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Rate of Expansion of an Inhomogeneous Branching Process of Brownian Particles

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Summary. Let X be the $(B^0, \{q_n(x)\})$-branching diffusion where B^0 is the $\exp\left(-\int_0^t k(B_s)\,ds\right)$-subprocess of $BM(R^1)$ and $q_n(x)$ is the probability that a particle dying at x produces n offspring, $q_0 \equiv q_1 \equiv 0$. Put $m(x) = \sum n q_n(x)$. We assume q_n, $n \geqq 2$, m and k are all continuous (but m is not necessarily bounded). If $k(x)m(x) \to 0$ as $|x| \to \infty$, then we prove that $R_t/t \to (\lambda_0/2)^{1/2}$, as $t \to \infty$, a.s. and in mean (of any order) where R_t is the position of the rightmost particle at time t and λ_0 is the largest eigenvalue of $(1/2)d^2/dx^2 + Q$, $Q(x) = k(x)(m(x)-1)$.

1. Introduction

Consider the following branching process. At time $t=0$ a single particle begins a standard Brownian motion $\{B_t, t \geqq 0\}$ on the line starting at $B_0 = 0$. The motion continues for a random time τ whose law is

$$P[\tau > t \mid B_s, s \geqq 0] = \exp\left(-\int_0^t k(B_s)\,ds\right)$$

where $k \geqq 0$, $k \not\equiv 0$, is a given continuous function. (τ may be realized as the first time the functional $A_t = \int_0^t k(B_s)\,ds$ reaches a random level η where η is an independent Exp(1)-distributed random variable.) At τ the particle splits into $n \geqq 2$ particles with probability $q_n(x)$ where $x = B_{\tau-}$. (We assume that $q_0(x) = q_1(x) = 0$, $\sum q_n(x) = 1$ for all x and that every q_n is continuous.) Each of the n new particles continues along independent Brownian paths starting from $x = B_{\tau-}$ and is also subject to the same killing and splitting rules. At time t there are Z_t particles located at positions $X_t^{(1)}, \ldots, X_t^{(r)}$, $r = Z_t$. It is well known that, under reasonable assumptions, $Z_t\{J\}$, the number of particles in an arbitrary interval J, tends to increase exponentially like $e^{\lambda_0 t}$, λ_0 defined below, as $t \to \infty$.

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See, for example [1] or [13]. In this paper we show that, under reasonable assumptions, the diameter of the process tends to increase linearly, like $(2\lambda_0)^{1/2}t$, as $t\to\infty$. More precisely, let R_t be the rightmost edge of the population at time t:

$$R_t=\max\{X_t^{(1)},\dots,X_t^{(r)}\},\qquad r=Z_t.$$

Let $m(x)$ be the expected number of offspring of a particle which dies at x:

$$m(x)=\sum_{n=2}^{\infty} n q_n(x)\geqq 2.$$

We assume m is finite and continuous not necessarily bounded. Put

$$Q(x)=k(x)(m(x)-1).$$

Theorem 1. *If, in addition to the preceding assumptions,*

$$Q(x)\to 0 \quad as \quad |x|\to\infty, \tag{1.1}$$

then

$$R_t/t \xrightarrow[L^r]{\text{a.s.}} \beta_0 \quad as \quad t\to\infty \ (any\ r>0), \tag{1.2}$$

where $\beta_0=(\lambda_0/2)^{1/2}$ and λ_0 is the largest positive eigenvalue of the boundary value problem

$$(1/2)(d^2u/dx^2)+Qu=\lambda u,\qquad u>0,\qquad \int_{-\infty}^{\infty} u^2\,dx<\infty. \tag{1.3}$$

Corollary. *Let L_t be the left-most edge and $D_t=R_t-L_t$ the diameter of the population at time t. Then $L_t/t\to-\beta_0$ and $D_t/t\to 2\beta_0$ a.s. and in mean (of any order) as $t\to\infty$.*

Notes. 1. Under our assumptions, $k(x)\to 0$ as $|x|\to\infty$. If k goes to 0 fast enough, for example if k vanishes off a compact interval, then the lifetimes of the individual particles have infinite expectation. That the conclusion of Theorem 1 should still obtain in these cases seems a little surprising. See Note 4. Note that when k vanishes off a compact interval J, say, then particles neither age nor reproduce outside of J.

2. Our result provides a little information about a family of solutions to the semilinear heat equation

$$\partial v/\partial t=(1/2)\,\partial^2 v/\partial x^2+k(x)(F(x,v)-v) \tag{1.4}$$

where

$$F(x,v)=q_2(x)\,v^2+q_3(x)\,v^3+\dots.$$

Put $v(t,x,y)=P_x[R_t\leqq y]$, then v as a function of t and x satisfies (1.4) with the initial condition $v(0,x,y)=1$, $x\leqq y$; $=0$, $x>y$. See [8]. From Theorem 1 it follows that for each fixed x as $t\to\infty$ $v(t,x,\beta t)\to 1$ or 0 according as $\beta>\beta_0$ or $\beta<\beta_0$. Moreover, for $\beta>\beta_0$, $a<2\beta_0(\beta-\beta_0)$, $1-v(t,x,\beta t)=o(e^{-at})$ as an examination of the proof in Step 1 in §3 will reveal.

3. The existence of the eigenvalue λ_0, under (1.1), is a standard result in the theory of ordinary differential equations. A proof is easily made using the results of Chap. 9 of [5]. (For the one sided case see Problem 2, p. 255 of [5].) The spectrum of the operator $(1/2)\,d^2/dx^2+Q$ consists of a continuous part, $(-\infty,0]$, and a discrete part $\lambda_0>\lambda_1>\ldots>0$ (the eigenvalues.) If Q does not satisfy (1.1) then there need be no eigenvalues. If, for example, Q is constant, say $Q\equiv\lambda_0>0$, the spectrum is $(-\infty,\lambda_0]$ and there is no solution $(\lambda;u)$ of (1.3).

4. In the homogeneous case when k is a constant and all of the q_n are constant, the distribution $v(t,x,y)=P_x[R_t\leqq y]$ satisfies

$$v(t,x,y)=v(t,0,y-x).$$

Suppose for example that $k(x)=\lambda_0$, a positive constant, and that $q_n(x)=0$, $n\neq 2$, $q_2(x)=1$ for all x. Then $v(t,x)=v(t,0,x)$ satisfies

$$\partial v/\partial t=(1/2)\,\partial^2 v/\partial x^2+\lambda_0(v^2-v)$$

with initial condition $v(0,x)=1_{[0,\infty)}(x)$. This and similar homogeneous semi-linear diffusion equations have been studied in great detail. For a sample of the literature, see [2, 3, 10] and [11]. In this case if α_t is the median of R_t, i.e., the solution to $v(t,\alpha_t)=1/2$, then, as $t\to\infty$, $R_t-\alpha_t$ has a nondegenerate limit distribution and furthermore $\alpha_t\sim 2\beta_0 t$, $\beta_0=(\lambda_0/2)^{1/2}$. It follows that $R_t/t\to 2\beta_0$ in probability (and almost surely with a little extra effort). Note that λ_0 though not an eigenvalue is still the largest point in the spectrum of $(1/2)\,d^2/dx^2+Q$ $=(1/2)\,d^2/dx^2+\lambda_0$. The lifetime of a particle in the homogeneous case is independent of the particle's path and has an exponential distribution with mean $1/\lambda_0$.

In [4] Biggins has studied the asymptotic linearity of a homogeneous branching random walk. In [12] Uchiyama proves a limit theorem for a quite different class of branching processes (but again with exponentially distributed particle lifetimes independent of paths) which implies a linear growth for their diameters.

One of the key estimates is the bound given in Step 1 of the proof of Theorem 1. To prove it we h-transform the expectation semigroup (M_t) into the transition semigroup of a conservative recurrent diffusion from which we quickly obtain uniform estimates of the expected number of particles to the right of βt at time t. This method is similar in spirit to the method of "associated distributions" so useful in large deviation theory and renewal theory. See Feller [6]. For another important estimate see Step 4. The idea behind it is to stop the particle production at time $t/2$ and run the process during the time interval $(t/2,t)$ as if we had $r=Z_{t/2}$ independent Brownian motions.

2. The Expectation Semigroup

The formal definition and construction of branching Markov processes and the derivation of their fundamental equations and basic properties may be found in Ikeda, Nagasawa, Watanabe [8].

Let Z_t = the total number of particles at time t. If at time t there are $Z_t = r$ particles, their positions will be denoted $X_t = [X_t^{(1)}, \ldots, X_t^{(r)}]$, an unordered r-tuple. These position variables are not independent but for any set A in the, rather complicated, state space of X we have

$$\begin{aligned} &P[X_{t+s} \in A \mid X_u, 0 \leqq u \leqq t, X_t = [x_1, \ldots, x_r]] \\ &= P_{x_1}[X \in A] P_{x_2}[X \in A] \ldots P_{x_r}[X \in A]. \end{aligned}$$

Here $P_x[\cdot]$ stands for probability given that a single particle starts at time 0 at x. Let $\tau_1, \tau_2, \ldots$ denote the successive splitting times: $\tau_1 = \inf\{t: Z_t \neq Z_0\}$, etc. We duly note here the unsurprising fact that under the assumptions of Theorem 1 there is no explosion; if $\tau_\infty = \lim \tau_n$, then $P_x[\tau_\infty = \infty] = 1$ for all x. This fact, whose proof we omit, is in this case a straightforward uniqueness result for nonlinear integral equations ($v = P_x[\tau_\infty > t]$ satisfies the nonlinear renewal equation called the S-equation in [8]).

For any function f put $Z_t(f) = \sum_{1 \leqq i \leqq Z_t} f(X_t^{(i)})$. For sets J $Z_t(J) = Z_t(1_J)$ is the number of particles in J at time t. The basic properties of the law of X exhibited in the last paragraph imply that the equation

$$M_t f(x) = E_x Z_t(f), \qquad t \geqq 0,$$

defines a positive but not contracting semigroup (M_t) on $bC(R)$.

Lemma 2.1. *Put* $e_Q(t) = \exp\left(\int_0^t Q(B_s)\, ds\right)$, $Q = (m-1)k$ *as in* §1. *Then*

$$M_t f(x) = E_x^B[e_Q(t) f(B_t)], \qquad t \geqq 0, \tag{2.1}$$

for any bounded measurable f. *In particular*

$$E_x Z_t(J) = E_x^B[e_Q(t), B_t \in J]$$

$$E_x Z_t = E_x^B e_Q(t) \leqq e^{\|Q\| t}, \qquad \|Q\| = \sup_x Q(x).$$

In these formulas and elsewhere P^B, E^B denote probabilities and expectations for Brownian motion.

Proof. We may suppose f is positive, bounded and continuous. Let $v(t,x)$ denote the lefthand side of (2.1). Then v satisfies the renewal equation $(*)$

$$v(t,x) = T_t^0 f(x) + \int_0^t \int_R K(x; dy, ds)\, m(y)\, v(t-s, y),$$

where $T_t^0 f(x) = E_x^B[f(B_t), \ \tau > t]$, $K(x; \ dy, \ ds) = P_x^B[B_{\tau-} \in dy, \ \tau \in ds]$ and $m = \sum n q_n$. (Here τ is as defined in §1 and obviously coincides in distribution with τ_1.) Call the righthand side of (2.1) $v_1(t,x)$. If we compute the Laplace transform of v_1 (in t), use Kac's formula and a formula for K (see [9], Problem 2, p. 184) and a little algebra, we find that v_1 also satisfies $(*)$. We obtain the equality in (2.1) by establishing a uniqueness result for solutions to $(*)$. The fine details are left to the reader. For a direct probabilistic proof of (2.1) in a special case (but with unbounded Q), see [9], §5.13.

Corollary. (i) $M_t\colon bC(R)\mapsto bC(R)$.

(ii) *The differential operator* $(1/2)\,d^2/dx^2+Q$ *is the (local) generator of* (M_t).

Lemma 2.2. *Let h be the unique solution to the eigenvalue problem* (1.3) *subject to* $h(0)=1$. *Then*

$$\lim_{x\to\pm\infty} h'(x)/h(x)=\mp 2\beta_0=\mp(2\lambda_0)^{1/2}, \tag{2.2}$$

and consequently $h(x)=\exp[-2\beta_0|x|(1+o(1))]$ *as* $|x|\to\infty$.

Proof. Fix ε, $0<\varepsilon<\lambda_0$ and choose $x_0>0$ so that $Q(x)<\varepsilon$ for $x>x_0$, see (1.1). Then $h''=2(\lambda_0-Q)h>0$ on (x_0,∞) so $h'\in\uparrow$ there. But $h\in L^2$ entails $h''\in L^2$ (Q is bounded) and this forces $h'\in L^2$. It follows from these considerations that $h'<0$ on (x_0,∞) and $r=(h')^2-2\lambda_0 h^2$ is integrable on (x_0,∞). But $r'=-4Qhh'\geqq 0$ on (x_0,∞) from which we conclude $r\leqq 0$ there and this implies $(*)$ $h'(x)/h(x)\geqq -2\beta_0$ for $x\geqq x_0$. We now put $r_\varepsilon=(h')^2-(2\lambda_0-2\varepsilon)h^2$. Then $r_\varepsilon\in L^1$ and $r'_\varepsilon\leqq 0$ on (x_0,∞) so $r_\varepsilon\geqq 0$ there. Hence $|h'|/h=-h'/h\geqq(2\lambda_0-2\varepsilon)^{1/2}$ on (x_0,∞). Since ε is arbitrary, we conclude from $(*)$ and the last inequality that (2.2) holds (as $x\to+\infty$ but the same argument with a sign change works at $-\infty$).

Remark 1. If we drop the assumption $Q(x)\to 0$ as $|x|\to\infty$ and require instead that $\int_{-\infty}^{\infty} Q\,dx<\infty$ (and that Q be bounded and continuous), then (2.2) remains valid. Indeed the estimate can be strengthened to

$$-h'(x)=2\beta_0 h(x)(1+o(1))=Ce^{-2\beta_0 x}(1+o(1)),\qquad x\to\infty, \tag{2.3}$$

for some constant C and a similar estimate at $-\infty$. The proof is different of course. See [7], Chap. XI, Corollary 9.2. (Continued in Remark 1 in § 3.)

3. Proof of Theorem 1

In what follows we sometimes write P, P^B, etc., for P_0, P_0^B, etc.

Step 1. Fix $z_1>0$. There exist finite positive constants t_1, C_1, d_1 and d_2 with $d_1z_1-d_2>0$ such that

$$P[R_t>(\beta_0+z)t]\leqq C_1e^{-(d_1z-d_2)t} \tag{3.1}$$

for all $z\geqq z_1$ and $t\geqq t_1$.

Proof. From Lemma 2.1 we obtain

$$P[R_t>b]\leqq EZ_t(b,\infty)=E^B[e_Q(t),B_t>b]=M_tf_b(0) \tag{3.2}$$

where f_b is the indicator of (b,∞). The generator of (M_t) is the operator $(1/2)\,d^2/dx^2+Q$ restricted to an appropriate dense subset of $C_0(R)$. The eigenfunction h of Lemma 2.2 is in the domain and it follows that for all t and x

$$M_t h(x)=e^{\lambda_0 t}h(x).$$

We define a new semigroup (M_t^h) by

$$M_t^h f = e^{-\lambda_0 t}(1/h) M_t(hf). \tag{3.3}$$

Then $M_t^h 1 = 1$. From Lemma 2.1 and its Corollary it is clear that $M_t^h f(x)$ is continuous in x whenever f is bounded and continuous and that $M_t^h f \to f$ as $t \to 0$. Using some bounds on h from (2.2), one can in fact show that M_t^h: $C_0(R) \to C_0(R)$. Thus (M_t^h) is the transition semigroup of a strong Markov process Y and an elementary computation shows that its local generator is

$$(1/2)\, d^2/dy^2 + (h'/h)\, d/dy,$$

so Y is a diffusion. Now Y has an invariant probability distribution given by $\pi(dy) = c h(y)^2 dy$, $c = (\int h^2 dx)^{-1}$. Furthermore for any $x \geqq 0$, $P_0^Y[Y_t > y] \leqq P_x^Y[Y_t > y]$ as a simple coupling argument shows (see also McKean's *Stochastic Integrals*, p. 58, Exercise 4). It follows that

$$c \int_y^\infty h(x)^2\, dx \geqq \int_0^\infty P_z^Y[Y_t > y]\, \pi(dz) \geqq \pi(0, \infty) P_0[Y_t > y]$$

for all $t \geqq 0$, $y \geqq 0$. Fix ε, $0 < \varepsilon < \beta_0$. Then by Lemma (2.2) we have for all $x \geqq x_0$ sufficiently large

$$h(x) \leqq e^{-(2\beta_0 - \varepsilon)x} \quad \text{and} \quad h(x)^{-1} \leqq e^{(2\beta_0 + \varepsilon)x}.$$

So, for some constant C_2 and all $y \geqq x_0$,

$$P^Y[Y_t > y] \leqq C_2 e^{-(4\beta_0 - 2\varepsilon)y}.$$

From (3.2) and (3.3) (and $h(0) = 1$), we have

$$P[R_t > b] \leqq e^{\lambda_0 t} M_t^h(f_b/h)(0) = e^{\lambda_0 t} \int_b^\infty h(y)^{-1} P^Y[Y_t \in dy],$$

which yields, on integrating by parts and applying the preceding bounds,

$$P[R_t > b] \leqq C_1 \exp(\lambda_0 t - (2\beta_0 - 3\varepsilon) b)$$

for all $t \geqq 0$, $b \geqq x_0$, C_1 independent of t and b. Setting $b = (\beta_0 + z)t$ and prechoosing $\varepsilon > 0$ sufficiently small we get (3.1) with the obvious choice of constants.

Step 2. For any interval J and any $\lambda < \lambda_0$ we have

$$\lim_{t \to \infty} Z_t(J) e^{-\lambda t} = \infty \quad \text{a.s.} \tag{3.4}$$

Proof. Case 1. We will assume that for some $N \geqq 2$ the offspring probabilities $\{q_n(x)\}$ satisfy

$$q_n(x) = 0 \qquad \text{for all } n > N \text{ and all } x. \tag{3.5}$$

Then $\sum_{n=2}^{\infty} n^2 q_n(x) \leqq N^2 < \infty$ and the assumptions A, B, and C of Theorem 3.2 in Watanabe [13], p. 222, are easily checked. (Assumption C is trivially satisfied since λ_0 is an isolated point in the spectrum.) We conclude that as $t \to \infty$

$$Z_t(J) e^{-\lambda_0 t} \qquad \int_J h(x)\, dx \quad \text{a.s.} \tag{3.6}$$

for every bounded interval J where h is our ubiquitous eigenfunction and W is a random variable which, since $q_0 \equiv 0$, may be shown to satisfy

$$P_x[W > 0] = 1 \qquad \text{for all } x. \tag{3.7}$$

(For the interested reader, if $v(x)$ denotes the lefthand side of (3.7), then v satisfies (1.4) with $\partial v/\partial t$ set $=0$. This quickly leads to (3.7).) Clearly (3.6) and (3.7) imply (3.4).

Case 2. We now drop the assumption (3.5). Let $N \geqq 2$ be fixed but arbitrary. We define a process of tagged particles with the following rules: (i) The initial particle is tagged. (ii) If the number of offspring of a tagged particle at its splitting time is more than N, then, at the split time, N of its offspring are selected at random and tagged, the remaining offspring being left untagged. (iii) If the number of offspring of a tagged particle is no more than N, then all of them are tagged at the split time. (iv) Finally, no offspring of an untagged particle is ever tagged. (Of course the random selecting of the offspring to be tagged must be done independently of the positions and future evolution of all the particles, so the formal construction will require enlarging the original sample space in the usual manner. This we leave to the reader.) If $Z_t^{(N)}(J)$ denotes the number of tagged particles in J at time t, then by construction

$$P[Z_t^{(N)}(J) \leqq Z_t(J) \text{ for all } t] = 1. \tag{3.8}$$

The tagged process is a branching Brownian motion process with killing rate function $k(x)$ and offspring probabilities $q_j^{(N)}(x) = q_j(x)$, $j < N$; $= q_N(x) + q_{N+1}(x) + \ldots$, $j = N$; $=0$, $j > N$. From (3.8) and *Case 1* we get (3.4) for any $\lambda < \lambda_0^{(N)}$ $=$largest eigenvalue of $(1/2)d^2/dx^2 + Q^{(N)}$, $Q^{(N)} = k(\sum n q_n^{(N)} - 1)$. But, as $N \uparrow \infty$, $Q^{(N)} \uparrow Q$ (uniformly), so $\lambda_0^{(N)} \uparrow \lambda_0$ by standard comparison results in the spectral theory of differential equations and thus (3.4) holds for any $\lambda < \lambda_0$. (That $\lambda_0^{(N)} \uparrow \lambda_0$ is easily verified from the variational formula

$$\lambda_0 = \sup \left\{ \int_{-\infty}^{\infty} Q f^2\, dx + \tfrac{1}{2} \int_{-\infty}^{\infty} f'' f\, dx \right\}$$

where the sup is taken over $f \in C_0^2$ with $\int_{-\infty}^{\infty} f^2\, dx = 1.$)

Step 3. Fix $b > a$, $s_1 < s_2$. Then

$$P[R_{s_1} \geqq b,\ R_t \leqq a \text{ for some } s_1 \leqq t \leqq s_2] \leqq 2(1 - G((b-a)/(s_2 - s_1)^{1/2})), \tag{3.9}$$

and

$$P[R_{s_2} \leqq a,\ R_t \geqq b \text{ for some } s_1 \leqq t \leqq s_2] \leqq 1 - G((b-a)/(s_2-s_1)^{1/2}), \tag{3.10}$$

where G is the standard Gaussian distribution function:

$$G(z) = \int_{-\infty}^{z} e^{-x^2/2}\, dx/(2\pi)^{1/2}.$$

Proof. The event in (3.9) implies that some Brownian path was to the right of b at time s_1 and then the same path and all of the ther paths were to the left of a at some time t in (s_1, s_2). Ignoring the other paths and using the Markov property gives

$$\begin{aligned}\text{Lh.S.}\,(3.9) &\leqq \max_{y \geqq b} P_y^B[\min_{0 \leqq s \leqq s_2 - s_1} B_s \leqq a] \\ &= 2P^B[B_{s_2-s_1} \leqq -(b-a)] = \text{Rh.S.}\,(3.9).\end{aligned}$$

A similar argument leads to (3.10). (Condition on the stopped field F_T where $T = \min\{t\colon R_t = b\}$. Note that $\{R_t\}$ is a continuous process.)

Step 4. Let J be any interval contained in $[0, \infty)$. Then for all $t, b, r \geqq 0$, we have

$$P[R_t \leqq b,\ Z_{t/2}(J) \geqq r] \leqq G(b(2/t)^{1/2})^r. \tag{3.11}$$

Proof. Put $u(t,y) = P_y[R_t \leqq b] = E_y I[X_t^{(1)} \leqq b] \dots I[X_t^{(Z_t)} \leqq b]$. By the Markov property and the independence of particles, we have

$$\begin{aligned}&P[R_t \leqq b \mid X_{t/2} = [x_1, \dots, x_n],\ Z_{t/2} = n] \\ &\quad = P_{x_1}[R_{t/2} \leqq b]\, P_{x_2}[R_{t/2} \leqq b] \dots P_{x_n}[R_{t/2} \leqq b] \\ &\quad \leqq \prod_{i\colon x_i \in J} u(t/2, x_i).\end{aligned}$$

Integrating this over the event $[Z_{t/2}(J) \geqq r]$ gives

$$P[R_t \leqq b,\ Z_{t/2}(J) \geqq r] \leqq \max_{x \in J} u(t/2, x)^r.$$

But again, if we ignore all of the processes branching off of the initial path, we obtain a single Brownian motion. Hence

$$P_y[R_t \leqq b] \leqq P_y^B[B_t \leqq b] = G((b-y)/t^{1/2}). \tag{3.12}$$

Since $J \subset [0, \infty)$ (3.11) follows immediately.

Step 5. $\limsup_{t \to \infty} R_t/t \leqq \beta_0$ a.s.

Proof. From Step 1 and Step 3, (3.10), we have for $\delta > 0$

$$\sum_{n=1}^{\infty} P[R_n > (\beta_0 + \delta)n] < \infty$$

and

$$\begin{aligned}&P[R_{n+1} \leqq (\beta_0+\delta)(n+1),\ R_t \geqq (\beta_0 + 2\delta)(n+1) \text{ for some } t \text{ in } (n, n+1)] \\ &\quad \leqq 1 - G(\delta(n+1)) = O(e^{-\delta^2 n^2/2}).\end{aligned}$$

Since $\sum e^{-\delta^2 n^2/2}<\infty$ it follows that

$$P[R_n \leqq (\beta_0+\delta)n, \max_{n \leqq s \leqq n+1} R_s \leqq (\beta_0+2\delta)(n+1) \text{ for all } n \text{ suff. large}]=1$$

by the Borel-Cantelli Lemma. This concludes Step 5 since δ is arbitrary.

Step . $\liminf_{t\to\infty} R_t/t \geqq \beta_0$ a.s.

Proof. Fix an interval $J\subset[0,\infty)$, say $J=[0,1]$, fix $\beta<\beta_0$ and note that

$$\begin{aligned}
&[R_t \leqq \beta t \text{ for some } t \geqq n] \subset A_n \cup B_n,\\
&A_n=[Z_{t/2}[0,1] \leqq e^{\lambda t/2} \text{ for some } t \geqq n],\\
&B_n=[R_t \leqq \beta t \text{ for some } t \geqq n,\ Z_{t/2}[0,1] \geqq e^{\lambda t/2} \text{ for all } t \geqq n].
\end{aligned}$$

If $\lambda<\lambda_0$, then $PA_n\to 0$ as $n\to\infty$ by Step 2. For $\delta>0$ we have

$$\begin{aligned}
PB_n \leqq & \sum_{k\geqq n} P[R_k \leqq (\beta+\delta)k,\ Z_{k/2}[0,1] \geqq e^{\lambda k/2}]\\
&+\sum_{k\geqq n} P[R_k \geqq (\beta+\delta)k,\ R_t \leqq \beta k \text{ for some } t\in(k,k+1)]\\
\leqq & \sum_{k\geqq n} G((\beta+\delta)(2k)^{1/2})^{r(k)}+\sum_{k\geqq n} 2(1-G(\delta k))
\end{aligned}$$

by (3.11) and (3.9) where $r(k)=e^{\lambda k/2}$. But

$$G((\beta+\delta)(2k)^{1/2})^{r(k)}=O(\exp(-c k^{-1/2} r(k) e^{-(\beta+\delta)^2 k}))$$

for some $c>0$, so the first sum above converges and then goes to 0 as $n\to\infty$ provided we keep $2(\beta+\delta)^2<\lambda<\lambda_0$. Also $1-G(\delta k)=O(e^{-\delta^2 k^2/2})$ so the second sum also goes to 0 as $n\to\infty$. What all this says is that for any $\beta<\beta_0$

$$P[R_t \leqq \beta t \text{ for some } t \geqq n] \leqq PA_n+PB_n \to 0$$

as $n\to\infty$. But this is exactly the assertion of Step 6.

Steps 5 and 6 obviously imply the a.s. convergence assertion of Theorem 1.

Step 7. Mean convergence. Put $R_t^*=(1/t)R_t-\beta_0$. We will show

$$E|R_t^*|\to 0, \quad \text{as } t\to\infty.$$

(The proof is easily modified to show $E|R_t^*|^r\to 0$ for any $r>0$.) An integration by parts gives $E|R_t^*|=\int_0^\infty P[|R_t^*|>y]\,dy=I_1+I_2+I_3$ where

$$\begin{aligned}
I_1&=\int_0^\infty P[R_t^*>y]\,dy=\int_{\beta_0}^\infty P[R_t>\beta t]\,d\beta,\\
I_2&=\int_{-\beta_0}^0 P[R_t^*<y]\,dy=\int_0^{\beta_0} P[R_t<\beta t]\,d\beta,\\
I_3&=\int_{-\infty}^{-\beta_0} P[R_t^*<y]\,dy=\int_{-\infty}^0 P[R_t<\beta t]\,d\beta.
\end{aligned}$$

Fix $z_1>0$ but otherwise arbitrary. Applying Step 1 gives

$$I_1 \leqq z_1 + C_1 \int_{z_1}^{\infty} \exp(-(d_1 z - d_2)t)\,dz$$
$$= z_1 + O(e^{-at}), \qquad a = d_1 z_1 - d_2 > 0.$$

Letting $t\to\infty$ and then $z_1\to 0$ shows that $I_1\to 0$. Next $I_2\to 0$ by the a.s. convergence of R_t/t. Finally, from (3.12) we obtain $I_3 \leqq \int_0^{\infty} [1-G(\beta t^{1/2})]\,d\beta = O(t^{-1/2})\to 0$ as $t\to\infty$. This concludes the proof of Theorem 1.

Remark 1. Using the estimate (2.3) we can obtain the following strengthening of (3.1) in the case that Q is integrable (and bounded and continuous): For some constants $C_0>0$ and $t_1>0$ we have

$$P[R_t > \beta_0 t + x] \leqq C_0 e^{-2\beta_0 x}$$

for all $t \geqq t_1$, all $x>0$. This estimate in turn allows us to conclude that, in addition to (1.2), for any $\alpha > (2\beta_0)^{-1}$

$$P[R_t \leqq \beta_0 t + \alpha \log t \text{ for all sufficiently large } t] = 1.$$

We omit the details.

4. Additional Remarks

(1) A multidimensional version of Theorem 1 is easily proved with very much the same methods. For example consider the case of Brownian particles in R^d. Suppose that the offspring probabilities q_n and the function k are spherically symmetric. Let $Q=(m-1)k$ as before and let D_t be the distance of that particle which is furthest from the origin at time t. If $Q(x)\to 0$ as $\|x\|\to\infty$, then $D_t/t\to(\lambda_0/2)^{1/2}$ as $t\to\infty$, where λ_0 is the largest positive eigenvalue of the boundary value problem

$$(1/2)u''(r) + (d-1)u'(r)/2r + \tilde{Q}(r)u(r) = \lambda u(r), \qquad r>0, \quad \int_0^{\infty} u(r)^2 r^{d-1}\,dr < \infty$$

(and $u'(0)=0$ in the $d=1$ case). ($\tilde{Q}(r)=Q(x)$ for $\|x\|=r$.)

(2) The assumptions of Theorem 1 can be weakened. Suppose B^0, the nonbranching part of X, is the $\exp(-A_t)$-subprocess of B where $A_t = \int L(t,x)\,k\{dr\}$, $L=$local time, and k is a measure with $k(-a,a)<\infty$ for every $a>0$. Under very general conditions the expectation semigroup is given by $M_t f(x) = E_x \exp(Q_t) f(B_t)$ where $Q_t = \int L(t,x)(m(x)-1)\,k\{dx\}$. As long as the generator of (M_t) has a largest positive eigenvalue with an eigenfunction which decays exponentially at $\pm\infty$ (this will be the case if k has bounded support, for example), then one can expect asymptotic linear increase in the diameter of the branching process. However this property of the generator of (M_t) is certainly not necessary. See Note 4 in § 1.

(3) When the function k of Theorem 1 is unbounded the conclusion of Theorem 1 is not true. We obtain an interesting class of examples with

$$k(x)=|x|^r$$

for some $r>0$. Let us suppose that $q_2(x)=1$ for all x (so $m\equiv 2$). In this case it is known that $P[\tau_\infty<\infty]=0$ or 1 according as $0\leqq r\leqq 2$ or $r>2$. (τ_∞ is the explosion time.) See [9], pp. 209–10. Let us suppose $0<r<2$. Proceeding almost exactly as in [9], pp. 207–9, one can show

$$P[R_t\geqq t^q \text{ i.o. as } t\uparrow\infty]=1 \qquad \text{for } q<q_0:=2/(2-r). \tag{4.1}$$

With a little more effort one can also show $E^B\exp\left(\int_0^t |B_s|^r\,ds\right)=e^{O(t^p)}$ where $p=(2+r)/(2-r)$. Therefore, see (3.2),

$$\begin{aligned}\sum_n P[R_n\geqq n^q] &\leqq \sum_n (E^B e_Q(n)^2)^{1/2}(P^B[B_n\geqq n^q])^{1/2}\\ &\leqq \sum_n \exp(c\,n^p-(1/4)n^{2q-1})<\infty\end{aligned}$$

whenever $q>(p+1)/2=q_0$. Thus $P[R_n\geqq n^q \text{ i.o.}]=0$, $q>q_0$. This and (4.1) and a slight modification of Step 5, § 3, enable us to conclude: Almost surely

$$\limsup R_t/t^q=0 \qquad \text{or } \infty$$

according as $q>q_0$ or $q<q_0$. (In fact $\limsup R_t/t^{q_0}<\infty$.)

The case $r=2$ is particularly interesting. In this case the expectation semigroup is ∞ for all t sufficiently large, [9], p. 204, and R_t tends to increase faster than any power of t.

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