_{t_1}]; \qquad B\left[\sum_{i=1}^n\lambda_i\varepsilon_{t_i}\right]=\prod_{i=1}^n B[\tilde\lambda_i\varepsilon_{(t_i-t_{i-1})}] \tag{2.j}$$

where the sequence $(\tilde\lambda_i,\ i=1,2,\ldots,n)$ is determined by:

$$\tilde\lambda_n=\lambda_n; \qquad \tilde\lambda_i=\lambda_i+C[\lambda_{i+1}\varepsilon_{(t_{i+1}-t_i)}]$$

(cf. Shiga and Watanabe [19], p. 40, formulae (1.20) and (1.21)).

We now turn to

Example 1 (and applications). In [11], P. Lévy showed that if (X_t, Y_t) is a $\mathbb{R}^2$-valued Brownian motion, starting from 0, then, for every $(x,y)\in\mathbb{R}^2$, and $b\in\mathbb{R}^*$:

$$\begin{aligned} &E\left[\exp ib\int_0^1 (X_u dY_u - Y_u dX_u) \mid X_1 = x, Y_1 = y\right] \\ &= E\left[\exp\left(-\frac{b^2}{2}\int_0^1 \rho_u^2 du\right)\Big| \rho_1 = \rho\right] \qquad (\rho = (x^2+y^2)^{1/2}) \\ &= \frac{b}{(\operatorname{sh} b)} \exp\left(\frac{\rho^2}{2}[1 - b\coth b]\right). \end{aligned}$$

We will extend this result by calculating, for every $d, x, t \geqq 0$:

$$Q_x^d\left[\exp\left(-\frac{b^2}{2}\int_0^t X_s ds\right)\Big| X_t\right]$$

with the help of formula (2.a).

To begin with, we show that, for every $\alpha > 0$, $b \neq 0$, one has:

$$\text{(2.k)} \quad \begin{aligned} &Q_x^d\left[\exp\left(-\alpha X_t - \frac{b^2}{2}\int_0^t X_s ds\right)\right] \\ &= [\operatorname{ch} bt + 2\alpha b^{-1}\operatorname{sh} bt]^{-d/2} \exp - \frac{xb}{2}\frac{[1+2\alpha b^{-1}\coth bt]}{[\coth(bt) + 2\alpha b^{-1}]}. \end{aligned}$$

From Theorem (2.1), we seek $\phi: [0,\infty) \to [0,1]$, solution of:

$$\text{(2.l)} \qquad \tfrac{1}{2}\phi'' = \frac{b^2}{2}\phi \quad \text{on } (0,t);\ \tfrac{1}{2}\phi^-(t) = -\alpha\phi(t);\ \phi(0) = 1$$

(ϕ^- denotes the left derivative of ϕ); having found this (unique) ϕ, the right hand side of (2.k) will be equal to:

$$(\phi(t))^{d/2}\exp\left[\frac{x}{2}\phi^+(0)\right].$$

Now, $s \to \operatorname{sh}[bs]$; $\operatorname{ch}[bs]$ span the solutions of (2.d) on $(0,t)$, and the condition $\phi(0)=1$ forces $\phi(s) = \operatorname{ch}[bs] + k\operatorname{sh}[bs]$, where k is determined by the other boundary condition:

$$b\operatorname{sh}(bt) + kb\operatorname{sh}[bt] = -2\alpha[\operatorname{ch} bt + k\operatorname{sh}[bt]].$$

This gives (2.k), after some trivial algebra. Using (2.i), one deduces now from (2.k), after some lengthy but straightforward calculations, that:

$$\text{(2.m)} \quad \begin{aligned} &Q_x^d\left[\exp\left(-\frac{b^2}{2}\int_0^t X_s ds\right)\Big| X_t = y\right] \\ &= \frac{bt}{\operatorname{sh}(bt)}\exp\left\{\frac{x+y}{2t}(1 - bt\coth bt)\right\}\frac{I_\nu\left[\frac{\sqrt{xy}\,b}{\operatorname{sh} bt}\right]}{I_\nu\left[\frac{\sqrt{xy}}{t}\right]}. \end{aligned}$$

The appearance of a product of 4 terms in the above formula suggested, and, in turn, will be explained by, the decomposition of Bessel bridges into a sum of four pieces (cf. Sect. 5). Also, a less computational proof of (2.m) is given in Sect. 6, with the explicitation (6.3) of the Radon-Nikodym density between the Bessel laws and the laws of radial parts of multidimensional Ornstein-Uhlenbeck processes.

For the moment, we remark that formula (2.m) allows the computation of the joint law of the stochastic (Paul Lévy) area $\mathfrak{A}_t = \int_0^t (X_s dY_s - Y_s dX_s)$, and the total (continuous) winding number $\theta_t - \theta_0 = \int_0^t \frac{(X_s dY_s - Y_s dX_s)}{|Z_s|^2}$ for the complex Brownian bridge $(Z_u, 0 \leq u \leq t)$, starting at $z_0 \neq 0$ (we note $a = |z_0|$). Since $Z_t = \rho_t e^{i\theta_t}$ $(\rho_t \equiv |Z_t|)$, all amounts in fact to calculating

$$J \stackrel{\text{def}}{=} E_{z_0}[\exp i\{\alpha \mathfrak{A}_t + \beta(\theta_t - \theta_0)\} | \rho_t = \rho].$$

Now, $\gamma_t \stackrel{\text{def}}{=} \int_0^t \frac{X_s dY_s - Y_s dX_s}{\rho_s}$ $(t \geq 0)$ is a real valued BM which is independent of $(\rho_t, t \geq 0)$, whence:

$$\begin{aligned} J &= E_{z_0}\left[\exp i \int_0^t \{\alpha\rho_s + \beta/\rho_s\}\, d\gamma_s \,\middle|\, \rho_t = \rho\right] \\ &= E_{z_0}\left[\exp - 1/2 \int_0^t \{\alpha\rho_s + \beta/\rho_s\}^2\, ds \,\middle|\, \rho_t = \rho\right] \\ &= \exp(-\alpha\beta)\, Q_{a'}^2 \left[\exp - 1/2 \int_0^t \{\alpha^2 X_s + \beta^2/X_s\}\, ds \,\middle|\, X_t = \rho'\right] \end{aligned}$$

where $a' = a^2$, $\rho' = \rho^2$. Now, from formula (2.c), one gets:

$$J = \exp(-\alpha\beta)\, Q_{a'}^{d_\beta}\left[\exp(-\alpha^2/2 \int_0^t X_s\, ds) \,\middle|\, X_t = \rho'\right] \frac{I_{|\beta|}\left(\frac{a\rho}{t}\right)}{I_0\left(\frac{a\rho}{t}\right)},$$

where $d_\beta = 2(|\beta| + 1)$, and the last equality originates from ([28], formula (4.9)). Finally, one gets, from formula (2.m):

$$\begin{aligned} &E_{z_0}[\exp i\{\alpha\mathfrak{A}_t + \beta(\theta_t - \theta_0)\} | \rho_t = \rho] \\ &= \exp(-\alpha\beta) \frac{\alpha t}{\operatorname{sh}(\alpha t)} \exp\left\{\frac{a^2 + \rho^2}{2t}[1 - \alpha t \coth \alpha t]\right\} \frac{I_{|\beta|}\left(\frac{a\rho\alpha}{\operatorname{sh}\alpha t}\right)}{I_0\left(\frac{a\rho}{t}\right)} \end{aligned} \tag{2.n}$$

Before embarking on other examples, it may be worthwhile to set up an algorithm for the "practical" computation of the constants $\phi_\mu(\infty)$ and $\phi_\mu^+(0)$ which appear in (2.a).

The most convenient presentation is to assume that the support of μ is compact and that the supremum σ_μ of this support is 1.

(The case where $\sigma_\mu < \infty$ is then obtained by rescaling, and $\sigma_\mu = \infty$ by a limit procedure.)

Let ϕ_0 and ϕ_1 be two linearly independent solutions of: $\frac{1}{2}\phi'' = \mu \cdot \phi$ on $(0, 1)$. Then, ϕ_μ is equal, on the interval $(0, 1)$, to: $\alpha\phi_0 + \beta\phi_1$, where the constants α and β are computed from the boundary conditions:

$$\phi_\mu(0) = 1; \quad \phi_\mu^+(1) = 0.$$

It is now easily deduced that:

$$\phi_\mu(\infty) = \phi_\mu(1) = \frac{1}{D} W(\phi_0, \phi_1)(1); \quad \phi_\mu^+(0) = \frac{N}{D}, \tag{2.o}$$

where:

$$\begin{aligned} &W(\phi_0, \phi_1)(1) = \phi_0(1)\,\phi_1'(1) - \phi_1(1)\phi_0'(1) \\ &D = \phi_0(0)\phi_1'(1) - \phi_0'(1)\phi_1(0); \quad N = \phi_0'(0)\phi_1'(1) - \phi_0'(1)\phi_1'(0). \end{aligned} \tag{2.p}$$

We are now ready to take up examples 2 and 3, which are extensions, in two different directions, of example 1. Our final aim, in these examples, is to compute $Q_x^d(\exp - I_\mu | X_1 = y)$, for suitable μ's.

It is shown, in paragraph (5.7), that we need only calculate $Q_x^d(e^{-I_\mu})$ and $Q_x^d(e^{-I_{\hat{\mu}}})$, where $\hat{\mu}$ is the image of μ under: $t \to (1-t)$. We feel free to, and will, rely on this result, since example 1 has been dealt with in a bare-handed fashion.

Example 2. Here, we take $\mu(ds) = \frac{k^2}{2} s^{2p-2} 1_{[0,1]}(s)\,ds$, where $k > 0$, and $2p > 1$, so that μ is a bounded measure on $[0, 1]$.

Denote $\nu = 1/2p$, and $\tilde{x} = \frac{k}{p} x^p$. In agreement with our previous notations, we may take (Petiau [13], p. 306):

$$\phi_0 x) = \sqrt{x}\, K_\nu(\tilde{x}), \quad \text{and} \quad \phi_1(x) = \sqrt{x}\, I_\nu(\tilde{x}).$$

Using the classical recurrence relations between Bessel functions (cf. Watson [22], or Petiau [13]), one gets:

$$\phi_0'(x) = -\frac{\tilde{x}}{2\nu\sqrt{x}} K_{\nu-1}(\tilde{x}), \quad \text{and} \quad \phi_1'(x) = \frac{\tilde{x}}{2\nu\sqrt{x}} I_{\nu-1}(\tilde{x}).$$

Finally, one obtains, after some straightforward, if tedious, calculations, from the algorithm (2.o) and (2.p):

$$\begin{aligned} &C(\mu) = \frac{c_\nu}{2} k^{2\nu} \left(\frac{I_{-\lambda}}{I_\lambda}\right)\left(\frac{k}{p}\right); \quad \hat{C}(\mu) = \left(\frac{k}{2}\right)\left(\frac{I_{-\lambda}}{I_{-\nu}}\right)\left(\frac{k}{p}\right) \\ &B^2(\mu) = \left[\Gamma(\nu)(k\nu)^{1-\nu} I_\lambda\left(\frac{k}{p}\right)\right]^{-1}; \quad \hat{B}^2(\mu) = \left[\Gamma(1-\nu)(k\nu)^\nu I_{-\nu}\left(\frac{k}{p}\right)\right]^{-1} \end{aligned} \tag{2.q}$$

where $\lambda = \nu - 1$, and $c_\nu = \dfrac{\pi \cdot \nu^{2\nu}}{\nu(\Gamma(\nu))^2 \sin(\nu\pi)}$.

Here are some comments relative to the formulae (2.q): in the following, P_x will denote the Wiener measure of real-valued BM $(B_t, t \geqq 0)$ starting at x, and $T_y = \inf\{t/B_t = y\}$.

(i) The expression $B^2(\mu)$ appears in Getoor-Sharpe ([4], Proposition (5.14), (b)), where it is shown that:

$$E_1\left(\exp -\frac{k^2}{2}\int_0^{T_0} B_s^{2p-2} 1_{(0,1)}(B_s)\,ds\right) = B^2(\mu),$$

the explanation for this equality being again the Ray-Knight theorem (R.K.1).

(ii) Also from (R.K.1), we get:

$$E_0\left[\exp -\frac{k^2}{2}\int_0^{T_1} ds B_s^{2p-2} 1_{(B_s\in(0,1))}\right] = \hat{B}^2(\mu).$$

The expression $\hat{B}^2(\mu)$ also appears in [4] (Proposition (5.14), (c)).

(iii) Using the same notation as in (R.K.2), we get, after rescaling, the following interpretation of $C(\mu)$:

$$E_0\left[\exp\left\{-\frac{k^2}{2}\int_0^{\tau_x} ds B_s^{2p-2} 1_{(B_s\in(0,a))}\right\}\right] = \exp\left[\left(-\frac{x c_\nu}{2} k^{2\nu}\right)\left(\frac{I_{1-\nu}}{I_{\nu-1}}\right)(2\nu k a^p)\right].$$

As a goes to $+\infty$, the right-hand side converges to $\exp\left(-\frac{x}{2} c_\nu k^{2\nu}\right)$, showing that the process $\left(\int_0^{\tau_x} ds B_s^{2p-2} 1_{(B_s \geqq 0)};\ x \geqq 0\right)$ is the one-side stable process with exponent ν, and rate $2^{\nu-1} c_\nu$, a fact which is found in Itô-McKean ([7], p. 226). □

Using now the same notations as in Proposition (5.10), we get:

$$\lambda(\mu) \stackrel{\text{def}}{=} \frac{2C(\mu)}{B^2(\mu)} = \Gamma(1-\nu)\,\nu^\nu k^{1+\nu} I_{-\lambda}\left(\frac{k}{p}\right),$$

and

$$a(\mu) = [B^2(\mu)\hat{B}^2(\mu)]^{-1} = \frac{\pi}{\sin(\nu\pi)}(k\nu)(I_\lambda I_{-\nu})\left(\frac{k}{p}\right)$$

From formulae (5.u), we now get (somewhat complicated) expressions for $B_0^2(\mu)$, $C_0(\mu)$ and $\hat{C}_0(\mu)$, and, are therefore able to write down, from corollary (5.9) expressions for $Q_x^d(e^{-I_\mu} | X_1 = y)$.

Example 3. We generalize now example 1, in another direction, by taking

$$\mu(dx) = \frac{p^2}{2}(\beta^2 e^{2px} + \lambda^2)\,dx.$$

From Petiau ([13], p. 308), we may take:

$$\phi_0(x) = I_\lambda(\beta e^{px}); \qquad \phi_1(x) = K_\lambda(\beta e^{px})$$

Using again the notations of the algorithm (2.o)-(2.p), we get:

$$W(\phi_0, \phi_1)(x) = [(I_\lambda K'_\lambda - K_\lambda I'_\lambda)(\beta e^{px})](\beta p e^{px}) = -(1/\beta e^{px})\beta p e^{px} = -p.$$

Consequently,

$$\phi_\mu(1) = -p/D; \quad \phi_\mu^+(0) = N/D,$$

where:

$$D = [I_\lambda(\beta) K'_\lambda(\beta e^p) - K_\lambda(\beta) I'_\lambda(\beta e^p)](\beta p e^p),$$
$$N = [I'_\lambda(\beta) K'_\lambda(\beta e^p) - K'_\lambda(\beta) I'_\lambda(\beta e^p)](\beta^2 p^2 e^p).$$

Changing μ in $\hat{\mu}$ amounts to changing p in $(-p)$ and β in (βe^p) in the previous formulae. Therefore, we get:

$$\hat{D} = [I_\lambda(\beta e^p) K'_\lambda(\beta) - K_\lambda(\beta e^p) I'_\lambda(\beta)](-\beta p), \quad \text{and} \quad \hat{N} = -N.$$

Finally, we obtain:

$$C(\mu) = -N/2D; \quad \hat{C}(\mu) = N/2\hat{D}; \quad B^2(\mu) = -p/D; \quad \hat{B}^2(\mu) = p/\hat{D},$$
$$\lambda(\mu)(=\hat{\lambda}(\mu)) = N/p; \quad a(\mu) = -\frac{D\hat{D}}{p^2}.$$

In the case where $\lambda = 0$, the previous formulae simplify, since: $I'_0 = I_1$, and $K'_0 = -K_1$. We get:

$$D = (-\beta p e^p)[I_0(\beta) K_1(\beta e^p) + K_0(\beta) I_1(\beta e^p)],$$
$$N = (-\beta^2 p^2 e^p)[I_1(\beta) K_1(\beta e^p) - I_1(\beta e^p) K_1(\beta)],$$
$$\hat{D} = (\beta p)[I_0(\beta e^p) K_1(\beta) + K_0(\beta e^p) I_1(\beta)]$$

and, to conclude this section, we note that:

$$E_0\left[\exp -\frac{p^2}{2} \int_0^{T_1} \beta^2 e^{2pB_s} ds\right] = \hat{B}^2(\mu).$$

3. Excursion Laws

We consider in this section a diffusion process – to be called the 0-diffusion – on the interval $[0, \infty)$, which is regular on $(0, \infty)$, with 0 as an absorbing boundary. We assume that the 0-diffusion has infinite lifetime, and take it to be defined on the canonical path space $C = C([0, \infty), [0, \infty))$ by laws $(P_x, 0 \leqq x < \infty)$, where P_x governs the co-ordinate process $(X_t, t \geqq 0)$ as the diffusion with starting point x.

Let T_y be the hitting time of y. We assume also that $P_x(T_0 < \infty) > 0$, $x > 0$. This last assumption is that 0 is an exit point for the 0-diffusion, following the classification of boundary points on p. 130 of Itô-McKean [7] as exit or non exit, entrance or non entrance.

Our purpose here is to describe a certain σ-finite measure Λ on C, to be called the *excursion law of the 0-diffusion.* Under Λ, the trajectories come in from zero according to an entrance law, then move according to the 0-diffusion. If 0 is an entrance boundary point, that is to say if there exists a diffusion which moves as the 0-diffusion up to time T_0 but then reenters $(0, \infty)$ instead of being absorbed at 0 (see (3.g) below), then the measure Λ we describe is simply a multiple of the Itô law for the excursions away from 0 of any such reentering diffusion, and our descriptions in this case are just variants of well established results due to McKean [10], Itô [6] and Williams [23, 24]. But we take pains here to bring out the fact, which is important for our applications, that these various descriptions still make perfectly good sense (and agree) even if 0 is not an entrance point for the 0-diffusion.

Let $s(x)$ be a scale function for the 0-diffusion. Since we assume the absorbing boundary point 0 can be reached with positive probability from $x>0$ we can take it that

$$s(0)=0, \qquad s(x)>0 \qquad \text{for } x>0,$$

and then s is defined uniquely up to a constant factor by the identity

$$P_x(T_y<\infty)=s(x)/s(y), \qquad 0<x<y<\infty.$$

The excursion law Λ can now be described in a preliminary way as

$$\Lambda=\lim_{\varepsilon\to 0} s(\varepsilon)^{-1} P_\varepsilon, \tag{3.a}$$

where the limit indicates weak convergence of finite measures on C away from neighbourhoods of the trajectory which is identically zero (denoted 0 below). To establish the existence of this limit and give much more precise descriptions of Λ, we follow Doob [2], McKean [10] and Williams [23], by considering now the diffusion on $[0, \infty)$ obtained by conditioning the 0-diffusion never to hit 0.

This diffusion, which will be referred to as the $\uparrow$ diffusion, has semi-group $(P^{t\uparrow}, t\geqq 0)$ obtained from the absorbing semi-group $(P^t, t\geqq 0)$ by the formula

$$P^{t\uparrow}(x, dy)=s(x)^{-1} P^t(x, dy) s(y), \tag{3.b}$$

and generator $G^\uparrow$ given in terms of the original generator G by

$$G^\uparrow=s^{-1} G s. \tag{3.c}$$

We shall write $P_{x\uparrow}$ for the probability law of the $\uparrow$-diffusion started at x. Though it will not occur in any of the applications in this paper, we note that the semi-group of the $\uparrow$-diffusion may be strictly sub-Markovian, indicating that the $\uparrow$-diffusion reaches ∞ in finite time a.s. Thus strictly speaking, we should define $P_x^\uparrow$ (note the slight change of notation!) on an enlarged trajectory space allowing such explosions, but we won't bother to do this as we shall only consider $P_x^\uparrow$ probabilities of events which are determined by the $\uparrow$-diffusion before the time T_b for some $b>x$, and this can be managed within our space C by simply stopping the paths at time T_b. Then for $F\in\mathscr{F}_{T_b}$ (i.e. the σ-field of pre-T_b events in C), for $x>0$, one has simply

(3.d) $$P_{x\uparrow}(F)=P_x(F\,|\,T_b<T_0),$$

a prescription which is consistent for varying b by the strong Markov property of the 0-diffusion.

Because we assume that 0 is an exit point for the 0-diffusion, 0 is an entrance but not exit point for the $\uparrow$-diffusion, *regardless* of whether 0 is entrance for the 0-diffusion. Indeed, the $\uparrow$ diffusion started at its entrance boundary point 0 and run up to the last time it hits a level $y>0$ is described by Theorem 2.5 of Williams [23] as the time reversal back from T_0 of the $\downarrow$ diffusion started at y, where the $\downarrow$ diffusion is the 0-diffusion conditioned on $(T_0<\infty)$. It follows that the distribution at time t of the $\uparrow$-diffusion started at its entrance point 0 is

(3.e) $$P^{t\uparrow}(0,dy)=\lim_{\varepsilon\to 0} P^{t\uparrow}(\varepsilon,dy)$$

where the limit exists in the sense of convergence in distribution of sub-probability measures on $[0,\infty)$.

(3.1) *First description of the measure Λ on C.*

(i) $\Lambda(X_0\neq 0)=\Lambda(\{0\})=0$, where 0 is the zero function in C.

(ii) $\Lambda(X_0=0,\ T_y<\infty)=1/s(y),\ y>0$.

(iii) Under the probability $\Lambda|(X_0=0,\ T_y<\infty)$,

the processes $(X_t,\ 0\leqq t\leqq T_y)$ and $(X_{T_y+s},\ s\geqq 0)$ are independent, the first being a $\uparrow$-diffusion started at its entrance boundary point 0 and run until it hits y, and the second being a 0-diffusion started at y.

Here for a measure M and a set B with $0<\mathrm{M(B)}<\infty$, $\mathrm{M|B}$ is the conditional probability measure

$$A\to M(A\,|\,B)=M(A\cap B)/M(B).$$

That such a measure Λ exists and is σ-finite on C is obvious once it is verified that the prescriptions of $\Lambda|(X_0=0,\ T_y<\infty)$ are consistent for different values of y, and this is a consequence of the strong Markov property of the $\uparrow$ diffusion and (3.d).

That Λ so defined is unique is trivial.

Note. As an embellishment of this description, it may be observed that for each $t>0$ the Λ conditional law of $(X_s, 0\leqq s\leqq t)$ given $X_t=x$ is identical to the $P_{0\uparrow}$ conditional law of $(X_s, 0\leqq s\leqq t)$ given $X_t=x$ (cf. McKean [10]).

(3.2) *Second description of the measure Λ on C.*

(i) $\Lambda(X_0\neq 0)=\Lambda(\{0\})=0$.

(ii) Under Λ, the process $(X_t, t>0)$ is Markovian with the 0-diffusion transition probabilities and entrance law $\lambda_t(dy)=\Lambda(X_t\in dy)$ defined by

$$\begin{aligned}\lambda_t(dy)&=P^{t\uparrow}(0,dy)/s(y), && t>0,\ y>0\\ \lambda_t\{0\}&=\infty, && t>0.\end{aligned}$$

That (λ_t) defined above is an entrance law for the semi-group (P^t) of the 0-diffusion is immediate from the definition of the semi-group $(P^{t\uparrow})$, but it is not obvious, until after matching with the first description, that (λ_t) brings the trajectories in continuously from zero. Still, it is clear that at most one measure fits the second description. We now take Λ to be defined by the first description, and offer a

Proof that Λ fits the second description. The only thing about which there is really any doubt is whether we have found the right entrance law in the second description. But, from the first description, if we take f bounded and continuous with compact support contained in $(0, \infty)$, and ε below the support of f,

$$\Lambda(f \circ X_t) = \int_0^t P_{0\uparrow}(T_\varepsilon \in du)\, P^{t-u}(\varepsilon, f)/s(\varepsilon)$$

$$= \int_0^t P_{0\uparrow}(T_\varepsilon \in du)\, P^{t-u\uparrow}(\varepsilon, f/s).$$

Letting $\varepsilon \to 0$, this last integral tends to $P^{t\uparrow}(0, f/s) = \lambda_t(f)$ because the fact that 0 is an entrance boundary point for the $\uparrow$ diffusion makes $(\varepsilon, t) \to P^{t\uparrow}(\varepsilon, g)$ jointly continuous in $(\varepsilon, t) \in [0, \infty)^2$ for bounded continuous g, and $T_\varepsilon \to 0$ as $\varepsilon \to 0$, $P_{0\uparrow}$ a.s.

These two views of Λ would suffice for the applications in later sections, but it would be a pity to pass on without recording what is surely the most beautiful description of Λ, due to Williams [24] (Sect. II.67). He describes Λ for the case of a BM absorbed at 0 (giving the Itô law for excursions away from 0 of a reflecting BM) but his description extends immediately to our setting by virtue of the first description above and the path decomposition at the maximum of Williams [23]. The reader may consult Rogers [17] for details in the Brownian case which are readily transferred. Here is

(3.3) *Williams's description of Λ.* (i) The Λ distribution of $M = \sup_t X_t$ is concentrated on $(0, \infty]$ with

$$\Lambda(M \geqq y) = 1/s(y), \qquad 0 < y < \infty.$$

(ii) Under Λ, conditional on $M = y$, for $0 < y < \infty$, the maximum is attained at an a.s. unique time R, $0 < R < T_0$ a.s., and the processes

$$(X_t, 0 \leqq t \leqq R) \quad \text{and} \quad (X_{T_0 - t}, 0 \leqq t \leqq T_0 - R)$$

are independent $\uparrow$-diffusions started at 0 and stopped when they first hit y.

(iii) Under Λ, conditional on $(M = \infty)$ (a possibility which can be ignored iff $s(\infty) = \infty$), the process X is the $\uparrow$ diffusion started at 0 and run forever.

(3.4) *Remarks and Interpretations.* If the 0-diffusion admits an extension after time T_0 which reenters $(0, \infty)$, it is immediate from the first description of Λ that Λ is some constant multiple of the Itô excursion law for the extension (see e.g. Rogers [17], paragraph 3), the constant being determined by the normalisation of local time at 0 for the extended diffusion and the choice of scale function s. In this case, the Λ distribution of T_0 is a multiple of the Lévy measure of the subordinator which is the inverse local time process, whence

(3.f) $$\Lambda(1 - e^{-\alpha T_0}) < \infty, \qquad \alpha > 0,$$

where this condition for some α is obviously equivalent to the condition for all α. (See Itô [6] for details). Conversely, if (3.f) holds, then from a Poisson point process of excursions with characteristic measure Λ one can make an extension of the 0-diffusion after time T_0 by sticking these excursions together. Thus, (3.f) is seen to be the necessary and sufficient condition for the existence of an extension after time T_0. On the other hand, by starting from the first description of Λ and arguing as in Sect. 4.10 and 6.2 of Itô-McKean [7], one finds that

$$\Lambda(1-e^{-\alpha T_0})=\lim_{\varepsilon\to 0} s(\varepsilon)^{-1}[1-P_\varepsilon(e^{-\alpha T_0})], \tag{3.g}$$

which shows that (3.f) holds and the 0-diffusion admits an extension after T_0 iff 0 is an entrance point for the 0-diffusion according to the criteria of Table 1 on p. 130 of Itô-McKean [7].

If 0 is not an entrance point for the 0-diffusion, one can still make a Poisson point process of excursions with characteristic measure Λ, but the excursions cannot be stuck one after another to form a reentering diffusion because between every two such excursions in the point process, the total length of the excursions which should go in between is a.s. infinite.

(3.5) *Examples.* Consider a Bessel process on $[0,\infty)$ with index ν (dimension $d=2\nu+2$), so the generator on $(0,\infty)$ agrees with the differential operator

$$\tfrac{1}{2}D^2+\frac{2\nu+1}{2x}D,$$

where $D=d/dx$. We assume $\nu=-\mu<0$ so that 0 is an exit point, and take the 0-diffusion to be the Bessel process with this negative index and absorption at 0. It is well known that $s(x)=x^{2\mu}$ serves as a scale function, and that 0 is an entrance point iff $\nu>-1$. For all $\nu<0$, one finds from (3.c) that the $\uparrow$ diffusion is simply a Bessel process with positive index $\mu=-\nu$. Taking $\nu=-\frac{1}{2}$, the 0-diffusion is ordinary Brownian motion on $[0,\infty)$ with absorption at 0, and the $\uparrow$-diffusion is the Bessel diffusion with index $\mu=+\frac{1}{2}$ (dimension 3). This is the case considered by Williams ([23], Sect. 3), and one has the interpretation of Λ as the Itô excursion law of reflecting BM. Taking $\nu=-1$ (dimension 0) for the 0-diffusion, the $\uparrow$-diffusion becomes a 4-dimensional Bessel process. There no longer exists any reflecting extension of the 0-diffusion. But, starting from a p.p.p. of trajectories in C governed by Λ, if, instead of vainly attempting to stick these trajectories one after the other, one squares the trajectories and adds them, one obtains something interesting, as will be seen in the next section.

4. The Levy-Ito Representation of $\mathbf{BESQ}_x^d$

Following Shiga and Watanabe [19], we consider a family of diffusions on $[0,\infty)$ $\{(Q_x^d, x\geqq 0), d\geqq 0\}$ indexed by a parameter d, and satisfying the additivity property

$$Q_x^d\oplus Q_y^f=Q_{x+y}^{d+f}, \qquad x,y,d,f\geqq 0. \tag{4.a}$$

Here Q_x^d is the law on $C=C([0,\infty),[0,\infty))$ of the d-diffusion starting at x, and for two distributions P and Q on C, $P\oplus Q$ is the law of the sum of two independent random processes, respectively P and Q distributed. According to Theorem 1.2 of [19], a family of diffusions admits this additivity property iff the d-diffusion governed by $(Q_x^d, x\geqq 0)$ has generator of the form

$$2\alpha x D^2+(2\beta x+\gamma d)D, \qquad \left(D=\frac{d}{dx}\right) \tag{4.b}$$

acting on twice differentiable functions on $(0,\infty)$ with compact support, where α, β, γ are real valued constants, $\alpha, \gamma \geqq 0$.

By a trivial rescaling of x and d, the study of such a family is reduced to the case $\alpha=\gamma=1$, which will be assumed henceforth. [We ignore the deterministic case $\alpha=0$, and note that the case $\gamma=0$ appears within the case $\gamma=1$ at $d=0$.] The law of the diffusion obtained in this case with generator

$$2xD^2+(2\beta x+d)D \tag{4.b'}$$

starting at the point x will be denoted ${}^{\beta}Q_x^d$. We note that ${}^{\beta}Q_x^d$ is the law of the square of a diffusion started at $\sqrt{x}$ with generator

$$\tfrac{1}{2}D^2+\left(\beta x+\frac{2d-1}{2x}\right)D. \tag{4.c}$$

For $\beta=0$ this is the usual Bessel process on $[0,\infty)$ with dimension parameter $d\geqq 0$. If d is an integer, for arbitrary β this is the radial part of a d-dimensional Ornstein-Uhlenbeck process with generator

$$\tfrac{1}{2}\sum_{i=1}^{d}\frac{\partial^2}{\partial x_i^2}+\beta\sum_{i=1}^{d} x_i\frac{\partial}{\partial x_i}, \tag{4.d}$$

which is a process in which the d co-ordinates perform independent one-dimensional Ornstein-Uhlenbeck motions governed by the generator

$$\tfrac{1}{2}D^2+\beta x D. \tag{4.e}$$

For $\beta=0$ the process generated by (4.e) is just Brownian motion, while $\beta<0$ corresponds to the usual O$-$U process with a restoring drift toward 0 of β times the distance from 0. In the case $\beta>0$ this one dimensional motion is transient with trajectories going off either to $+\infty$ or $-\infty$ at an exponential rate.

We focus now on the family of diffusions $({}^{\beta}Q_x^d, x\leqq 0, d\geqq 0)$ with generator (4.b), where β is a fixed real number. We note that, regardless of the value of β, the formula (4.b) for the generator implies that the boundary point 0 is

exit not entrance	for $d=0$
entrance and exit	for $0<d<2$
entrance not exit	for $d\geqq 2$.

Thus for $d=0$ or $d\geqq 2$, the laws ${}^{\beta}Q_x^d$ are completely specified by the generator (4.b) above, but for $0<d<2$ it must be stipulated that 0 is an instantaneously reflecting boundary.

Our object here is to firstly provide a Lévy-Hinčin formula in C for the infinitely divisible laws ${}^\beta Q_x^d$, and secondly to show that for each fixed β one can construct a C valued process with nice sample paths which has these laws as its marginals and independent increments in x and d. It would certainly be possible to do this by a double application of a general theorem asserting the existence of a Lévy-Hinčin formula and a nice process corresponding to any suitably regular infinitely divisible law on C. We shall encounter such laws in our study of bridges in the next section, but will not take up here the question of just how regular is "suitably regular". Doubtless this is known, but we do not know the reference. What interests us most is the specific form of the Lévy measures ${}^\beta M$ and ${}^\beta N$ which appear.

(4.1) **Theorem.** *Let β be a fixed real number.*

(i) *There exist unique measures ${}^\beta M$ and ${}^\beta N$ on C, each with zero mass on the trajectory 0, such that for every random variable I on C of the form $\int X_t \mu(dt)$ for a positive Radon measure μ on $(0, \infty)$, and every $\alpha > 0$:*

$$(4.\mathrm{f}) \qquad {}^\beta Q_x^d e^{-\alpha I} = \exp\{(x\,{}^\beta M + d\,{}^\beta N)(e^{-\alpha I} - 1)\}.$$

(ii) *${}^\beta M$ is the excursion law for the zero dimensional diffusion $({}^\beta Q_x^0, x \geqq 0)$ normalised so that its entrance law is given by the formula*

$$(4.\mathrm{g}) \qquad {}^\beta M(X_t \in dx) = dx\,[C_\beta(t)]^2 \exp[-e^{-\beta t} C_\beta(t) x],$$

where $C_\beta(t) = (\beta/2)\operatorname{cosech} \beta t, \quad \beta > 0$
$\qquad = 1/2t, \qquad \beta = 0.$

(iii) $${}^\beta N = \int_0^\infty {}^\beta M_u\, du,$$

where ${}^\beta M_u$ is the ${}^\beta M$ distribution of $(X_{(t-u)+}, t \geqq 0)$.

(iv) *There exists a C-valued process $(Y_x^d, x \geqq 0, d \geqq 0)$ such that Y_x^d has law ${}^\beta Q_x^d$,*

$$(4.\mathrm{h}) \qquad Y_x^d = Y_x + Y^d, \qquad x \geqq 0,\ d \geqq 0,$$

where $Y_x = Y_x^0$, $Y^d = Y_0^d$, and $(Y_x, x \geqq 0)$ and $(Y^d, d \geqq 0)$ are independent processes with stationary independent increments, each of them having trajectories which are increasing and right continuous with left limits in C.
Moreover, a process (Y_x^d) has these properties iff

$$(4.\mathrm{i}) \qquad \begin{aligned} Y_x(0) = x, \ Y_x &= \sum_{v \leqq x} \Delta_v \qquad && on\ (0, \infty), \\ Y^d &= \sum_{b \leqq d} \Delta^b && on\ [0, \infty), \end{aligned}$$

for all $x \geqq 0$ and $d \geqq 0$ a.s., where $(\Delta_v, v > 0)$ and $(\Delta^b, b > 0)$ are two independent C-valued Poisson point processes with characteristic measures ${}^\beta M$ and ${}^\beta N$ respectively.

We shall prove this theorem first in the case $\beta=0$. This will be done in this section, in which the superscript β will also vanish in the notation. The case of general β will then be treated in Sect. 6 by a simple reduction to the case $\beta=0$ using a transformation of space and time.

Proof of the Theorem. Taking $s(x)=x$ as the scale function of BESQ^0, it is a trivial matter to check from (3.b), (3.e), (3.2), and the formula (2.i) for the transition probabilities of BESQ^0 that one obtains the entrance law (4.g) for the excursion law M of BESQ^0. It is now immediate from (2.h) that M and N defined by (iii) satisfy (4.f) for $\mu=\alpha\varepsilon_t$, $\alpha>0$, $t>0$. The extension of the formula (4.f) to a measure $\mu=\sum_{i=1}^{n}\alpha_i\varepsilon_{t_i}$ with finite support now follows easily from the recurrence formula (2.j) and the fact that M is Markov with the BESQ^0 semigroup. The uniqueness of M and N is already evident, and the extension of (4.f) to general μ is an immediate consequence of part (iv), which we now proceed to establish. We start by showing that the process (Y_x^d) defined by (4.h) by adding the points of independent p.p.p.s with characteristic measures M and N has the desired properties.

It is clear from our finite dimensional form of (4.f) that Y_x^d has the same finite dimensional distributions as Q_x^d, but it must be argued that the trajectories Y_x and Y^d defined by (4.i) are a.s. continuous for each $x>0$ and $d>0$. Once this has been done, it is easy to see that these processes in x and d are in fact increasing and right continuous with left limits in C. To see the a.s. continuity of Y_x, note first that for each $t>0$, this function on $[t,\infty)$ is the sum of the restrictions to $[t,\infty)$ of the continuous functions Δ_v for $0<v\leqq x$. But one only has to add those Δ_v with $\Delta_v(s)>0$ for some $s>t$. From the definition of M in (ii), one finds that $M(X_s>0$ for some $s\geqq t)=M(X_t>0)$ $=1/2t<\infty$, so the number of terms to be added is a Poisson random variable with parameter $x/2t$, hence a.s. finite. Thus Y_x is a.s. continuous on $(0,\infty)$, and since it has the finite dimensional distributions of Q_x, and $Y_x(0)=x$ by definition, the a.s. continuity of Y_x for each x is established.

The a.s. continuity of Y^d for each d is more delicate because the formula

$$N(X_t\in dx)=\frac{dx}{2tx}e^{-x/2t},\qquad x>0$$

shows that $N(X_t>0)=\infty$, so for each $t>0$ and $d>0$ there are a.s. infinitely many strictly positive terms in the sum for $Y^d(t)$. Still, Y^d is a.s. uniformly continuous on bounded intervals of rationals because it has the finite dimensional distributions of Q_0^d, and letting $\bar{Y}^d$ be the continuous extension to $[0,\infty)$ of Y^d on the rationals, we have

$$Y^d\leqq\bar{Y}^d\qquad\text{a.s.,}$$

because Y^d is a limit from below of continuous functions. Also, for each fixed rational $s\geqq 0$, we have the decomposition:

$$Y^d(s+\cdot)=U_s^d(\cdot)+V_s^d(\cdot)$$

where

$$U_s^d(\cdot)=\sum_{0\leqq b\leqq d}\Delta^b(s+\cdot)1(\Delta^b(s)>0),$$
$$V_s^d(\cdot)=\sum_{0\leqq b\leqq d}\Delta^b(s+\cdot)1(\Delta^b(s)=0),$$

and the processes U^d and V^d are independent.

A glance at the definition of N reveals that in terms of finite dimensional distributions this is a decomposition of the BESQ^d process $Y_0^d(s+\cdot)$, starting at $Y_0^d(s)$, into the sum of a BESQ^0 process starting at $Y_0^d(s)$ and an independent BESQ^d process starting at 0, in keeping with the additivity property (4.a). But the argument used earlier to prove that Y_x has continuous paths now shows that the BESQ^0 process U_s^d has continuous paths, whence

$$0\leqq \bar{Y}^d(s+\cdot)-Y^d(s+\cdot)\leqq \bar{V}_s^d(\cdot),$$

where $\bar{V}_s^d$ is the continuous extension to $[0,\infty)$ of V_s^d, so $\bar{V}_s^d$ is a BESQ_0^d with trajectories in C. Now taking $s=k/n$ for $k=1,\dots,n-1$, one finds that for $\delta>0$,

$$P[\sup_{0\leqq t\leqq 1}(\bar{Y}^d(t)-Y(t))\geqq\delta]\leqq P\left(\bigcup_{k=0}^{n-1}\left\{\sup_{0\leqq u\leqq 1/n}\bar{V}_{k/n}^d(u)\geqq\delta\right\}\right). \tag{4.j}$$

But another look at the definition of N and the processes $\bar{V}_s^d$ shows that the n events in the union on the right are independent with equal probability

$$Q_0^d(\sup_{0\leqq u\leqq 1/n} X_u\geqq\delta)$$

which is less than

$$\frac{4(\mathrm{d}+1)}{\sqrt{2\pi}}\frac{1}{n\delta}\exp\left(-\tfrac{1}{2}\delta^2n^2\right),$$

by an obvious reduction to $d=1$, followed by a standard estimate for the maximum of one dimensional Brownian motion using the reflection principle. Thus the probability on the left of (4.j) is less than

$$1-\left(1-\frac{c}{n}e^{-\frac{1}{2}\delta^2n^2}\right)^n$$

where $c=c(\delta,d)$ is a constant, and this tends to 0 as $\mathrm{n}\to\infty$. The conclusion is that $\bar{Y}^d=Y^d$ a.s. on $[0,1]$, hence too on $[0,t]$ for any $t>0$, proving that $Y^d\in C$ a.s. It is now a trivial matter to verify that the process Y_x^d in (4.h) has the stated properties. To complete the proof of (iv), it only remains to be seen that every process Y_x^d with these properties comes from adding the points of two independent C-valued Poisson point processes with characteristic measures M and N. But the points are recovered as the jumps of Y_x^d, these form Poisson point processes by a theorem of Itô [6], and the finite dimensional distributions force the characteristic measures to be M and N. □

Interpretations with Brownian Motion

By virtue of the Ray-Knight theorem, much of the structure of the above results can be seen embedded in the local times of Brownian paths. Indeed, the reader who is familiar with local time and excursion theory will be able to rederive virtually everything above from within this framework. Let P_0 govern a reflecting BM on $[0, \infty)$, and let $l(s, t)$ be a version of the local time at $s \geqq 0$ before time $t \geqq 0$ chosen in accordance with Trotter's theorem to be P_0 almost surely jointly continuous in (s, t), and normalised as occupation measure. Let $(\tau_x, x \geqq 0)$ be the right continuous inverse of $(l(0, t), t \geqq 0)$, and let $(Y_x, x \geqq 0)$ be the C-valued process defined by

$$Y_x(\cdot) = l(\cdot, \tau_x).$$

By the Ray-Knight theorem, Y_x has law Q_x^0, and by the strong Markov property of P_0 and continuity of $l(\cdot, \cdot)$, the process $(Y_x, x \geqq 0)$ has independent increments and trajectories which are RCLL. We thus obtain a complete Brownian local time representation of initial half of the decomposition (4.h). Since the jump $\Delta_x = Y_x - Y_{x-}$ is given by

$$\Delta_x(\cdot) = l(\cdot, \tau_x) - l(\cdot, \tau_{x-}),$$

the p.p.p. $(\Delta_x, x > 0)$ with characteristic measure M may be identified with a transformation of the Itô point process of excursions away from 0 of the reflecting Brownian motion; Δ_x is the local time process in the space variable of the excursion between times τ_{x-} and τ_x. It follows that, if Λ is the Itô excursion law of the reflecting BM (for our present normalisation of local time, which means that one should take $s(x) = 2x$ in the formulae of Sect. 3), one has

(4.2) **Theorem.** *M is the Λ distribution of a Λ a.e. continuous version of the local time process $l(\cdot, T_0)$.*

The dimension half of the decomposition (4.h) seems to be much less well represented in the Brownian local time process, and we can only see fragments of the full decomposition with d an even integer. Still, these fragments are important clues to the structure of BESQ^d processes for arbitrary d, as will be seen in the next section.

The Ray-Knight theorem (R.K.1) gives us a BESQ_0^2 run up to time 1, and, starting at $s > 0$ instead of 1 gives a BESQ_0^2 run up to time s. But, to get, in the Brownian set-up, a BESQ_0^2 run forever, you have to reverse time from T_0, which turns the BM_1 into a BES_0^3. Thus we have the following variant, due to Williams ([24]; Theorem 65, p. 38) of the Ray-Knight theorem:

(4.3) BESQ_0^2 *is the* BES_0^3 *distribution of a continuous version of the local time process* $l(\cdot, \infty)$.

Let $L_r = \sup\{t: X_t = r\}$, $F_t = \inf_{s \geqq t} X_s$, so $(L_r, r \geqq 0)$ is the right continuous inverse of the continuous increasing process $(F_t, t \geqq 0)$. As shown in [14] (see also [17]), the BES_0^3 law of $(X - F, F)$ is the same as the BES_0^1 law of $(X, \frac{1}{2} l)$, where $l(\cdot) = l(\cdot, 0)$ is the local time of X at 0 considered above. Put together with (4.2) and (4.3), this gives an explanation of (4.1) (iii), and one finds that for BES_0^3, one obtains a p.p.p. with expectation measure $2N$ from the random measure which puts mass 1 at each of the countable number of non-zero trajectories among $l(\cdot, L_r) - l(\cdot, L_{r-})$, $r>0$. (Since the trajectory (point) indexed by r leaves 0 at time r if it is not identically zero, the points in the p.p.p. have been listed in order of their departure times from zero.)

5. Bessel Bridges

Motivated largely by the desire to understand the remarkable product form of the conditional Laplace transform (2.m), we explore in this section the additive structure of squared Bessel bridges. To see where we are headed, the reader should look ahead at Theorem (5.8) after a glance at the definitions below.

For $d, x, y, t \geqq 0$, let $Q_{x\to y}^{d,t}$ be the *d-dimensional squared Bessel bridge from x to y over time t*, that is the Q_x^d *conditional distribution of* $(X_s, 0 \leqq s \leqq t)$ *given* $X_t = y$, viewed as a probability on $C([0,t],[0,\infty))$, and chosen to be weakly continuous in y wherever possible (that is to say for $y \geqq 0$ if $d>0$, but only for $y>0$ if $d=0$: see the special discussion and definitions for the case $d=0$ in (5.3) below). Thus $Q_{x\to y}^{d,t}$ governs an inhomogeneous diffusion process with transition probabilities which could be written down using (2.i). Since Q_x^d is the $Q_{x/t}^d$ law of $(tX(u/t), u \geqq 0)$, $Q_{x\to y}^{d,t}$ is the $Q_{x/t, y/t}^{d,1}$ law of $(tX(u/t), u \geqq 0)$, which reduces the study of the bridges over an arbitrary time interval t to the case $t=1$. We write simply $Q_{x\to y}^d$ instead of $Q_{x\to y}^{d,1}$, and call this the *standard d-dimensional squared Bessel bridge from x to y*.

(5.1) *Representations in Terms of Unconditioned Processes*. For $x, d \geqq 0$, $Q_{x\to 0}^d$ is the Q_x^d distribution of the process

$$(1-u)^2 X\left(\frac{u}{1-u}\right), \quad 0 \leqq u < 1. \tag{5.a}$$

More generally, for $d, x, y \geqq 0$ (*except* if $d=x=0$, $y>0$ – see (5.c) below), $Q_{x\to y}^d$ is the law of the process in (5.a) when X is the square of a d-dimensional Bessel process started at $\sqrt{x}$ with drift $\sqrt{y}$, as defined in [16] following Watanabe [21] – see Theorem (5.8) of [16] for a proof.

(5.2) *Reversals*. Let $\hat{P}$ be the P distribution of $(X(1-t), 0 \leqq t \leqq 1)$. It is easy to see that for $d>0$

$$Q_{x\to y}^d = \hat{Q}_{y\to x}^d. \tag{5.b}$$

We now *define* $Q_{0\to y}^0$ as $\hat{Q}_{y\to 0}^0$ for $y>0$, and then (5.b) will hold for all $d, x, y \geqq 0$.

(5.3) *The special case* $d=0$. The case $d=0$ differs from the case $d>0$ in that the bridge $Q_{x\to 0}^0$, which plays a fundamental role in the sequel, is not the weak limit as $y \downarrow 0$ of the bridges $Q_{x\to y}^0$, but rather the elementary conditional probability law of Q_x^0 given the event $(X_1 = 0)$, which, from (2.g), has Q_x^0 probability $e^{-x/2} > 0$. Thus, under $Q_{x\to 0}^0$, the trajectory hits 0 strictly before time 1 at a time T_0 with distribution

$$Q^0_{x\to 0}(T_0 \in dt) = dt \frac{x}{2t} \exp\frac{x}{2}\left(1-\frac{1}{t}\right), \qquad 0<t<1,$$

and after T_0, the trajectory stays at 0.

Because BESQ^0 may be described as BESQ^4 conditioned to hit 0 in the sense of Sect. 3, the $Q^0_{x\to 0}$ conditional law of the process $(X_s,\ 0\leqq s\leqq t)$ given $T_0=t$ is $Q^{4,t}_{x\to 0}$, the 4-dimensional bridge from x to 0 over time t. Similarly, one finds that

(5.c) $$Q^0_{x\to y} = Q^4_{x\to y}, \qquad x, y>0,$$

so for $x>0$ the limit of $Q^0_{x\to y}$ as $y\downarrow 0$ is $Q^4_{x\to 0}$ [under which the paths arrive at 0 at time 1], and not $Q^0_{x\to 0}$ [under which the paths arrive before time 1 in the manner made precise above].

We note that the law $Q^0_{0\to y}$ has no sense as a conditional probability except for $y=0$, when it is the trivial law with mass one on the zero trajectory. This entitles us to make the arbitrary definition $Q^0_{0\to y} = \hat{Q}^0_{y\to 0}$ for $y>0$, and together with the property (5.c), this ensures that the reversal property (5.b) holds without exception.

(5.4) *First Additivity Property.* From the representation (5.a), it is obvious that the bridges to zero inherit the additivity properties of the (Q^d_x) family:

(5.d) $$Q^c_{w\to 0} \oplus Q^d_{x\to 0} = Q^{c+d}_{w+x\to 0}.$$

It follows easily that an analogue of Theorem (4.1) obtains for the laws $\{Q^d_{x\to 0},\ d\geqq 0,\ x\geqq 0\}$, with the measures M and N replaced by M_0 and N_0. Thus for a measure μ on $[0, 1]$, $I=\int X\,d\mu$,

(5.d′) $$Q^d_{x\to 0} \exp(-\alpha I) = A_0(\alpha)^x\, B_0(\alpha)^d$$

where $A_0(\alpha) = \exp M_0(e^{-\alpha I}-1)$ and B_0 is obtained in the same way from N_0. By time reversal, (5.e) holds also for the $0\to x$ bridges instead of the $x\to 0$ bridges, the only change on the right side being that $A_0(\alpha)$ is replaced by $\hat{A}_0(\alpha)$, the $A_0(\alpha)$ for $\hat{\mu}$, the reversal of μ (i.e. the image of μ under the map $t\to 1-t$). More explicit methods for calculating $A_0(\alpha)$ and $B_0(\alpha)$ will be given later in (5.7).

(5.5) *The Ray-Knight-Williams representations for $d=0, 2, 4$.* While not logically necessary for our eventual proof of the main decomposition theorem of the $Q^d_{x\to y}$ bridge, (5.8) below, these representations were the key to our discovery of that theorem, and they provide striking illustrations of the decomposition in terms of Brownian local time. Let $l_s(t)$ be a bicontinuous version of the local time at s before t for a BM $(B_t, t\geqq 0)$ started at $B_0=1$, and governed by probability P_1.

We are only interested in s with $0\leqq s\leqq 1$. If R and T are two random times $R\leqq T$, we write $l.(R)$ for the process $l_s(R)$, $0\leqq s\leqq 1$, and $l.(R, T)$ for the process $l.(T)-l.(R)$. By the Ray-Knight theorem (R.K.1), for all $b\leqq 0$, we have

(5.e) *$Q^2_{x\to y}$ is the conditional law of $l.(T_b)$ given $l_0(T_b)=x$ and $l_1(T_b)=y$, where $T_b=\inf\{t/B_t=b\}$.*

Let L_1 be the last passage time of $(B_t, t \geqq 0)$ at 1 before T_0. Then

(5.f) a) $l_1(L_1)$ *is exponentially distributed with rate* 1/2.
b) $Q^0_{0\to y}$ *is the conditional law of* $l.(L_1)$ *given* $l_1(L_1)=y$.
c) $Q^2_{0\to 0}$ *is the law of* $l.(L_1, T_0)$.
d) *the processes* $l.(L_1)$ *and* $l.(L_1, T_0)$ *are independent.*

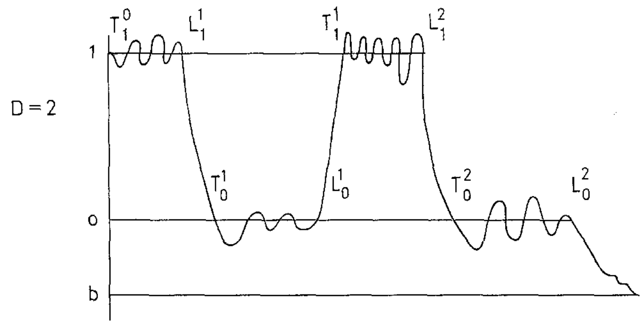
Here, a) is quite trivial, c) is a consequence of (5.e), while d) is a consequence of the last exit decomposition at L_1, and b) is forced by the reversal of the additivity property (5.d) in view of c), d) and (5.e) with $b=x=0$. But, here is a more illuminating proof of b) which should put the reader in the right frame of mind for further developments: simply put the Ray-Knight presentation of Q^0_y as in (4.l) together with the consequence of Itô's excursion theory that, conditional on $l_1(T_0)=y$, the process of excursions below 1 before $l_1(T_0)$ has the same law as the process of excursions below 1 before $\tau_y = \inf\{t\colon\ l_1(t)=y\}$ conditioned on none of these excursions reaching zero, which means conditioning on $l_0(\tau_y)=0$. (The Itô theory also yields the last exit decomposition required for d)).

Parts c) and d) of (5.f) were stated by Williams in Theorem 4 of [26], together with a much deeper decomposition of the process $l.(L_1)$ than that provided by a) and b) above. According to Williams, if R is the (a.s. unique) time that $(B_t, 0 \leqq t \leqq L_1)$ attains its minimum value V, then V is uniformly distributed on $[0,1]$, and conditional on $V=v$ the processes $l.(R)$ and $l.(R, L_1)$ are independent with identical law which is the Q^2_0 law of $X((\cdot - v)^+)$, (see Williams [23] for the idea of the proof). Thus, given $V=v$, the law of $l_0(L_1)$ is the Q^4_0 law of $X((\cdot - v)^+)$, where V is uniformly distributed on $[0,1]$, a description of the process $l.(L_1)$ which is readily seen to be equivalent to the description of (5.f) a) and b) above, by virtue of the characterisation of $Q^0_{y\to 0}$ in terms of 4-dimensional bridges given in (5.3). Now fix a number $b<0$, and consider the downcrossings and upcrossings of the interval $[0,1]$ by the BM B started at 1, up to the time T_b. Let $D \geqq 1$ be the (random) number of downcrossings of the interval, and $U=D-1$ the number of upcrossings. We declare that, for $1 \leqq k \leqq D$, the k-th downcrossing *begins* at time L^k_1 at the start of the k-th excursion away from 1 which reaches 0, and *ends* at time T^k_0, the first instant this excursion reaches 0. Similarly, for $1 \leqq k \leqq U$ we declare that the k-th upcrossing *begins* at time L^k_0 at the start of the k-th excursion away from 0 which reaches 1, and *ends* at time T^k_1 when this excursion first reaches 1. By convention, $T^0_1=0$, and L^0_0 is the last time at zero before the first hit of $b<0$. Thus

$$0 = T^0_1 < \underbrace{L^1_1 < T^1_0}_{\substack{\text{first}\\ \text{downcrossing}}} < \underbrace{L^1_0 < T^1_1}_{\substack{\text{first}\\ \text{upcrossing}}} < \underbrace{L^2_1 < T^2_0}_{\substack{\text{second}\\ \text{downcrossing}}} < \dots$$

$$\dots < \underbrace{L^D_1 < T^D_0}_{\substack{\text{last}\\ \text{downcrossing}}} \underbrace{{} < L^D_0}_{\substack{\text{last 0}\\ \text{before } T_b}}$$

where the superscripts indicate the number of the crossing, the subscripts indicate the position of the motion at the time in question, and crossings start at times L and end at times T



We now have a decomposition of the local time process into four components

$$(5.g)\qquad l.(T_b)=\left\{\sum_{k=1}^{D} l.(T_0^k, L_0^k)\right\}+\left\{\sum_{k=1}^{D} l.(T_1^{k-1}, L_1^k)\right\}$$
$$+\{l.(L_1^1, T_0^1)\}+\left\{\sum_{k=1}^{U} l.(L_0^k, T_1^k)+\sum_{k=1}^{U} l.(L_1^{k+1}, T_0^{k+1})\right\},$$

where the first component represents the contribution of excursions away from zero which fail to reach 1, the second component corresponds to excursions away from 1 which fail to reach 0, the third component comes from the first downcrossing, the fourth component from subsequent upcrossings and downcrossings. Now the reader who is familiar with excursion theory will quickly see, using (5.f), that *conditional* on $l_0(T_b)=x$ and $l_1(T_b)=y$ (when $l.(T_b)$ has law $Q^2_{x\to y}$ by (5.e)), the four component processes on the right are *mutually independent*, with respective laws:

$$Q^0_{x\to 0},\ Q^0_{0\to y},\ Q^2_{0\to 0}\quad \text{and}\quad \sum_{n=0}^{\infty} p_{x,y}(n)\, Q^{4n}_{0\to 0}$$

where the probability distribution $(p_{x,y}(n))$ on $\mathbb{N}$ appearing in the final mixture is the *conditional distribution of the number of upcrossings U given $l_0(T_b)=x$ and $l_1(T_b)=y$*. An application of (5.j) below reveals that $p_{x,y}$ is in fact the Bessel distribution with parameters $\nu=0$ and $z=\sqrt{xy}$, to be defined in the next paragraph (5.6). It was this decomposition of $Q^2_{x\to y}$ into independent components which led us to the general decomposition of $Q^d_{x\to y}$ in Theorem (5.8) below. The reader will find that the case $d=0$ of this Theorem (and part of the case $d=4$) can be interpreted in terms of Brownian local time almost exactly as above, but we do not know of any such interpretations for other d's.

(5.6) *The Bessel distribution on* $\mathbb{N}=\{0,1,2,\dots\}$

Recall that the modified Bessel function $I_\nu(z)$ may be defined by the series

$$(5.h)\qquad I_\nu(z)=(\tfrac{1}{2}z)^\nu \sum_{n=0}^{\infty}\frac{(z/2)^{2n}}{n!\,\Gamma(\nu+n+1)},$$

where the index ν is a fixed real number.

(See Watson [22], p. 77). For ν real, $z>0$, we define the *Bessel* (ν, z) *distribution on* $\mathbb{N}$, $b_{\nu,z}$, to be the probability obtained by normalising the terms of this series by their sum. That is

$$(5.\mathrm{i}) \qquad b_{\nu,z}(n)=(z/2)^{2n+\nu}/n!\,\Gamma(n+\nu+1)\,I_\nu(z).$$

We define $b_{\nu,0}$ to achieve continuity at $z=0$: $b_{\nu,0}$ is the unit mass at zero. The appearance of this family of distributions here is a consequence of the following elementary fact (cf. Feller [3], p. 58, example II.7(a))

(5.j) *For* $\alpha, \lambda, y>0$, $\nu>-1$, $z=2\sqrt{\alpha\lambda y}$, *the Bessel* (ν, z) *distribution is the conditional distribution of* U *given*

$$X_*+X_1+X_2+\ldots+X_U=y$$

when X_* *has gamma* $(\nu+1,\alpha)$ *distribution,* $X_1, X_2, \ldots$ *have gamma* $(1,\alpha)$ *(i.e. exponential* (α)*) distribution,* U *has Poisson* (λ) *distribution, and* U, $X_*, X_1, X_2, \ldots$ *are mutually independent.*

Recall that the gamma (m, α) law has density at y equal to

$$(5.\mathrm{k}) \qquad \alpha^m y^{m-1} e^{-\alpha y}/\Gamma(m), \qquad 0<y<\infty.$$

From formulae (5.h) and (5.i), the generating function of the Bessel (ν, z) distribution is:

$$(5.\mathrm{l}) \qquad \sum_{n=0}^{\infty} b_{\nu,z}(n)\,x^n=x^{-\nu/2}\,I_\nu(z\sqrt{x})/I_\nu(z).$$

We note also that, for all ν and z, this distribution is *not* infinitely divisible. Indeed, if $p(n)$ is an infinitely divisible distribution on the positive integers, it is easy to see, by decomposing jumps of the associated compound Poisson process according to whether or not they are jumps of size one, that:

$$p(n)\geqq e^{-m}\mu^n/n!, \qquad n=1,2,\ldots$$

where m, the total mass of the Lévy measure, must be finite, and μ, the mass of the Lévy measure at 1, must be strictly positive if $p(1)>0$. Now, the Bessel (ν, z) law obviously fails to have this property. In view of the next theorem, this suggests, though it does not prove, that the law $Q^d_{x\to y}$ is not infinitely divisible when both x and y are non-zero.

Recall that $P\oplus Q$ is the law of the sum of independent P and Q distributed processes, and that $Q^0_{0\to y}$ is, by convention, the reversal of $Q^0_{y\to 0}$.

(5.8) **Theorem.** *For all* $d, x, y\geqq 0$,

$$Q^d_{x\to y}=Q^0_{x\to 0}\oplus Q^0_{0\to y}\oplus Q^d_{0\to 0}\oplus \sum_{n=0}^{\infty} b_{\nu,z}(n)\,Q^{4n}_{0\to 0}$$

where $\nu=\frac{1}{2}d-1$, $z=\sqrt{xy}$, *and the last term on the right is the mixture of the laws* $Q^{4n}_{0\to 0}$, $n=0,1,\ldots$ *with weights given by the Bessel* (ν, z) *law.*

Before the proof, we state the following corollary, which results from (5.l):

(5.9) **Corollary.** *Let* $I=\int X\,d\mu$ *where* μ *is a measure on* $[0,1]$, *let* $A_0(\alpha)$ *and* $B_0(\alpha)$ *be as in* (5.e), *and let* $\hat{A}_0(\alpha)$ *be the* $A_0(\alpha)$ *for the image of* μ *under the map:* $s\to 1-s$. *Then*

$$Q^d_{x\to y}(e^{-\alpha I})=A_0(\alpha)^x\hat{A}_0(\alpha)^y B_0(\alpha)^2\, I_\nu(\sqrt{xy}\,B_0(\alpha)^2)/I_\nu(\sqrt{xy})$$

where $\nu=\frac{1}{2}d-1$.

Proof of Theorem (5.8). Note first that if $xy=0$ then both the Bessel mixture and one or the other of $Q^0_{x\to 0}$ and $Q^0_{0\to y}$ are trivial, and, after a reversal if necessary, the theorem then reduces to the first additivity property (5.4). Assume therefore that $x,y>0$. Let $Y^d_x=(Y^d_x(s),\ 0\leqq s\leqq 1)$ be a process with law Q^d_x constructed as in Theorem (4.1) as:

$$Y^d_x=Y^0_x+Y^d_0,$$

where $(Y^0_v, v\geqq 0)$ has independent increments with Lévy measure M, and Y^d_0 is independent with law Q^d_0, where all laws are restricted to $[0,1]$ in the obvious way. Then we have the further independent decomposition

$$Y^d_x=Y^0_{x0}+Y^0_{x+}+Y^d_0 \tag{5.m}$$

where

$$Y^0_{x+}=Z_1+Z_2+\ldots+Z_U \tag{5.n}$$

is the sum of the jumps of the process $(Y^0_v,\ 0\leqq v\leqq x)$ which are in the set $(X_1>0)$ of trajectories that are strictly positive at time 1, there being an a.s. finite number U of these jumps which is Poisson distributed with parameter $x/2$ by (4.g), while Y^0_{x0} is the sum of the infinite number of jumps which are zero at time 1. Since Y^0_{x0} has law $Q^0_{x\to 0}$, by the description of $Q^0_{x\to 0}$ in terms of M_0 in (5.4), the first term of the decomposition (5.8) has appeared already without yet conditioning on $Y^d_x(1)=y$. It thus only remains to be seen that

(5.o) *the law of* $Y^0_{x+}+Y^d_0$ *on* $[0,1]$ *given*

$$Y^0_{x+}(1)+Y^d_0(1)=y \quad \textit{is } Q^0_{0\to y}\oplus Q^d_{0\to 0}\oplus\sum_{n=0}^{\infty} b_{\nu,z}(n)\,Q^{4n}_{0\to 0}.$$

Now, conditional on $U=n>0$, the trajectories $Z_1,\ldots,Z_n$ are independent with common distribution M conditional on $X_1>0$, and if we condition further on $Z_i(1)=z_i$ for $1\leqq i\leqq n$, we find from the note below (3.2) and the fact that BESQ^4 is BESQ^0 conditioned to never hit zero that the processes Z_i have laws

$$Q^4_{0\to z_i}=Q^0_{0\to z_i}\oplus Q^4_{0\to 0}$$

by the reversal of the additivity property (5.d) and the definition of $Q^0_{0\to y}$ as $\hat{Q}^0_{y\to 0}$. Thus, by another application of this additivity in reverse,

(5.p) *the law of* Y^0_{x+}, *given* $U=n$ *and* $Y^0_{x+}(1)=z$, *is* $Q^0_{0\to z}\oplus Q^{4n}_{0\to 0}$.

Similarly,

(5.q) *the law of* Y^d_0, *given* $U=n$ *and* $Y^d_0(1)=w$, *is* $Q^d_{0\to w}=Q^0_{0\to w}\oplus Q^d_{0\to 0}$.

Since Y_0^d is independent of U and $Z_1, \dots, Z_U$, we find that

(5.r) *the law of* $Y_{x+}^0 + Y_0^d$ *given* $Y_{x+}^0(1) + Y_0^d(1) = y$ *and* $U = n$ *is*

$$Q_{0\to y}^0 \oplus Q_{0\to 0}^d \oplus Q_{0\to 0}^{4n}.$$

Finally, to obtain (5.o) and complete the proof, it thus suffices to see that the conditional law of U given $Y_{x+}^0(1) + Y_0^d(1) = y$ is $b_{v,z}$, and this is a consequence of (5.j), since $Y_0^d(1)$ has a gamma $\left(\frac{d}{2}, \frac{1}{2}\right)$ law, independently of $Y_{x+}^0(1)$ and U, U is Poisson with parameter $x/2$, and, given $U=n$, $Y_{x+}^0(1)$ is the sum of n independent gamma $(1, \frac{1}{2})$ variables by (5.n) and (5.j).

(5.7) *Calculating* $A_0(\alpha)$ *and* $B_0(\alpha)$. Making some slight (but obvious) change in the notations used in Corollary (5.9), we may write:

$$Q_{x\to y}^d(e^{-I_\mu}) = A_0(\mu)^x \hat{A}_0(\mu)^y B_0(\mu)^2 \frac{I_v(\sqrt{xy}\, B_0(\mu)^2)}{I_v(\sqrt{xy})}$$

for any positive and finite (for simplicity) measure μ on $[0, 1]$. Using now the explicit form of the transition densities of BESQ^d:

$$q^d(t, x, y) = \frac{1}{2t}\left(\frac{y}{x}\right)^{v/2} \exp -\left(\frac{x+y}{2t}\right) I_v\left(\frac{\sqrt{xy}}{t}\right), \tag{2.i}$$

where $v = \frac{d}{2} - 1$, we get:

$$Q_x^d(e^{-I_\mu}) = \frac{A_0(\mu)^x}{2x^{v/2}} B_0(\mu)^2 \exp\left(-\frac{x}{2}\right) \int dy\, y^{v/2} \exp\left(-\frac{y}{2}\hat{G}_0(\mu)\right) I_v(\sqrt{xy}\, B_0^2(\mu))$$

where $\hat{G}_0(\mu) = 1 + 2\hat{C}_0(\mu)$, and $\hat{C}_0(\mu) = -\log \hat{A}_0(\mu)$. We define $t = \hat{G}_0(\mu)^{-1}$, and x' by the formula: $\frac{\sqrt{x'}}{t} = \sqrt{x}\, B_0^2(\mu)$. From the fact that: $\int q^d(t, x', y)\, dy = 1$, we obtain:

$$\begin{aligned} Q_x^d(e^{-I_\mu}) &= \frac{1}{2x^{v/2}} A_0(\mu)^x B_0(\mu)^2 \exp(-x/2)(2t)(x')^{v/2} \exp(x'/2t) \\ &= \exp\left(-\frac{x}{2} G_0(\mu)\right) B_0(\mu)^d \frac{1}{(\hat{G}_0(\mu))^{d/2}} \exp\left(\frac{x}{2}\frac{B_0(\mu)^4}{\hat{G}_0(\mu)}\right) \end{aligned}$$

On the other hand, we noted, in (2.d), that there exist $A(\mu)$ and $B(\mu)$ such that: $Q_x^d(e^{-I\mu}) = A(\mu)^x B(\mu)^d$.

Identifying the coefficients of x and d in these two expressions for $Q_x^d(e^{-I_\mu})$, we obtain, after dropping the notation μ:

$$2C = G_0 - \frac{B_0^4}{\hat{G}_0} \tag{5.s}$$

and:

(5.t) $$B = B_0/(\hat{G}_0)^{1/2}.$$

We write (5.s) and (5.t) in the more convenient form:

(5.t') $\hat{G}_0 = B_0^2/B^2$; (5.s$'$) $2C\hat{G}_0 = \hat{G}_0 G_0 - B_0^4$.

Of course, there are duals to these two relations, when replacing μ by $\hat{\mu}$. In particular, one gets: $G_0 = \hat{B}_0^2/\hat{B}^2$. However, since $Q_{0\to 0}^d = \hat{Q}_{0\to 0}^d$, the identity: $B_0 = \hat{B}_0$ follows.

Now, if we denote: $\lambda = 2C/B^2$, and $a = 1/B^2\hat{B}^2$, (5.s$'$) transforms into:

$$\lambda B_0^2 = B_0^4(a-1),$$

finally yielding:

(5.u) $$B_0^2 = \lambda/(a-1), \quad \text{and} \quad 1 + 2C_0 = 2C\left(\frac{a}{a-1}\right)$$

(remark that, as a by-product of (5.u), we get: $\lambda = \hat{\lambda}$).

These results are summed up in the following

(5.10) **Proposition.** *Let μ be a positive finite measure on $[0,1]$, and $\hat{\mu}$ its image under the time reversal: $t \to (1-t)$.*

Define the constants C and B through the equality:

$$Q_x^d(e^{-I_\mu}) = \exp(-xC) \cdot B^d,$$

and their duals $\hat{C}$ and $\hat{B}$ associated with $\hat{\mu}$.

Put $\lambda = 2C/B^2$, and $a = 1/B^2\hat{B}^2$. Then, $\lambda = \hat{\lambda}$ and:

(5.u) $$B_0^2 = \lambda/(a-1); \quad 1 + 2C_0 = 2C\left(\frac{a}{a-1}\right),$$

where the constants B_0 and C_0 are defined through:

$$Q_{x\to 0}^d(e^{-I_\mu}) = e^{-xC_0} \cdot B_0^d.$$

As an example, we take $\mu(=\hat{\mu}) = \dfrac{b^2}{2}dx$ (on $\mathscr{B}[0,1]$). Then, from formula (2.k), one gets: $B = (ch\, b)^{-\frac{1}{2}}$ and $C = b/2 \coth b$, which yield, using (5.u):

$$B_0^2 = b/(\operatorname{sh} b), \quad \text{and} \quad G_0 = 1 + 2C_0 = b \coth b,$$

in agreement with formula (2.m).

(5.8) *More remarks about the Bessel bridges.* (i) For integer d, the squared Bessel bridges are, of course, the squares of the radial parts of d-dimensional Brownian bridges. In particular, for $d=1$, $Q_{0\to 0}^1$ is the law of the square of the standard one-dimensional Brownian bridge - see e.g. Billingsley ([1], p. 65). As noted by Williams [24], the bridge $Q_{0\to 0}^3$ is the law of the square of the standard Brownian excursion of Itô-McKean ([7], p. 79). In terms of the Itô excursion law Λ for reflecting Brownian motion described in Sect. 3, this is to say that Λ conditional on $(T_0=t)$ is the $Q_{0\to 0}^{3,t}$ bridge. More generally, if Λ is the excursion law of BESQ^{2-a} $(a>0)$, then Λ conditional on $T_0=t$ is the $Q_{0\to 0}^{2+a,t}$ bridge.

(ii) Fix $x, y>0$. Then, from formula (2.c), the probabilities $(Q_{x\to y}^d,\ d\geqq 2)$ are mutually absolutely continuous, and the same formula enables us to obtain an expression for the joint Laplace transform, under

$$Q_{x\to y}^d \text{ of } I_\mu \quad\text{and}\quad C_1\equiv\int_0^1 ds/X_s.$$

Indeed, one gets, denoting $\delta=\frac{d}{2}-1$:

$$Q_{x\to y}^d(\exp\{-\alpha I_\mu-(v^2/2)\,C_1\})=A_0(\alpha\mu)^x\hat{A}_0(\alpha\mu)^y B_0(\alpha\mu)^2\,\frac{I_\lambda(zB_0(\alpha\mu)^2)}{I_\delta(z)}$$

where $z=\sqrt{xy}$, and $\lambda=(v^2+\delta^2)^{1/2}$.

(iii) From (5.1), one easily obtains a decomposition – similar to that produced in Theorem (5.8) for $Q_{x\to y}^d$ – of the distribution ${}^yQ_x^d$ of the square of the d-dimensional Bessel process, started at $\sqrt{x}$, with drift $\sqrt{y}$.

6. Ornstein-Uhlenbeck Processes

Let $({}^\beta Q_x^d,\ x\geqq 0)$ govern the diffusion with generator

$$2xD^2+(2\beta x+d)D, \tag{6.a}$$

where β is real, $d\geqq 0$.

There are two methods for reducing the study of this family of diffusions to the case $\beta=0$ of BESQ^d: transformation of space-time and change of law.

(6.1) *Transformation of space-time.* For all real $\beta\neq 0$, $x\geqq 0$, $d\geqq 0$, ${}^\beta Q_x^d$ is the Q_x^d law of the process:

$$e^{2\beta t}X\left(\frac{1-e^{-2\beta t}}{2\beta}\right), \tag{6.b}$$

as well as the $Q_{2|\beta|x}^d$ law of the process:

$$\frac{e^{2\beta t}}{2|\beta|}X(|e^{-2\beta t}-1|). \tag{6.b'}$$

Proof. For $d=1$, (6.b) follows at once from the resolution of the O-U (stochastic) differential equation:

$$dU_t=dB_t+\beta U_t dt;\qquad U_0=\sqrt{x},$$

where (B_t) is a real valued BM_0. This yields:

$$U_t=e^{\beta t}\left(\sqrt{x}+\int_0^t e^{-\beta s}dB_s\right),$$

from which (6.b) is deduced, using time-substitution. (6.b′) is then deduced using a second time-rescaling.

The result for general $d \geqq 0$ now ensues from the additivity property and continuity of the laws ${}^\beta Q_x^d$, as in the proof of Theorem (3.1) of Shiga and Watanabe ([19]).

From the formula for the transition probabilities of BESQ^d:

$$q^d(t,x,y)=\frac{1}{2t}\left(\frac{y}{x}\right)^{\nu/2}\exp\left[-\frac{x+y}{2t}\right] I_\nu\left(\frac{\sqrt{xy}}{t}\right),$$

one obtains immediately from (6.1) the corresponding formula for $\beta \neq 0$:

$$\text{(6.c)}\quad {}^\beta q^d(t,x,y)=\frac{\beta}{2\operatorname{sh}\beta t}\left(\frac{y}{x}\right)^{\nu/2}\exp\beta\left[(1-\nu)t+\frac{xe^{\beta t}+ye^{-\beta t}}{2\operatorname{sh}\beta t}\right] I_\nu\left(\frac{\beta\sqrt{xy}}{\operatorname{sh}\beta t}\right).$$

We now complete, for $\beta \neq 0$, the:

(6.2) *Proof of Theorem* (4.1). Let T_β now denote the transformation of the trajectory space C defined by

$$T_\beta\omega(t)=\frac{e^{2\beta t}}{2|\beta|}\,\omega(|e^{-2\beta t}-1|),$$

so (6.1) just says that the $Q^d_{2|\beta|x}$ law of T_β is ${}^\beta Q_x^d$. Let M and N be as in (4.1) for $\beta=0$, let Y_x^d be defined as in (4.h) using (4.i), except that $(\Delta_v, v>0)$ is now a p.p.p. with characteristic measure $2|\beta|M$, but $(\Delta^b, b>0)$ admits, as before, the characteristic measure N. Then Y_x^d has law $Q^d_{2|\beta|x}$, and we find that

$${}^\beta Y_x^d=T_\beta\circ Y_x^d=\sum_{v\leqq x} T_\beta\circ\Delta_v+\sum_{a\leqq d} T_\beta\circ\Delta^a,$$

where ${}^\beta Y_x^d$ has the same regularity properties as Y_x^d. It follows that (4.1) (iv) and (i) hold with ${}^\beta M$, the $2|\beta|M$ distribution of T_β, and ${}^\beta N$ the N distribution of T_β, and finally parts (ii) and (iii) are easily checked.

(6.3) *Change of law.* Let $\mathscr{F}_t=\sigma(X_s, 0\leqq s\leqq t)$. Then

$$\frac{d^\beta Q_x^d}{dQ_x^d}=\exp\left\{\frac{\beta}{2}[X_t-x-dt]-\frac{\beta^2}{2}\int_0^t X_s\,ds\right\}\quad \text{on } \mathscr{F}_t.$$

Proof. Itô's formula tells us that, for any real β

$$L_t^\beta \overset{\text{def}}{=} \exp\left\{\frac{\beta}{2}[X_t-x-dt]-\frac{\beta^2}{2}\int_0^t X_s\,ds\right\}$$

is a local Q_x^d martingale. For $\beta\leqq 0$, (L_t^β) is uniformly bounded on compacts of $[0,\infty)$, and therefore, is a martingale. This last assertion is equally verified, for $\beta\geqq 0$, since one easily checks, as an application of formula (2.k), that $Q_x^d(L_t^\beta)=1$, for any $t\geqq 0$.

Now, (6.3) follows as a routine application of the Cameron-Martin-Girsanov theorem. □

Using the fact that the density in (6.3) is the exponential of an (explicit) affine functional of X, the reader will easily verify the following

(6.4) **Corollary.**

$$\frac{d\,{}^{\beta}M}{dM}=\frac{d\,{}^{\beta}N}{dN}=\exp\left\{\frac{\beta}{2}X_t-\tfrac{1}{2}\beta^2\int_0^t X_s\,ds\right\} \quad \text{on } \mathscr{F}_t.$$

We note also that formula (2.m) is an immediate consequence of (6.c) and (6.3), though this derivation hides the significance of each factor as brought out by the deeper structure of the Bessel bridges in Sect. 5. This structure in turn extends easily to the O-U case by the change of law formula (6.3). Indeed, from (6.3), one gets:

$$(6.d)\qquad {}^{\beta}Q^{d,t}_{x\to y}=\frac{\exp\left(-\frac{\beta^2}{2}\int_0^t X_s\,ds\right)}{Q^{d,t}_{x\to y}\left\{\exp-\frac{\beta^2}{2}\int_0^t X_s\,ds\right\}}\cdot Q^{d,t}_{x\to y},$$

where ${}^{\beta}Q^{d,t}_{x\to y}$ denotes the bridge for $(X_u,\ 0\leqq u\leqq t)$ obtained by conditioning ${}^{\beta}Q^{d}_{x}$ by $(X_t=y)$, and the denominator is given explicitly by formula (2.m). From (6.d), one deduces that ${}^{\beta}Q^{d,t}_{x\to y}$ is the law of $tX(\cdot/t)$ under ${}^{\beta t}Q^{d}_{x/t\to y/t}$, where we write ${}^{\gamma}Q^{d}_{x\to y}$ for ${}^{\gamma}Q^{d,1}_{x\to y}$.

Finally, (6.d) also implies that Theorem (5.8) and Corollary (5.9) hold for ${}^{\beta}Q^{d}_{x\to y}$ instead of $Q^{d}_{x\to y}$, the only change being that $z=\sqrt{xy}$ is modified into $z=\dfrac{\beta\sqrt{xy}}{\operatorname{sh}\beta}$.

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