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An Invariance Principle for the Local Time of a Recurrent Random Walk

Naresh C. Jain* and William E. Pruitt*
University of Minnesota, School of Mathematics, Minneapolis, Minnesota 55455, USA

Summary. Let (S_j) be a lattice random walk, i.e. $S_j = X_1 + \ldots + X_j$, where $X_1, X_2, \ldots$ are independent random variables with values in the integer lattice $\mathbb{Z}$ and common distribution F, and let $L_n(\omega, k) = \sum_{j=0}^{n-1} \chi_{\{k\}}(S_j(\omega))$, the local time of the random walk at k before time n. Suppose $EX_1 = 0$ and F is in the domain of attraction of a stable law G of index $\alpha > 1$, i.e. there exists a sequence $a(n)$ (necessarily of the form $n^{1/\alpha} l(n)$, where l is slowly varying) such that $S_n/a(n) \to G$. Define $g_n(\omega, u) = \frac{c(n)}{n} L_n(\omega, [uc(n)])$, where $c(n) = a(n/\log \log n)$ and $[x]$ = greatest integer $\leqq x$. Then we identify the limit set of $\{g_n(\omega, \cdot): n \geqq 1\}$ almost surely with a nonrandom set in terms of the I-functional of Donsker and Varadhan. The limit set is the one that Donsker and Varadhan obtain for the corresponding problem for a stable process. Several corollaries are then derived from this invariance principle which describe the asymptotic behavior of $L_n(\omega, \cdot)$ as $n \to \infty$.

1. Introduction

Let $X_1, X_2, \ldots$ be real-valued independent identically distributed random variables on a probability space $(\Omega, \mathscr{F}, P)$ with a common distribution function F. Let

$$S_n = X_1 + \ldots + X_n, \quad n \geqq 1, \ S_0 = 0.$$

We assume F to be in the domain of attraction of a stable law G of index $\alpha, 0 < \alpha \leqq 2$, which has a strictly positive density on $\mathbb{R}$ and satisfies the scaling property, i.e., if $y(t)$, $t \geqq 0$, is the stable process with stationary and independent increments and $y(1)$ has distribution G then $c^{-1/\alpha} y(ct)$, $t \geqq 0$, has the same finite

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dimensional distributions as $y(t)$, $t \geqq 0$. If $\alpha > 1$ we assume $EX_1 = 0$. Under these conditions there exists a function

$$a(t) = t^{1/\alpha} l(t), \qquad t > 0, \tag{1.1}$$

where l is a slowly varying function near ∞, such that

$$\frac{S_n}{a(n)} \to G \tag{1.2}$$

in the sense of weak convergence of the corresponding measures.

We need to introduce some more notation to describe our results. Let

$$M = \{\mu \colon \mu \text{ is a subprobability measure on } \mathbb{R}\}.$$

M is given the topology of vague convergence, i.e. $\mu_n \to \mu$ in M means $\int f\,d\mu_n \to \int f\,d\mu$ for f continuous with compact support. For $\mu \in M$ the I-functional of Donsker and Varadhan [1] corresponding to the semigroup generated by G is given by

$$I(\mu) = -\inf_{u \in \mathscr{U}} \int \left(\frac{Lu}{u}\right)(x)\,d\mu(x), \tag{1.3}$$

where $\mathscr{U}$ denotes the class of strictly positive C^∞ functions on $\mathbb{R}$ which are constant outside of a compact interval (which depends on the function), and L denotes the infinitesimal generator of the Markov semigroup generated by G. If $\mu \in M$ and μ has a density f on $\mathbb{R}$ (with respect to Lebesgue measure) then $I(f)$ will denote $I(\mu)$. This functional plays a crucial role in the probability estimates and helps in the evaluation of the limit constants. Now define

$$\mathscr{A} = \left\{f \colon f \geqq 0,\ f \text{ uniformly continuous on } \mathbb{R}, \quad \int_{-\infty}^{\infty} f(x)\,dx \leqq 1\right\}, \tag{1.4}$$

and

$$\mathscr{B} = \{f \in \mathscr{A} \colon I(f) \leqq 1\}. \tag{1.5}$$

Assume now that the random walk takes values in the integer lattice $\mathbb{Z}$ and $\alpha > 1$. In this case EX_1 exists and equals zero by our assumption, so the random walk is recurrent. Let

$$\Sigma = \{x \in \mathbb{Z} \colon P[S_n = x] > 0 \text{ for some } n \geqq 1\}.$$

We assume without loss of generality that $\Sigma = \mathbb{Z}$; this amounts to relabeling the state space [9].

The local time at $k \in \mathbb{Z}$ before time n is defined to be the number of visits to k before time n, i.e.

$$L_n(\omega, k) = \sum_{j=0}^{n-1} \chi_{\{k\}}(S_j(\omega)), \qquad \omega \in \Omega, \tag{1.6}$$

where χ_A denotes the indicator of the set A. We also need to introduce

$$c(n) = a(n/\log\log n), \qquad n > e, \tag{1.7}$$

and for $\omega \in \Omega$

$$g_n(\omega, u) = \frac{c(n)}{n} L_n(\omega, [uc(n)]), \qquad u \in \mathbb{R}. \tag{1.8}$$

Here (and later) if $z \in \mathbb{R}$ then $[z]$ denotes the greatest integer less than or equal to z.

We prove an invariance principle (Theorem 3.5) which says that the limit points of $\{g_n(\omega, \cdot): n \geqq 1\}$, in the sense of uniform convergence on compacts, almost surely equals the set $\mathscr{B}$. The main tools involved in the proof are some techniques used in [1], some of the main results of [4], and Theorem 2 of [5]. The results in [5] deal with the local time of a recurrent lattice random walk in a more general situation where F need not be attracted to a limit (stable) law; these results in turn depend on some estimates obtained in [3]. In the latter more general situation we of course get less precise results.

Some preliminary results are given in Sect. 2 which are derived under the sole assumption (1.2) with G having a strictly positive density and satisfying the scaling property. Section 3 contains the main results. Theorem 3.1 is a refinement of a result in [4]; we need it to prove the invariance principle (Theorem 3.5).

Theorems 3.6 and 3.7 are corollaries of Theorem 3.5 which we shall describe now in a special case. Let G be symmetric with characteristic function $\varphi_G(t) = e^{-|t|^\alpha}$ if $1 < \alpha < 2$, and $\varphi_G(t) = e^{-t^2/2}$ when $\alpha = 2$; then for $k \in \mathbb{Z}$

$$\limsup_n \frac{c(n)}{n} L_n(\omega, k) = \limsup_n \frac{c(n)}{n} \max_{m \in \mathbb{Z}} L_n(\omega, m) = d_\alpha$$

a.s., where $d_\alpha = \Gamma(1/\alpha)\, \Gamma(1-1/\alpha)/(\pi(\alpha-1)^{1-1/\alpha})$ if $1 < \alpha < 2$, and $d_2 = \sqrt{2}$. If $l(t) \equiv 1$ in (1.1), then $c(n)/n = n^{1/\alpha - 1}(\log\log n)^{-1/\alpha}$, so for $\alpha = 2$ we have in this case $c(n)/n = (n \log\log n)^{-1/2}$. This result was obtained by Kesten [6] for $\alpha = 2$. For some related results see also [8], where the case of a simple random walk is considered and the main tool used is the Skorohod embedding. The special case of Theorem 3.7 states that for $-\infty < a < b < \infty$, j a positive integer,

$$\limsup_n \frac{c(n)^{j-1}}{n^j} \sum_{ac(n) \leqq k \leqq bc(n)} L_n^j(\omega, k) = \sup_{f \in \mathscr{B}} \int_a^b f^j(t)\, dt,$$

a.s.; the limit constant is positive and finite.

We would like to note here that a weak invariance principle for the local time of a recurrent lattice random walk is established in [7] via the Skorohod weak invariance principle.

2. Some Preliminary Results

For the results of this section we assume only that F satisfies (1.2) and G satisfies the scaling property and has a strictly positive density on $\mathbb{R}$. We do not assume that F is lattice. These results are analogues of results in [1], §2, where local times of stable processes are considered.

Let T_x and S_θ, $x \in \mathbb{R}$, $\theta > 0$, be transformations of the real line given by

$$T_x(y) = x + y$$

and

$$S_\theta(y) = \theta^{1/\alpha} y,$$

where α is the index of G. If $\mu \in M$, we define $\mu_\theta \in M$ by

$$\mu_\theta = \theta \mu S_\theta^{-1}, \quad 0 < \theta \mu(\mathbb{R}) \leqq 1. \tag{2.1}$$

The I-functional on M is defined in (1.3). For later reference we observe that I is translation invariant, i.e. $I(\mu) = I(\mu T_x^{-1})$ for $\mu \in M$. Furthermore, the scaling property of G is inherited by I in the following form: if $\mu \in M$ and G has index α, then for $\theta > 0$

$$I(\theta \mu S_\theta^{-1}) = \theta I(\mu S_\theta^{-1}) = I(\mu).$$

The translation invariance of I implies the translation invariance of the set $\mathscr{B}$ defined in (1.5) in the sense that if f is in the set, so is the function $f_x(\cdot) = f(\cdot + x)$, $x \in \mathbb{R}$.

For $\omega \in \Omega$, $x \in \mathbb{R}$, and A a Borel subset of $\mathbb{R}$ we write

$$\bar{L}_n^x(\omega, A) = \frac{1}{n} \sum_{j=0}^{n-1} \chi_A \left(x + \frac{S_j(\omega)}{c(n)} \right), \tag{2.2}$$

and if $x = 0$ we will simply drop the superscript x. Thus for each $\omega \in \Omega$, $\bar{L}_n^x(\omega, \cdot)$ is a probability measure on $\mathbb{R}$. For $K > 0$ and $k \in \mathbb{Z}$, define

$$I_{k,n} = [(-K + k - 1) c(n), (K + k + 1) c(n)]$$

and

$$\begin{aligned} T_k^{(n)} &= \inf\{j \geqq 0 \colon S_j \in I_{k,n}\}, \\ &= \infty, \text{ if the above set is empty.} \end{aligned} \tag{2.3}$$

We now prove four lemmas which culminate in Lemma 2.4. This in turn is used in the proof of Theorem 3.1 which is needed in the proof of the invariance principle.

Lemma 2.1. *With $I_{k,n}$ and $T_k^{(n)}$ defined as above, we have*

$$\sum_{k \in \mathbb{Z}} P[T_k^{(n)} \leqq n] = O(\log \log n), \quad \text{as } n \to \infty.$$

Proof. For given K, ω, j, and n, $S_j(\omega)$ can belong to at most $2K+3$ intervals $I_{k,n}$. Therefore we have

$$\begin{aligned} (2K+3) 2n &\geqq \sum_{k \in \mathbb{Z}} \sum_{j=0}^{2n-1} \chi_{[\omega: S_j(\omega) \in I_{k,n}]}(\omega) \\ &\geqq \sum_k \chi_{[\omega: T_k^{(n)}(\omega) \leq n]}(\omega) \sum_{j=T_k^{(n)}(\omega)}^{2n-1} \chi_{[\omega: S_j(\omega) \in I_{k,n}]}(\omega). \end{aligned}$$

Now taking expectations and using the Markov property of the random walk we get

$$\begin{aligned}(2K+3)2n &\geqq \sum_k \sum_{p=0}^{n} P[T_k^{(n)}=p] \sum_{j=p}^{2n-1} P[S_j\in I_{k,n}|\, T_k^{(n)}=p] \\ &\geqq \sum_k \sum_{p=0}^{n} P[T_k^{(n)}=p] \inf_{x\in I_{k,n}} \sum_{j=0}^{2n-p-1} P[S_j+x\in I_{k,n}] \\ &\geqq \sum_k P[T_k^{(n)}\leqq n] \inf_{x\in I_{k,n}} \sum_{j=0}^{n-1} P[S_j+x\in I_{k,n}] \\ &\geqq \inf_{0\leqq z\leqq 2K+2} \sum_{j\in\Gamma_n} P\{S_j\in[-zc(n),(2K+2-z)c(n)]\} \sum_k P[T_k^{(n)}\leqq n]\end{aligned}$$

where $\Gamma_n=[n/(2\log\log n), n/\log\log n]$. Since G has strictly positive density, for $j\in\Gamma_n$ it is clear that

$$\inf_{0\leqq z\leqq 2K+2} P\{S_j\in[-zc(n),(2K+2-z)c(n)]\}\geqq c>0.$$

Therefore $\sum_k P[T_k^{(n)}\leqq n]=O(\log\log n)$ and the lemma is proved.

For $A\subset M$, let

$$C(A)=\{v\in M:\ v=(\mu T_x^{-1})_\theta=\theta\mu T_x^{-1}S_\theta^{-1} \text{ for some } \mu\in A, \text{ some } x, \text{ and some } 0<\theta<\infty\}. \tag{2.4}$$

Lemma 2.2. *If $\lambda\in M$ then given $\varepsilon>0$ there exists a vague neighborhood N of λ such that*

$$\inf_{\beta\in C(N)} I(\beta)\geqq I(\lambda)-\varepsilon. \tag{2.5}$$

Proof. By the lower semicontinuity of I (see [1]) there exists a vague neighborhood N of λ such that

$$I(\beta)\geqq I(\lambda)-\varepsilon, \beta\in N. \tag{2.6}$$

Let v denote the left side in (2.5). Then there exist θ_n, x_n and $\beta_n\in N$ such that

$$\lim_{n\to\infty} I(\theta_n\beta_n T_{x_n}^{-1}S_{\theta_n}^{-1})=v.$$

By the scaling property of I we have $I(\theta_n\beta_n T_{x_n}^{-1}S_{\theta_n}^{-1})=I(\beta_n T_{x_n}^{-1})$, and since I is translation invariant this last quantity equals $I(\beta_n)$. Therefore by (2.6)

$$v=\lim_n I(\beta_n)\geqq I(\lambda)-\varepsilon.$$

If $V\subset M$, let $V_x=\{\beta T_x^{-1}:\ \beta\in V\}$ and let $D_V=\bigcup_x V_x$.

Lemma 2.3. *Let $\lambda\in M$ and $\varepsilon>0$ be given. Then there exists a neighborhood V of λ such that*

$$\limsup_n \frac{1}{\log\log n}\log P[\bar{L}_n\in D_V]<-I(\lambda)+\varepsilon, \tag{2.7}$$

where $\bar{L}_n$ is defined in (2.2).

Proof. If $\lambda(\mathbb{R})=0$ then $I(\lambda)=0$ and there is nothing to prove, so let $\lambda(\mathbb{R})=a>0$. Let N be a neighborhood of λ such that

$$\inf_{\beta\in C(N)} I(\beta)\geqq I(\lambda)-\varepsilon, \tag{2.8}$$

where $C(N)$ is defined by (2.4). Such a neighborhood exists by Lemma 2.2. Then we can find

$$W=\{\beta\colon |\int f_i\,d\beta-\int f_i\,d\lambda|<2\delta,\ i=1,\ldots,r\}$$

where $\delta<a/8$ and the f_i are continuous functions with supports in a compact interval $[-K,K]$ (neighborhoods such as W form a base for the vague topology) such that $\bar{W}\subset N$ and without loss of generality we may assume that

$$0\leqq f_1\leqq 1,\ \int f_1\,d\lambda\geqq 3a/4. \tag{2.9}$$

Let

$$V=\{\beta\colon |\int f_i\,d\beta-\int f_i\,d\lambda|<\delta,\ i=1,\ldots,r\}. \tag{2.10}$$

We will show that (2.7) is satisfied by this V. We have

$$P[\bar{L}_n\in D_V]\leqq\sum_{k\in\mathbb{Z}} P[\bar{L}_n\in\bigcup_{|x-k|\leqq\frac{1}{2}} V_x], \tag{2.11}$$

and

$$\begin{aligned}P[\bar{L}_n\in &\bigcup_{|x-k|\leqq\frac{1}{2}} V_x]\\ &=P\left\{\bigcup_{|x-k|\leqq\frac{1}{2}}\left[\left|\frac{1}{n}\sum_{j=0}^{n-1} f_i\left(-x+\frac{S_j}{c(n)}\right)-\int f_i\,d\lambda\right|<\delta,\ i=1,\ldots,r\right]\right\}.\end{aligned} \tag{2.12}$$

Let $T_k^{(n)}$ be defined as in (2.3). Then since the f_i have support in $[-K,K]$, $T_k^{(n)}\geqq l$ implies

$$f_i\left(-x+\frac{S_j}{c(n)}\right)=0,\quad 0\leqq j<l,\ |x-k|\leqq\tfrac{1}{2},\ 1\leqq i\leqq r. \tag{2.13}$$

Therefore, if $T_k^{(n)}\geqq n\left(1-\frac{a}{2}\right)$ and $|x-k|\leqq\frac{1}{2}$, then

$$\begin{aligned}\left|\frac{1}{n}\sum_{j=0}^{n-1} f_1\left(-x+\frac{S_j}{c(n)}\right)\right| &\leqq\frac{1}{n}\left(n-1-n\left(1-\frac{a}{2}\right)+1\right)\\ &=\frac{a}{2},\end{aligned} \tag{2.14}$$

where $0\leqq f_1\leqq 1$ is used. Since $\delta<a/8$, by (2.9) and (2.14) the event in (2.12) cannot occur if $T_k^{(n)}\geqq n\left(1-\frac{a}{2}\right)$. Therefore

$$P[\bar{L}_n \in \bigcup_{|x-k|\leqq\frac{1}{2}} V_x] \leqq \sum_{l=0}^{[n(1-\frac{a}{2})]} P\{\bigcup_{|x-k|\leqq\frac{1}{2}} [\bar{L}_n \in V_x] \cap [T_k^{(n)}=l]\}$$

$$=\sum_{l}\sum_{z\in I_{k,n}} P\left[\left|\frac{1}{n}\sum_{j=l}^{n-1} f_i\left(-x+\frac{z}{c(n)}+\frac{\hat{S}_j}{c(n)}\right)-\int f_i\,d\lambda\right|<\delta,\ 1\leqq i\leqq r,\right.$$
$$\left.\text{some } |x-k|\leqq\tfrac{1}{2},\ T_k^{(n)}=l,\ S_l=z\right], \quad (2.15)$$

where the summation on l is for the same values as above and $\hat{S}_j = X_{l+1}+\ldots+X_j$. Now by the independence of S_r, $1\leqq r\leqq l$, and $\hat{S}_j$ the last quantity equals

$$\sum_{l}\sum_{z\in I_{k,n}} P[T_k^{(n)}=l, S_l=z]$$
$$\cdot P\left[\left|\frac{1}{n}\sum_{j=0}^{n-l-1} f_i\left(-x+\frac{z}{c(n)}+\frac{S_j}{c(n)}\right)-\int f_i\,d\lambda\right|<\delta, 1\leqq i\leqq r, \text{ some } |x-k|\leqq\tfrac{1}{2}\right]. \quad (2.16)$$

If $|x-k|\leqq\frac{1}{2}$ and $z\in I_{k,n}$, then $-x+\frac{z}{c(n)}\in[-K-\frac{3}{2}, K+\frac{3}{2}]$. Therefore

$$P[\bar{L}_n\in \bigcup_{|x-k|\leqq\frac{1}{2}} V_x] \leqq \sum_{l}\sum_{z\in I_{k,n}} P[T_k^{(n)}=l, S_l=z]\,p_n, \quad (2.17)$$

where

$$p_n = \max_{0\leqq l\leqq[n(1-\frac{a}{2})]} P\left[\left|\frac{1}{n}\sum_{j=0}^{n-l-1} f_i\left(u+\frac{S_j}{c(n)}\right)-\int f_i\,d\lambda\right|<\delta,\ 1\leqq i\leqq r,\right.$$
$$\left.\text{some } u\in[-K-\tfrac{3}{2}, K+\tfrac{3}{2}]\right]. \quad (2.18)$$

Summing on z first, then on l in (2.17) gives

$$P[\bar{L}_n\in \bigcup_{|x-k|\leqq\frac{1}{2}} V_x] \leqq p_n P\left\{T_k^{(n)}\leqq\left[n\left(1-\frac{a}{2}\right)\right]\right\}. \quad (2.19)$$

Now summing on $k\in\mathbb{Z}$ and using Lemma 2.1 we get

$$P[\bar{L}_n\in D_V]\leqq p_n q_n, \quad (2.20)$$

where $q_n = O(\log\log n)$. It follows that

$$\limsup_n \frac{1}{\log\log n}\log P[\bar{L}_n\in D_V] \leqq \limsup_n \frac{1}{\log\log n}\log p_n. \quad (2.21)$$

Let γ denote the right side in (2.21). Then there exist sequences of non-negative integers (n_s) and (l_s), $l_s\leqq\left[n_s\left(1-\frac{a}{2}\right)\right]$, such that $n_s\to\infty$ and

$$\lim_{s\to\infty} \frac{1}{\log\log n_s} \log P\left[\left|\frac{1}{n_s}\sum_{j=0}^{n_s-l_s-1} f_i\left(u+\frac{S_j}{c(n_s)}\right)-\int f_i\,d\lambda\right|<\delta,\right.$$

$$\left. 1\leqq i\leqq r,\ \text{some } u\in[-K-\tfrac{3}{2}, K+\tfrac{3}{2}]\right]=\gamma. \tag{2.22}$$

Along some subsequence (again denoted by n_s) we have

$$\frac{n_s-l_s}{n_s}\to\theta_0, \quad \frac{a}{2}\leqq\theta_0\leqq 1. \tag{2.23}$$

This implies that $c(n_s-l_s)/c(n_s)\to\theta_0^{1/\alpha}$ as $s\to\infty$. Since the f_i are continuous with compact support, we get

$$\gamma\leqq\limsup_s \frac{1}{\log\log(n_s-l_s)}\log P\left[\left|\frac{\theta_0}{n_s-l_s}\sum_{j=0}^{n_s-l_s-1} f_i\left(u+\frac{S_j}{c(n_s-l_s)}\theta_0^{1/\alpha}\right)\right.\right.$$

$$\left.\left. -\int f_i\,d\lambda\right|<2\delta,\ 1\leqq i\leqq r,\ \text{some } u\in[-K-\tfrac{3}{2}, K+\tfrac{3}{2}]\right]. \tag{2.24}$$

Therefore

$$\gamma\leqq\limsup_n \frac{1}{\log\log n}\log P\left[\left|\frac{\theta_0}{n}\sum_{j=0}^{n-1} f_i\left(u+\frac{S_j}{c(n)}\theta_0^{1/\alpha}\right)-\int f_i\,d\lambda\right|\leqq 2\delta,\right.$$

$$\left. 1\leqq i\leqq r,\ \text{some } u\in[-K-\tfrac{3}{2}, K+\tfrac{3}{2}]\right]. \tag{2.25}$$

With θ_0 as in (2.23), let

$$\Gamma=\{\beta\in M: |\theta_0\int f_i\,d\beta T_u^{-1}S_{\theta_0}^{-1}-\int f_i\,d\lambda|\leqq 2\delta,\ 1\leqq i\leqq r,$$
$$\text{some } u\in[-K-\tfrac{3}{2}, K+\tfrac{3}{2}]\}.$$

The set Γ is closed and (2.25) is the same as

$$\gamma\leqq\limsup_n \frac{1}{\log\log n}\log P[\bar{L}_n\in\Gamma]. \tag{2.26}$$

By Theorem 3.2 in [4] we get

$$\gamma\leqq-\inf_{\beta\in\Gamma} I(\beta), \tag{2.27}$$

and since $\Gamma\subset C(N)$ (if $\beta\in\Gamma$, then $\nu=(\beta T_u^{-1})_{\theta_0}\in\bar{W}$ for some $u\in[-K-\frac{3}{2}, K+\frac{3}{2}]$, but then $\beta=(\nu T^{-1}_{-u\theta_0^{-1/\alpha}})_{\theta_0^{-1}}$, so $\beta\in C(\bar{W})\subset C(N)$) we have $\gamma\leqq-\inf_{\beta\in C(N)} I(\beta)\leqq -I(\lambda)+\varepsilon$ by (2.5). This proves the lemma.

Lemma 2.4. *If A is a closed subset of M, then*

$$\limsup_n \frac{1}{\log\log n}\log P[\bar{L}_n\in D_A]\leqq-\inf_{\beta\in A} I(\beta). \tag{2.28}$$

Proof. Let $\varepsilon>0$ be given. Then by Lemma 2.3 each λ in A has a neighborhood N_λ such that

$$\limsup_n \frac{1}{\log\log n} \log P[\bar{L}_n \in D_{N_\lambda}] \leqq -I(\lambda)+\varepsilon. \tag{2.29}$$

Since A is compact in the vague topology a finite number of such neighborhoods $N_1, \ldots, N_r$ (corresponding to $\lambda_1, \ldots, \lambda_r$, respectively) cover A. Thus

$$P[\bar{L}_n \in D_A] \leqq r \max_{1 \leqq j \leqq r} P[\bar{L}_n \in D_{N_j}].$$

Therefore

$$\limsup_n \frac{1}{\log\log n} \log P[\bar{L}_n \in D_A] \leqq -\min_{1 \leqq j \leqq r} I(\lambda_j)+\varepsilon$$
$$\leqq -\inf_{\lambda \in A} I(\lambda)+\varepsilon. \tag{2.30}$$

This proves the lemma.

3. The Main Results

Theorem 3.1 below is a refinement of Theorem 5.1 [4]; it is proved under the assumptions of Sect. 2. Below $\bar{A}$ denotes the closure of A with respect to the vague topology.

Theorem 3.1. *Let* $C_G = \{\beta \in M: I(\beta) \leqq 1\}$. *Let* $\bar{L}_n^x(\omega, \cdot)$ *be defined by* (2.2). *Then for almost all* ω

$$\bigcap_{m=1}^{\infty} \overline{\bigcup_{n \geqq m} \{\bar{L}_n^x(\omega, \cdot): x \in \mathbb{R}\}} \subset C_G \tag{3.1}$$

and

$$\bigcap_{m=1}^{\infty} \overline{\bigcup_{n \geqq m} \{\bar{L}_n(\omega, \cdot)\}} \supset C_G. \tag{3.2}$$

Proof. We need only prove (3.1) since (3.2) is contained in Theorem 5.1 [4]. Let N_1 be an open neighborhood of C_G. Since I is lower semicontinuous on M, we have $\inf_{\lambda \in N_1^c} I(\lambda) = \theta > 1$. Let $0 < \gamma < 1$ be such that $\theta\gamma > 1$ and let $j_n = [\exp(n^\gamma)]$. Let $\varepsilon > 0$ be such that $\gamma(\theta - \varepsilon) > 1$. By Lemma 2.4 we have

$$P[\bar{L}_{j_n}^x \in N_1^c \text{ for some } x \in \mathbb{R}] \leqq \exp\{-(\log\log j_n)(\theta-\varepsilon)\} \tag{3.3}$$

for all n sufficiently large. The right side in (3.3) equals $n^{-\gamma(\theta-\varepsilon)}(1+o(1))$ so summed on n it converges. By the Borel-Cantelli lemma

$$P[\bar{L}_{j_n}^x \in N_1^c \text{ for some } x \in \mathbb{R}, \text{ i.o.}] = 0. \tag{3.4}$$

Therefore

$$P\left[\omega: \bigcap_{m=1}^{\infty} \overline{\bigcup_{j_n \geqq m} \{\bar{L}^x_{j_n}(\omega, \cdot): x \in \mathbb{R}\}} \subset \bar{N}_1\right] = 1.$$

Now, if $j_{n-1} \leqq p_n < j_n$, then $p_n/j_n \to 1$ and $c(p_n)/c(j_n) \to 1$; consequently for any continuous f with compact support $[-z, z]$, $x_n \in \mathbb{R}$, $y_n = x_n c(j_n)/c(p_n)$, and $\omega \in \Omega$, by subtracting and adding terms it is easily seen that

$$\begin{aligned}\int f\,d\bar{L}^{x_n}_{j_n}(\omega,\cdot)-\int f\,d\bar{L}^{y_n}_{p_n}(\omega,\cdot)\\ =\frac{1}{p_n}\sum_{j=0}^{p_n-1}\left\{f\left(x_n+\frac{S_j(\omega)}{c(j_n)}\right)-f\left(y_n+\frac{S_j(\omega)}{c(p_n)}\right)\right\}+o(1)\end{aligned}\tag{3.5}$$

as $n\to\infty$. We also have

$$\left(x_n+\frac{S_j(\omega)}{c(j_n)}\right)-\left(y_n+\frac{S_j(\omega)}{c(p_n)}\right)=\left(x_n+\frac{S_j(\omega)}{c(j_n)}\right)\left(\frac{c(p_n)-c(j_n)}{c(p_n)}\right).$$

Let $\varepsilon>0$ be prescribed and $\delta>0$ be such that if $|x-y|<\delta$ then $|f(x)-f(y)|<\varepsilon$. Pick n_0 such that $2z|c(p_n)-c(j_n)|/c(p_n)<\delta$ and $|c(p_n)-c(j_n)|/c(p_n)<\frac{1}{2}$ for $n\geqq n_0$. If $|x_n+(S_j(\omega)/c(j_n))|>2z$ and $n\geqq n_0$, then the corresponding summand on the right side in (3.5) is zero because each argument of f is then strictly bigger than z; on the other hand, if $|x_n+(S_j(\omega)/c(j_n))|\leqq 2z$ and $n\geqq n_0$, then the two arguments of f differ by less than δ, so the summand is less than ε in absolute value. It follows that the left side in (3.5) tends to zero as $n\to\infty$. Therefore $\{\bar{L}^x_{p_n}: x\in\mathbb{R}\}$ and $\{\bar{L}^x_{j_n}: x\in\mathbb{R}\}$ have the same vague limit points and

$$P\left[\bigcap_{m=1}^{\infty}\overline{\bigcup_{n\geqq m}\{\bar{L}^x_n: x\in\mathbb{R}\}}\subset\bar{N}_1\right]=1.\tag{3.6}$$

Since we can pick $N_j\supset\bar{N}_{j+1}$, N_j open, $j\geqq 1$, such that $\bigcap_j N_j=C_G$, (3.6) implies (3.1).

Assume from now on that (S_n) is a lattice random walk. The next theorem is proved under more general hypotheses in [5] (Theorem 2).

Theorem 3.2. *If* (1.2) *holds and* $\alpha>1$, *then given* $\varepsilon>0$ *there is a* $\delta>0$ *such that*

$$P\left[\omega:\limsup_n\frac{c(n)}{n}\sup_{|x-y|\leqq\delta c(n)}|L_n(\omega,x)-L_n(\omega,y)|>\varepsilon\right]=0.\tag{3.7}$$

We now define

$$g_n(\omega,u)=\frac{c(n)}{n}\sum_{j=0}^{n-1}\chi_{\{[uc(n)]\}}(S_j(\omega)),\qquad u\in\mathbb{R},\tag{3.8}$$

and

$$\begin{aligned}h_n(\omega,u)&=\frac{c(n)}{n}\sum_{j=0}^{n-1}\chi_{\{k\}}(S_j(\omega)),\qquad \text{if } u=k/c(n),\ k\in\mathbb{Z},\\ &=\text{linear elsewhere.}\end{aligned}\tag{3.9}$$

Theorem 3.2 can be rephrased as Theorem 3.3 and Theorem 3.4 is an easy corollary of Theorem 3.3.

Theorem 3.3. *If* $g_n(\omega,\cdot)$ *and* $h_n(\omega,\cdot)$ *are defined by* (3.8) *and* (3.9), *respectively, then given* $\varepsilon>0$ *there exists* $\delta>0$ *such that*

$$P[\omega:\limsup_n\sup_{|u-v|<\delta}|g_n(\omega,u)-g_n(\omega,v)|>\varepsilon]=0,\tag{3.10}$$

and

$$P[\omega:\limsup_n\sup_{|u-v|<\delta}|h_n(\omega,u)-h_n(\omega,v)|>\varepsilon]=0.\tag{3.11}$$

Theorem 3.4. *There exists a set Ω_0 with $P(\Omega_0)=1$ such that if $\omega\in\Omega_0$, then*

$$\lim_{\delta\to 0}\limsup_{n}\sup_{|u-v|<\delta}|g_n(\omega,u)-g_n(\omega,v)|=0 \tag{3.12}$$

and

$$\lim_{\delta\to 0}\limsup_{n}\sup_{|u-v|<\delta}|h_n(\omega,u)-h_n(\omega,v)|=0. \tag{3.13}$$

Remark. Theorem 3.4 implies that if $\omega\in\Omega_0$, then $(g_n(\omega,\cdot))$ and $(h_n(\omega,\cdot))$ are uniformly equicontinuous on $\mathbb{R}$ and $g_n(\omega,\cdot)-h_n(\omega,\cdot)\to 0$ uniformly.

With $\mathscr{A}$ as in (1.4) we let $\mathscr{T}$ denote the topology on $\mathscr{A}$ given by uniform convergence on compact subsets of $\mathbb{R}$. Now we are ready to prove the main result.

Theorem 3.5. *There exists Ω_0 with $P(\Omega_0)=1$ such that if $\omega\in\Omega_0$, then*

(i) *the set*

$$R(\omega)=\{h_n(\omega,x+\cdot)\colon n\geqq 1, x\in\mathbb{R}\}$$

is a relatively compact subset of $\mathscr{A}$;

(ii) *the set of limit points of $R(\omega)$ is contained in $\mathscr{B}$; and*

(iii) *the set of limit points of*

$$S(\omega)=\{h_n(\omega,\cdot)\colon n\geqq 1\}$$

contains the set $\mathscr{B}$.

Proof. For any ω the function $h_n(\omega,\cdot)$ is nonnegative, continuous, has compact support and satisfies

$$\int_{-\infty}^{\infty} h_n(\omega,u)\,du=1,\qquad n\geqq 1. \tag{3.14}$$

Therefore the set $R(\omega)$ is contained in $\mathscr{A}$. Now let Ω_0 be picked so that $P(\Omega_0)=1$ and (3.1) and (3.2) of Theorem 3.1 and (3.12) and (3.13) of Theorem 3.4 are satisfied for $\omega\in\Omega_0$. If $\{h_n(\omega,x_n)\colon n\geqq 1\}$ is an unbounded set, then along a subsequence (n_j) we have $h_{n_j}(\omega,x_{n_j})=K_j\to\infty$ as $j\to\infty$. Then by (3.13) we have $h_{n_j}(\omega,u)>K_j/2$ for $|u-x_{n_j}|<\delta$, for some $\delta>0$, $j\geqq j_0$. This contradicts (3.14), so for $\omega\in\Omega_0$ the set $\{\|h_n(\omega,\cdot)\|_\infty\colon n\geqq 1\}$ is bounded. This fact and (3.13) imply (via Ascoli's Theorem) that the set $R(\omega)$ is relatively compact in $\mathscr{A}$.

Let $m_i\in\mathbb{Z}$ and let (n_i) be a sequence of positive integers tending to infinity. Let $x_i=m_i/c(n_i)$ and

$$k_i(\omega,\cdot)=g_{n_i}(\omega,x_i+\cdot). \tag{3.15}$$

We will first show that if φ is continuous with compact support on $\mathbb{R}$, then

$$\lim_i\left(\int_{-\infty}^{\infty}\varphi(u)k_i(\omega,u)\,du-\int_{-\infty}^{\infty}\varphi(u)\,d\bar{L}_{n_i}^{-x_i}(\omega,u)\right)=0. \tag{3.16}$$

To see this, note that

$$\int_{-\infty}^{\infty} \varphi(u) k_i(\omega, u) du = \sum_{r\in\mathbb{Z}} \int_{r/c(n_i)}^{(r+1)/c(n_i)} \varphi(u) k_i(\omega, u) du$$
$$= \sum_{r\in\mathbb{Z}} \frac{c(n_i)}{n_i} \sum_{j=0}^{n_i-1} \chi_{\{r+m_i\}}(S_j(\omega)) \int_{r/c(n_i)}^{(r+1)/c(n_i)} \varphi(u) du$$
$$= \sum_{r\in\mathbb{Z}} \frac{1}{n_i} \sum_{j=0}^{n_i-1} \chi_{\{r+m_i\}}(S_j(\omega)) \left(\varphi\left(\frac{r}{c(n_i)}\right) + \varepsilon_{i,r}\right)$$

where $\varepsilon_{i,r} \to 0$ uniformly in r as $i \to \infty$. We have

$$\sum_r \frac{1}{n_i} \sum_{j=0}^{n_i-1} \chi_{\{r+m_i\}}(S_j(\omega)) \sup_r |\varepsilon_{i,r}| = \sup_r |\varepsilon_{i,r}| \to 0$$

as $i \to \infty$. Also

$$\sum_r \frac{1}{n_i} \sum_{j=0}^{n_i-1} \chi_{\{r+m_i\}}(S_j(\omega)) \varphi\left(\frac{r}{c(n_i)}\right) = \frac{1}{n_i} \sum_{j=0}^{n_i-1} \varphi\left(\frac{S_j(\omega)}{c(n_i)} - x_i\right)$$
$$= \int_{-\infty}^{\infty} \varphi(u) d\bar{L}_{n_i}^{-x_i}(\omega, u),$$

which shows (3.16).

To check (ii) it will be convenient to work with the set $R_1(\omega) = \{g_n(\omega, x + \cdot): n \geqq 1, x \in \mathbb{R}\}$. The members of $R_1(\omega)$ are step functions and it is clear from the remark after Theorem 3.4 that the set of limit points under $\mathcal{T}$ of $R_1(\omega)$, for $\omega \in \Omega_0$, is the same as the set of limit points of $R(\omega)$. Let $\omega \in \Omega_0$ and let

$$\hat{g}_i(\omega, \cdot) = g_{n_i}(\omega, x_i + \cdot) \to f \tag{3.17}$$

as $i \to \infty$ in the sense of $\mathcal{T}$; $f \in \mathcal{A}$ since $(g_n(\omega, \cdot))$ is uniformly equicontinuous. There is no loss of generality if we take $x_i = m_i/c(n_i)$, $m_i \in \mathbb{Z}$, because if each x_i is replaced by a nearest such number in (3.17) then the limit point would still be f. We want to show $I(f) \leqq 1$. If $\nu \in M$ with density f, then (3.16) applied with $\hat{g}_i(\omega, \cdot)$ in place of $k_i(\omega, \cdot)$ shows that under (3.17) the set $\{\bar{L}_{n_i}^{-x_i}(\omega, \cdot): i \geqq 1\}$ has ν as its limit point. By Theorem 3.1 we then have $I(f) = I(\nu) \leqq 1$. This proves (ii).

To prove (iii), let $f \in \mathcal{B}$. Let $\nu \in M$ with density f. By Theorem 3.1, if $\omega \in \Omega_0$ then there exists a sequence (n_i) along which $\bar{L}_n(\omega, \cdot) \to \nu$ vaguely. By (3.16) with $x_i = 0$, we then conclude

$$\lim_i \int_{-\infty}^{\infty} \varphi(u) g_{n_i}(\omega, u) du = \int_{-\infty}^{\infty} \varphi(u) f(u) du \tag{3.18}$$

for φ continuous with compact support. Since $\omega \in \Omega_0$, $\{g_{n_i}(\omega, \cdot): i \geqq 1\}$ is relatively compact in $\mathcal{A}$ and so along a subsequence $g_{n_i}(\omega, \cdot) \to k(\cdot) \in \mathcal{A}$ in the sense of $\mathcal{T}$. Therefore by (3.18) we get

$$\int_{-\infty}^{\infty} \varphi(u) k(u) du = \int_{-\infty}^{\infty} \varphi(u) f(u) du$$

for each continuous φ with compact support, which shows $k=f$ and (iii) is proved.

The following theorems are corollaries of Theorem 3.5 which describe the asymptotic behavior of $L_n(\omega, \cdot)$. Let

$$\theta_G = \sup\{f(0): f \in \mathscr{B}\}. \tag{3.19}$$

If G is symmetric stable with characteristic function $\varphi_G(u) = e^{-|u|^\alpha}$, $1<\alpha<2$, then θ_G is computed in [1] to be $\Gamma(1/\alpha)\Gamma(1-1/\alpha)/(\pi(\alpha-1)^{1-1/\alpha})$. When $\alpha=2$ and G is $N(0,1)$, θ_G is shown [1] to be $\sqrt{2}$.

Theorem 3.6. *If $\alpha>1$, then for almost all ω and $k \in \mathbb{Z}$*

$$\limsup_n \frac{c(n)}{n} L_n(\omega, k) = \limsup_n \frac{c(n)}{n} \max_m L_n(\omega, m) = \theta_G, \tag{3.20}$$

where θ_G is defined in (3.19). *This quantity is positive and finite.*

Remark. The constant θ_G is the same one that occurs in the corresponding behavior of the local time of a stable process $y(t)$ for which $y(1)$ has distribution G with $\alpha>1$; see [1].

Proof. Let $\Phi(f)=f(0)$. Φ is a continuous functional on $\mathscr{A}$ (topology $\mathscr{T}$). Let Ω_0 be as in Theorem 3.5 and for $\omega \in \Omega_0$, $k \in \mathbb{Z}$, let $f_n(\omega, u) = h_n(\omega, u+k/c(n))$. Then by Theorem 3.5 the sequence $\{f_n(\omega, \cdot): n \geqq 1\}$ is relatively compact in $\mathscr{A}$ and has limit set $\mathscr{B}$. $\left(\text{Note that } \{f_n(\omega,\cdot)\} \text{ and } \{h_n(\omega,\cdot)\} \text{ have the same limit set since } \frac{k}{c(n)} \to 0\right)$. Thus

$$\limsup_n \Phi(f_n(\omega, \cdot)) = \sup\{f(0): f \in \mathscr{B}\} = \theta_G.$$

On the other hand,

$$\begin{aligned}\limsup_n \Phi(f_n(\omega, \cdot)) &= \limsup_n h_n(\omega, k/c(n)) \\ &= \limsup_n \frac{c(n)}{n} L_n(\omega, k).\end{aligned}$$

The quantity θ_G is clearly positive and it is finite because $\{\|h_n(\omega, \cdot)\|_\infty : n \geqq 1\}$ is bounded as shown in the proof of Theorem 3.5.

To prove the second equality, let $\omega \in \Omega_0$ and observe that

$$\frac{c(n)}{n} \max_k L_n(\omega, k) = \sup_x h_n(\omega, x).$$

If β denotes $\limsup_n \sup_x h_n(\omega, x)$, then there exists (x_n) such that

$$\limsup_n h_n(\omega, x_n) = \beta.$$

Let $f_n(\omega, u) = h_n(\omega, u + x_n)$ and $\Phi(f) = f(0)$ as before. Again by Theorem 3.5 the set $\{f_n(\omega, \cdot): n \geqq 1\}$ is relatively compact in $\mathscr{A}$ and

$$\limsup_n \Phi(f_n(\omega, \cdot)) = \limsup_n h_n(\omega, x_n) = \beta.$$

This shows that $\beta \leqq \theta_G$ by Theorem 3.5, but clearly $\beta \geqq \limsup_n h_n(\omega, 0) = \theta_G$. Therefore $\beta = \theta_G$ and the theorem is proved.

For the next theorem Ω_0 is any set of probability 1 that satisfies Theorem 3.5.

Theorem 3.7. *Let φ be any continuous function on $\mathbb{R}$. Then for $\omega \in \Omega_0$ the following assertions hold:*

(i) *If (k_n) is an integer sequence such that*

$$\lim_n \frac{k_n}{c(n)} = a \in \mathbb{R} \tag{3.21}$$

then

$$\limsup_n \varphi\left(\frac{c(n)}{n} L_n(\omega, k_n)\right) = \sup_{0 \leqq t \leqq \theta_G} \varphi(t), \tag{3.22}$$

where θ_G is defined in (3.19). *In particular,*

$$\limsup_n \frac{c(n)}{n} L_n(\omega, k_n) = \theta_G. \tag{3.23}$$

In the above, if $k_n = k$, $n \geqq 1$, then $a = 0$, and the conclusion holds.

(ii) *If $-\infty < a < b < \infty$, then*

$$\limsup_n \frac{1}{c(n)} \sum_{ac(n) \leqq k \leqq bc(n)} \varphi\left(\frac{c(n)}{n} L_n(\omega, k)\right) = \sup_{f \in \mathscr{B}} \int_a^b \varphi \circ f(t)\,dt, \tag{3.24}$$

and

$$\limsup_n \inf_{ac(n) \leqq k \leqq bc(n)} \varphi\left(\frac{c(n)}{n} L_n(\omega, k)\right) = \sup_{f \in \mathscr{B}} \inf_{a \leqq t \leqq b} \varphi \circ f(t). \tag{3.25}$$

Proof. If $f \in \mathscr{A}$, let

$$\Phi(f) = \varphi \circ f(0).$$

This defines a continuous function on $\mathscr{A}$. Let

$$\Delta = \left\{\Phi\left(h_n\left(\omega, \cdot + \frac{k_n}{c(n)}\right)\right): n \geqq 1\right\} = \left\{\varphi \circ h_n\left(\omega, \frac{k_n}{c(n)}\right): n \geqq 1\right\}. \tag{3.26}$$

Since $k_n/c(n) \to a$ and $(h_n(\omega, \cdot))$ is a uniformly equicontinuous family, the set of limit points of Δ is the same as the set of limit points of Δ_1, where

$$\Delta_1 = \{\Phi(h_n(\omega, \cdot + a)): n \geqq 1\}. \tag{3.27}$$

By Theorem 3.5 the set of limit points of Δ_1 is the set $\{\Phi(f(\cdot + a)): f \in \mathscr{B}\}$, but the translation invariance of $\mathscr{B}$ implies that the limit set of Δ_1 is $\{\Phi(f): f \in \mathscr{B}\}$.

Therefore the limit set of $\Delta=\left\{\varphi\left(\frac{c(n)}{n}L_n(\omega,k_n)\right): n\geqq 1\right\}$ is the set $\{\varphi\circ f(0): f\in\mathscr{B}\}=\{\varphi(t): 0\leqq t\leqq\theta_G\}$, which implies (3.22).

To prove (3.24), let $\Phi(f)=\int_a^b\varphi\circ f(t)\,dt$ for $f\in\mathscr{A}$. Again Φ is a continuous function on $\mathscr{A}$. Since $h_n-g_n\to 0$ uniformly on $\mathbb{R}$, the set of limit points of $\{\Phi(h_n(\omega,\cdot)): n\geqq 1\}$ is the same as the set of limit points of $\{\Phi(g_n(\omega,\cdot)): n\geqq 1\}$. Now let r_n and s_n be integers such that

$$\frac{r_n}{c(n)}\leqq a<\frac{r_n+1}{c(n)},\qquad \frac{s_n}{c(n)}\leqq b<\frac{s_n+1}{c(n)}.$$

Then

$$\begin{aligned}\Phi(g_n(\omega,\cdot))&=\int_a^b\varphi\circ g_n(\omega,u)\,du\\&=\sum_{k=r_n+1}^{s_n-1}\int_{k/c(n)}^{(k+1)/c(n)}\varphi\circ g_n(\omega,u)\,du+\int_a^{(r_n+1)/c(n)}\varphi\circ g_n(\omega,u)\,du\\&\quad+\int_{s_n/c(n)}^{b}\varphi\circ g_n(\omega,u)\,du.\end{aligned}$$

We have

$$\sum_{k=r_n+1}^{s_n-1}\int_{k/c(n)}^{(k+1)/c(n)}\varphi\circ g_n(\omega,u)\,du=\sum_{k=r_n+1}^{s_n-1}\frac{1}{c(n)}\varphi\left(\frac{c(n)}{n}L_n(\omega,k)\right).$$

Also,

$$\begin{aligned}\left|\int_a^{(r_n+1)/c(n)}\varphi\circ g_n(\omega,u)\,du\right|&\leqq\int_{r_n/c(n)}^{(r_n+1)/c(n)}|\varphi\circ g_n(\omega,u)|\,du\\&=\frac{1}{c(n)}\left|\varphi\left(\frac{c(n)}{n}L_n(\omega,r_n)\right)\right|\end{aligned}$$

and by (3.20) this last expression tends to 0 as $n\to\infty$; likewise we have

$$\lim_n\int_{s_n/c(n)}^{b}\varphi\circ g_n(\omega,u)\,du=0.$$

Therefore, again using (3.20), we have

$$\Phi(g_n(\omega,\cdot))=\frac{1}{c(n)}\sum_{ac(n)\leqq k\leqq bc(n)}\varphi\left(\frac{c(n)}{n}L_n(\omega,k)\right)+o(1).$$

Since the limit points of $\{\Phi(g_n(\omega,\cdot)): n\geqq 1\}$ consist of the set $\{\Phi(f): f\in\mathscr{B}\}$ $=\left\{\int_a^b\varphi\circ f(u)\,du: f\in\mathscr{B}\right\}$, (3.24) follows.

The proof of (3.25) goes along the same lines (one defines $\Phi(f)$ $=\inf_{a\leqq u\leqq b}\varphi(f(u))$) and is left to the reader.

Remark. In (3.25) if we take $\varphi(x)=x$, then

$$\limsup_n \frac{c(n)}{n} \inf_{ac(n)\leqq k\leqq bc(n)} L_n(\omega,k)=\sup_{f\in\mathscr{B}} \inf_{a\leqq u\leqq b} f(u), \tag{3.28}$$

and since $\inf_{a\leqq u\leqq b} f(u)\leqq\frac{1}{b-a}\int_a^b f(u)\,du\leqq\frac{1}{b-a}$, it follows that the right side in (3.28) is $\leqq\frac{1}{b-a}$ and it is clearly positive. If $k_n=O(c(n))$ replaces (3.21) as the hypothesis of Theorem 3.7(i) then by the previous theorem θ_G is still an upper bound for the left side, and by (3.28) the lower bound is positive. It seems plausible that if $k_n=O(c(n))$ in (3.23) then the statement remains valid.

References

1. Donsker, M.D., Varadhan, S.R.S.: On laws of the iterated logarithm for local times. Comm. Pure Appl. Math. **30**, 707–753 (1977)
2. Feller, W.: An Introduction to Probability Theory and Its Applications, Vol. II. New York: Wiley 1966
3. Griffin, P., Jain, N.C., Pruitt, W.: Approximate local limit theorems for laws outside domains of attraction. Ann. Probability **12**, 45–63 (1984)
4. Jain, N.C.: A Donsker-Varadhan type of invariance principle. Z. Wahrscheinlichkeitstheorie verw. Gebiete **59**, 117–138 (1982)
5. Jain, N.C., Pruitt, W.: Asymptotic behavior of the local time of a recurrent random walk. Ann. Probability **12**, 64–85 (1984)
6. Kesten, H.: An iterated logarithm law for local time. Duke Math. J. **32**, 447–456 (1965)
7. Kesten, H., Spitzer, F.: A limit theorem related to a new class of self similar processes. Z. Wahrscheinlichkeitstheorie verw. Gebiete **50**, 5–25 (1979)
8. Révész, P.: Local time and invariance (preprint)
9. Spitzer, F.: Principles of Random Walk. New York: Van Nostrand 1964

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