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Relative Stability of Trimmed Sums

R.A. Maller

CSIRO Division of Mathematics and Statistics, Wembley, W.A. 6014, Australia

Summary. This paper gives extensions of Mori's strong law for ${}^{(r)}S_n = S_n - X_n^{(1)} - \dots - X_n^{(r)}$, where $S_n = X_1 + X_2 + \dots + X_n$, X_i are iidrv's and $(X_n^{(i)})$ is (X_i) arranged in decreasing order of absolute magnitude. The methods differ from Mori's. Continuity of the distribution of the X_i is assumed throughout. Necessary and sufficient conditions for relative stability (${}^{(r)}S_n/B_n \to \pm 1$ a.s. for some B_n), including a generalised condition of Spitzer's and a dominated ergodic theorem, are proved. A one-sided version of the relative stability results is also given. A theorem of Kesten's is generalised to show that if $({}^{(r)}S_n - A_n)/B_n$ is bounded almost surely for constants $A_n, B_n \uparrow +\infty$ then $({}^{(r)}S_n - \alpha_n)/B_n \xrightarrow{P} 0$ for some α_n. A corollary to this is that if $|{}^{(r)}S_n|/B_n$ is bounded away from 0 and $+\infty$ a.s. then ${}^{(r)}S_n$ is relatively stable. This generalises a result of Chow and Robbins, apart from the continuity assumption.

1. Introduction and Statement of Results

Let X_i be a sequence of i.i.d. random variables, let $S_n = X_1 + X_2 + \dots + X_n$, and let $(X_n^{(i)})$ be $(X_i)_{i=1}^n$ arranged in decreasing order of absolute magnitude. Let ${}^{(0)}S_n = S_n$, and if r is a fixed integer $\geqq 1$ and $n > r$, let ${}^{(r)}S_n = S_n - X_n^{(1)} \dots - X_n^{(r)}$ be the (lightly)-trimmed sum.

The almost sure properties of ${}^{(r)}S_n$ were studied first by Feller [7]. In 1976 Mori [22] gave the following elegant analogue to the law of large numbers for ${}^{(r)}S_n$: $({}^{(r)}S_n - \alpha_n)/n \to 0$ *a.s.* $(n \to \infty)$ *for some constants* α_n *if and only if*

$$\int_0^\infty x^r H^{r+1}(x)\,dx < +\infty, \tag{1.1}$$

where $H(x) = P(|X_1| > x)$. This shows that the almost sure behaviour of S_n can be "improved" by trimming off a fixed number of extreme observations, in the sense that (1.1) may converge when $E|X_1| = +\infty$. Mori's result has been generalised in [23] and [11].

One of the purposes of the present paper is to generalise this result to the case of *relative stability* of ${}^{(r)}S_n$, which we first explain for the case of S_n. The sample sum S_n is relatively stable (in probability) if there are positive constants $B_n \uparrow +\infty$ for which S_n/B_n converges in probability to a finite non zero constant (which may be taken as $+1$ or -1 by rescaling B_n); write this as $S_n/B_n \xrightarrow{P} \pm 1$ $(n \to +\infty)$. Conditions on the distribution function F of the X_i for this to occur have been given by Hintčin [13], Rogozin [27] and Maller [17]. The relative stability *almost surely* of S_n is not of interest since it is known that if S_n/B_n is bounded almost surely for some $B_n \to +\infty$, then $E|X_1| < +\infty$ and so $S_n/n \to EX_1$ a.s.; see Chow and Robbins [4], and [17].

It is shown in Lemma 5.4 below that ${}^{(r)}S_n/B_n \xrightarrow{P} \pm 1$ $(n \to +\infty)$ if and only if $S_n/B_n \xrightarrow{P} \pm 1$ $(n \to +\infty)$; that is, trimming a fixed number of observations from the sample sum has no effect on its convergence in probability behaviour. What concerns us here is the almost sure relative stability of ${}^{(r)}S_n$. We show that trimming does have an effect on this mode of convergence.

Mori's technique for the proof of (1.1) uses a clever form of truncation and appeal to either a strong law of large numbers for non-identically distributed independent r.v.'s due to Prohorov, or to Prohorov's inequality. These methods do not generalise easily to the case of relative stability. The alternative methods presented here are nearer in spirit to the classical techniques for sums of iidrv's, in that they utilise ordinary truncation and a maximal inequality. An inequality due to Bennett takes the place of Prohorov's. The methods can be applied to obtain other results on the almost sure and iterated logarithm behaviour of the trimmed sum, as we hope to show elsewhere.

We make a blanket assumption of the continuity of the distribution of the X_i for the proofs in this paper. For some of the results this restriction can be dispensed with as in [22] and [24].

Some further notation is required: $F(x) = P(X_1 < x)$ is the *continuous* distribution of the X_i, and we always assume $H(x) = 1 - F(x) + F(-x) > 0$ for $x > 0$. Let

$$v(x) = \int_{-x}^{x} y\, dF(y), \qquad A(x) = \int_0^x G(y)\, dy,$$

$$G(x) = 1 - F(x) - F(-x), \qquad V(x) = \int_{-x}^{x} y^2\, dF(y). \tag{1.2}$$

The sample sum S_n is relatively stable (in probability) if and only if ([17]) $xH(x)/v(x) \to 0$ (equivalently, [18], $V(x)/x\,v(x) \to 0$), and if it is then $v(x) \sim A(x)$, $v(x)$ and $A(x)$ are of constant sign (positive if $S_n/B_n \xrightarrow{P} +1$, negative otherwise) for x large enough and are slowly varying as $x \to +\infty$. The sequence B_n for which $S_n/B_n \to \pm 1$ may be chosen to satisfy $B_n = n|v(B_n)|$ or $B_n = n|A(B_n)|$ for n large enough, and is regularly varying with index 1 as $n \to +\infty$.

Let ${}^{(r)}S_n(t)$ denote the polygonal function obtained by interpolating linearly between the points $(k/n, {}^{(r)}S_k/B_n)$ for $0 \leq k \leq n$ (where ${}^{(r)}S_k \equiv 0$ for $0 \leq k \leq r$). Our main result is:

Theorem 1. *The following are equivalent for* $r = 1, 2, \ldots$ and some positive $B_n \uparrow +\infty$:

$$ {}^{(r)}S_n/B_n \to \pm 1 \text{ a.s. } (n \to +\infty); \tag{1.3} $$

$$ \sup_{0 \leqq t \leqq 1} |{}^{(r)}S_n(t) \mp t| \to 0 \text{ a.s. } (n \to +\infty); \tag{1.4} $$

$$ \sum_{n \geqq 1} n^{-1} P\{|{}^{(r)}S_n \mp B_n| > \varepsilon B_n\} < +\infty \quad \text{for every } \varepsilon > 0; \tag{1.5} $$

$$ E \sup_{n>r} |{}^{(r)}S_n|/B_n < +\infty \quad \text{and} \quad B_n \sim n|v(B_n)|; \tag{1.6} $$

$$ \int_{x_0}^{\infty} \frac{x^r H^{r+1}(x)}{|v(x)|^{r+1}} dx < +\infty \quad \text{for some } x_0 \geqq 0. \tag{1.7} $$

The upper signs or the lower signs are to be taken together throughout.

Remarks. (i) (1.4) can be motivated by: ${}^{(r)}S_n(t)$ is close to ${}^{(r)}S_{[nt]}/B_n$, and the latter converges to $\pm t$ if ${}^{(r)}S_n$ is relatively stable, since then B_n is regularly varying with index 1. For ordinary relative stability, the following weak version can be proved: $S_n/B_n \xrightarrow{P} \pm 1$ if and only if $\sup_{0 \leqq t \leqq 1} |S_n(t) \mp t| \xrightarrow{P} 0$; see Rogozin [26] for applications.

(ii) Condition (1.6) is a "dominated ergodic" result for relative stability. For versions of this type of result for S_n see Gut [10], Teicher [30], and the papers referenced therein; for applications of uniform integrability see Klass [14, 15]. By methods similar to the proof of the equivalence of (1.6) and (1.7), it is possible to prove the following (cf. Theorem 3.2 of [9]):

if $0 < \alpha < 2$, $p > 0$ and $r = 0, 1, 2, \ldots$ the following are equivalent:

$$ E \sup_{n>r} [n^{-1/\alpha} |{}^{(r)}S_n - \alpha_n|]^p < +\infty; $$

$$ E \sup_{n>r} [n^{-1/\alpha} |X_n^{(r+1)}|]^p < +\infty; $$

and

$$ \int_0^{\infty} x^{(r+1)\alpha - 1} H^{r+1}(x)\, dx < +\infty \quad \text{if } r+1 > p/\alpha $$

or

$$ \int_1^{\infty} x^{p-1} \log x\, H^{r+1}(x)\, dx < +\infty \quad \text{if } r+1 = p/\alpha $$

or

$$ \int_0^{\infty} x^{p-1} H^{r+1}(x)\, dx < +\infty \quad \text{if } r+1 < p/\alpha. $$

The proof of these is omitted (continuity of F is not required, incidentally). A version of (1.6) with $\{|{}^{(r)}S_n|/B_n\}^p$ replacing $|{}^{(r)}S_n|/B_n$ can be proved if the appropriate change is made in (1.7).

(iii) The requirement that $B_n \sim n|v(B_n)|$ in (1.6) cannot be omitted since it is clearly possible to have $E \sup_{n>r} |{}^{(r)}S_n|/B_n < +\infty$ for constants B_n which have nothing to do with relative stability. Also (1.6) does not hold for $r = 0$ since then (as in the S_n/n case) an extra logarithmic term is required in (1.7).

(iv) On the other hand (1.3)–(1.5) and (1.7) are still equivalent for $r=0$ but only to $0<|EX_1|\leqq E|X_1|<+\infty$. Condition (1.5) is for $r=0$ a generalisation of a result of Spitzer [29] for the strong law of large numbers.

(v) A distribution for which (1.7) holds but $E|X_1|=+\infty$ is easily given.

(vi) "One-sided" versions of Theorem 1 consist of conditions under which $\left(S_n-\sum_{i=1}^{r} M_n^{(i)}\right)\Big/B_n\to 1$ a.s., where $M_n^{(r)}$ is the rth largest of X_i. For these, we assume $0<F(0)<1$. Then $M_n^{(r)}\uparrow+\infty$ a.s., so $\left(S_n-\sum_{i=1}^{r} M_n^{(i)}\right)\Big/B_n\to 1$ implies the dominance of the positive part of X_i over the negative part. We state the simplest version of such a result; it can be expanded as before.

Theorem 2. $(S_n-M_n^{(1)})/B_n\to 1$ a.s. *for some positive* $B_n\uparrow+\infty$ *if and only if*

$$\int_{x_0}^{\infty}\frac{x[1-F(x)]^2}{\left[\int_0^x u\,dF(u)\right]^2}dx<+\infty \quad \textit{and} \quad \int_{x_0}^{\infty}\frac{x|dF(-x)|}{\int_0^x u\,dF(u)}<+\infty. \tag{1.8}$$

If these hold, B_n *may be chosen to satisfy* $B_n=n\int_0^{B_n} x\,dF(x)$.

Remarks. (vii) $S_n/B_n\xrightarrow{P}1$, if (1.8) holds, as is shown in the proof of Theorem 2. Thus one-sided trimming has no effect on convergence of S_n of this type.

(viii) The integral

$$\int_1^{\infty}\left[\int_0^x u\,dF(u)\right]^{-1} x|dF(-x)|<+\infty \qquad \text{if and only if } X_n^-\Big/\sum_{i=1}^{n}X_i^+\to 0 \quad \text{a.s.},$$

where $X_i^+=\max(X_i,0)$, $X_i^-=|X_i|-X_i^+$ (see Erickson [6]).

When $E|X_1|=+\infty$, these are equivalent to $\sum_{i=1}^{n}X_i^-\Big/\sum_{i=1}^{n}X_i^+\to 0$ a.s. (Pruitt [25, Lemma 8.1]).

(ix) One sided versions of Mori's theorem can be proved in a similar way to Theorem 2; e.g., $(S_n-M_n^{(1)}-\alpha_n)/n\to 0$ a.s. for some α_n if and only if $\int_0^{\infty}x^2[1-F(x)]\,dF(x)<+\infty$ and $EX_1^-<+\infty$. We omit the details of these.

The next theorem generalises a result of Kesten [12, Lemma 4, p. 728] on the almost sure boundedness of S_n. Our proof of Theorem 3 uses a method of Rosalsky and Teicher [28], who generalised Kesten's theorem to the case of triangular arrays.

Theorem 3. *If there are constants* $A_n, B_n, B_n>0$, $B_n\uparrow+\infty$, $r=0,1,2,\ldots$, *for which* $\limsup_{n\to+\infty}|{}^{(r)}S_n-A_n|/B_n<+\infty$ a.s., *then* $(S_n-\alpha_n)/B_n\xrightarrow{P}0$ where $\alpha_n=n\,v(B_n)$.

This theorem has the following corollary, related to relative stability:

Corollary. *There are positive constants* $B_n\uparrow+\infty$, $r=0,1,2,\ldots$, *for which*

$$0<\liminf_{n\to+\infty}|{}^{(r)}S_n|/B_n\leqq\limsup_{n\to+\infty}|{}^{(r)}S_n|/B_n<+\infty \quad \text{a.s.} \tag{1.9}$$

if and only if (1.3) *holds.*

The remainder of the paper is laid out as follows: the three theorems and the corollary to Theorem 3 are proved in the next three sections. Sect. 5 contains some technical results and lemmas which are used throughout the other sections.

2. Proof of Theorem 1.

Throughout these proofs we consider only the case ${}^{(r)}S_n/B_n \xrightarrow{P} +1$; the other case can be handled similarly.

Suppose first that (1.3) holds (with a+sign). Then ${}^{(r)}S_n/B_n \xrightarrow{P} 1$, so by Lemma 5.4, S_n is relatively stable. Since $X_{n+1}/B_{n+1} \xrightarrow{P} 0$ we have $B_{n+1} \sim B_n$, so $({}^{(r)}S_{n+1} - {}^{(r)}S_n)/B_n \to 0$ a.s. The following is easily proved:

$$\sup_{r<j\leqq n} |{}^{(r)}S_j - {}^{(r)}S_{j-1}| = |X_n^{(r+1)}|, \quad 1 \leqq r < n, \tag{2.1}$$

from which follows $X_n^{(r+1)}/B_n \to 0$ a.s., i.e. (5.3) and hence (1.7). (See Lemma 5.4 below) Thus (1.3) implies (1.7).

Now suppose (1.7) holds. This implies the relative stability of S_n as follows. From the mean value theorem, using the continuity of F, and of v,

$$\begin{aligned}\sum_{n\geqq n_0} \xi_n^r H^{r+1}(\xi_n)|v(\xi_n)|^{-r-1} &= \sum_{n\geqq n_0} \int_{n-1}^{n} x^r H^{r+1}(x)|v(x)|^{-r-1}\,dx \\ &= \int_{x_0}^{\infty} x^r H^{r+1}(x)|v(x)|^{-r-1}\,dx < +\infty\end{aligned}$$

for some $\xi_n \varepsilon [n-1, n]$ and some n_0, x_0. Using a result of Loève [16, p. 277] there is then a sequence $n_i \uparrow +\infty$ with $n_{i+1}/n_i \to 1$ for which $\xi_{n_i}^{r+1} H^{r+1}(\xi_{n_i})|v(\xi_{n_i})|^{-r-1} \to 0$. Since $\xi_n \geqq \xi_{n-1}$ and $\xi_{n_{i+1}} \sim \xi_{n_i}$, we have $y_i H(y_i)/|v(y_i)| \to 0$ for a nondecreasing sequence y_i satisfying $y_{i+1} \sim y_i$. (Here and throughout the paper $\sim$ connects quantities whose ratio converges to 1). Now for x large choose $i = i(x)$ so that $y_i < x \leqq y_{i+1}$; then

$$\frac{xH(x)}{|v(x)|} \leqq \frac{y_{i+1}}{|v(y_i)|} \frac{H(y_i)}{|v(x)|} \frac{|v(y_i)|}{|v(x)|} = o(1)\frac{|v(y_i)|}{|v(x)|} = o(1) \qquad (x \to +\infty)$$

because

$$\left|\frac{v(x)}{v(y_i)} - 1\right| = \left|\int_{y_i}^{x} y\,dG(y)\right| \Big/ |v(y_i)| \leqq xH(y_i)/|v(y_i)| \leqq \frac{y_i H(y_i)}{|v(y_i)|} \frac{y_{i+1}}{y_i} \to 0 \quad \text{as } x \to +\infty.$$

Thus $xH(x)/v(x) \to 0$ as $x \to +\infty$, and this is equivalent to relative stability of S_n. (See the discussion after (1.2)). This means $S_n/B_n \to +1$, say, where $B_n > 0$ is a nondecreasing sequence which is regularly varying with index 1; in fact ([21]) B_n satisfies a condition strong enough (Condition (2.1) of [3]) that we can assume $n(B_{n+1}/B_n - 1) \to 1$ (Theorem 4 of [3]), which will be required shortly.

Still assuming that (1.7) holds, we now show that (1.5) does. Let B_n be the sequence whose existence was deduced in the previous paragraph. We want to apply the bound of Lemma 5.5, so we truncate and re-center S_n. Define for $0 \leqq j \leqq n$, $n \geqq 1$,

$$S_j^n = \sum_{i=1}^{j} X_i I(|X_i| \leqq \varepsilon B_n), \qquad \varepsilon > 0,\ S_0^n = 0$$

where I denotes the indicator function. From $(S_n - B_n)/B_n \xrightarrow{P} 0$ a symmetrisation argument ([16, p. 259]) gives [median $(S_n) - ES_n^n]/B_n \to 0$, and since $ES_n^n = n\,v(\varepsilon B_n) \sim B_n$, also $(S_n - ES_n^n)/B_n \xrightarrow{P} 0$. As in Mori [22] it follows that

$$\{|{}^{(r)}S_j - S_j^n| > (r+1)\,\varepsilon B_n\} \subseteq \{|X_j^{(r+1)}| > \varepsilon B_n\} \tag{2.2}$$

so

$$\Sigma\, n^{-1} P\{|{}^{(r)}S_n - B_n| > (5r+3)\,\varepsilon B_n\} \leqq \Sigma\, n^{-1} P\{|S_n^n - B_n| > 2(2r+1)\,\varepsilon B_n\} + \Sigma\, n^{-1} P\{|X_n^{(r+1)}| > \varepsilon B_n\}.$$

The second series on the right converges by (5.6). To deal with the first series on the right, we use Lemma 5.5, with $M = 2\varepsilon B_n$, $s_n^2 = n V(\varepsilon B_n)$, $t = 2(2r+1)\,\varepsilon B_n$ to give

$$\Sigma\, n^{-1} P\{|S_n^n - ES_n^n| > 2(2r+1)\,\varepsilon B_n\} \ll \Sigma\, n^{2r}\, V^{2r+1}(\varepsilon B_n)\, B_n^{-4r-2}$$

where the notation $\ll$ is used instead of the 0 notation and summations are taken over values of $n > n_0$, n_0 large. Since $B_{n+1} - B_n \sim n^{-1} B_n$, the last summation is

$$\ll \Sigma \int_{\varepsilon B_n}^{\varepsilon B_{n+1}} y^{-4r-3}\, V^{2r+1}(y)\, [B^{-1}(y)]^{2r+1}\, dy.$$

Also $B^{-1}(y) \sim y/v(y)$, as shown in the proof of Lemma 5.4, so we have to prove the convergence of the integral

$$I = \int_{y_0}^{\infty} y^{-2r-2}\, v^{-2r-1}(y)\, V^{2r+1}(y)\, dy, \tag{2.3}$$

given (1.7).

This we do by successively integrating by parts, continually using the facts $yH(y)/v(y) \to 0$ and $V(y)/yv(y) \to 0$, which follow from relative stability. We merely sketch this procedure here, discarding arbitrary constants so as to simplify the notation. At the first stage we obtain

$$I \ll \int_{y_0}^{\infty} y^{-2r+1}\, v^{-2r-1}(y)\, V^{2r}(y)\, dh(y) - \int_{y_0}^{\infty} y^{-2r}\, v^{-2r-2}(y)\, V^{2r+1}(y)\, dG(y)$$

where $dh(y) = -dH(y)$, $G(y) = 1 - F(y) - F(-y)$. Now G is not monotone but $|dG(y)| \leqq dh(y)$ and $V(y)/y\,v(y) \to 0$, so the second integral is of smaller order than the first. Thus we need only deal with the first, and this we denote I_1 and integrate by parts, (integrating $-dH(y)$ to $H(y)$), to obtain, apart from a constant,

$$I_1 = 2r \int_{y_0}^{\infty} y^{-2r+3} v^{-2r-1}(y) V^{2r-1}(y) H(y) dh(y)$$
$$-(2r-1) \int_{y_0}^{\infty} y^{-2r} v^{-2r-1}(y) V^{2r}(y) H(y) dy$$
$$+(2r+1) \int_{y_0}^{\infty} y^{-2r+2} v^{-2r-2}(y) V^{2r}(y) H(y) dG(y).$$

From this we discard the (negative) second integral and note that the third integral is of smaller order than the first, so we again need only consider the first integral; call it I_2. Integrating by parts and again discarding a negative term and a constant, we have

$$I_2 \leqq 2r(2r-1) \int_{y_0}^{\infty} y^{-2r+5} v^{-2r-1}(y) V^{2r-2}(y) H^2(y) dh(y)$$
$$+2r(2r+1) \int_{y_0}^{\infty} y^{-2r+4} v^{-2r-2}(y) V^{2r-1}(y) H^2(y) dG(y)$$

where again we ignore the second integral because it is of smaller order than the first. This procedure can be repeated r times to obtain

$$I_r \leqq (2r)! \int_{y_0}^{\infty} y v^{-2r-1}(y) V^r(y) H^r(y) dh(y) \ll \int_{y_0}^{\infty} y^{r+1} v^{-r-1}(y) H^r(y) dh(y).$$

A final integration by parts on this last integral (call it I_r') gives

$$I_r' \ll \int_{y_0}^{\infty} y^r v^{-r-1}(y) H^{r+1}(y) dy - \int_{y_0}^{\infty} y^{r+2} v^{-r-2}(y) H^{r+1}(y) dG(y).$$

Since $yH(y)/v(y) \to 0$, the second integral is of smaller order than I_r'. The first integral converges by (1.7). Thus we have shown that (1.7) implies (1.5) for continuous F.

Now let (1.5) hold. We show that this implies $S_n/B_n \xrightarrow{P} 1$. By Lemma 2 of Egorov [5], there is a sequence $\varepsilon_n \downarrow 0$ such that $\Sigma n^{-1} P\{|{}^{(r)}S_n - B_n| > \varepsilon_n B_n\} < +\infty$, hence there is a sequence $n_i \uparrow +\infty$ with $n_{i+1} \sim n_i$ such that $P\{|{}^{(r)}S_{n_i} - B_{n_i}| > \varepsilon_{n_i} B_{n_i}\} \to 0$. This means ${}^{(r)}S_{n_i}/B_{n_i} \xrightarrow{P} 1$, so from Lemma 5.4, $S_{n_i}/B_{n_i} \xrightarrow{P} 1$, $n_i H(B_{n_i}) \to 0$, and $B_{n_i} \sim n_i A(B_{n_i})$.

We want to show $B_{n_{i+1}} \sim B_{n_i}$, and to do this we let $B_{n_i} = C_i n_i A(B_{n_i})$ where $C_i \to 1$. Choosing i so large that

$$\left| \frac{C_{i+1}}{C_i} \frac{n_{i+1}}{n_i} - 1 \right| \leqq \delta, \quad B_{n_i} \leqq (1+\delta) n_i A(B_{n_i}) \quad \text{and} \quad n_i H(B_{n_i}) \leqq \delta/(1+\delta)^2,$$

we have

$$\left|\frac{B_{n_{i+1}}}{B_{n_i}}-1\right|=\left|\frac{C_{i+1}n_{i+1}A(B_{n_{i+1}})}{C_i n_i A(B_{n_i})}-1\right|\leqq\left|\frac{C_{i+1}n_{i+1}}{C_i n_i}-1\right|+\frac{C_{i+1}n_{i+1}}{C_i n_i}\left|\frac{A(B_{n_{i+1}})}{A(B_{n_i})}-1\right|$$

$$\leqq\delta+(1+\delta)\left|\int_{B_{n_i}}^{B_{n_{i+1}}} G(u)\,du\right| \Big/ |A(B_{n_i})|$$

$$\leqq\delta+(1+\delta)n_i H(B_{n_i})\left|\frac{B_{n_{i+1}}}{B_{n_i}}-1\right|\frac{B_{n_i}}{n_i A(B_{n_i})}\leqq\delta+\delta\left|\frac{B_{n_{i+1}}}{B_{n_i}}-1\right|$$

which gives $\left|\frac{B_{n_{i+1}}}{B_{n_i}}-1\right|\leqq\delta/(1-\delta)$. Thus $B_{n_{i+1}}\sim B_{n_i}$, and $B_{n_i}H(B_{n_i})/A(B_{n_i})\to 0$, and standard methods give $xH(x)/A(x)\to 0$, which, since $A(x)=xG(x)+v(x)$ and $|G|\leqq H$, means $xH(x)/v(x)\to 0$. So F is relatively stable and $S_n/b_n\xrightarrow{P}1$ for some b_n. Since we must have $b_{n_i}\sim B_{n_i}$ and $n_{i+1}\sim n_i$, this implies $b_n\sim B_n$ and so $S_n/B_n\xrightarrow{P}1$ as required.

Still assuming that (1.5) holds, we now show that ${}^{(r)}S_n/B_n\to 1$ a.s. If $x>0$ and $\lambda_k=[\lambda^k]$, $\lambda>1$,

$P\{|{}^{(r)}S_n-B_n|>xB_n$ infinitely often$\}$

$$=\lim_m P\Big(\bigcup_{j>m}\{|{}^{(r)}S_j-B_j|>xB_j\}\Big)\leqq\lim_m P\Big(\bigcup_{k>\log m}\ \bigcup_{j=\lambda_k+1}^{\lambda_{k+1}}\{|{}^{(r)}S_j-B_j|>xB_j\}\Big)$$

$$\leqq\lim_m\sum P\{\sup_{\lambda_k<j\leqq\lambda_{k+1}}|{}^{(r)}S_j-B_j|>xB_{\lambda_k}\}$$

$$\leqq c\lim_m\sum_k\sum_{n=\lambda_{k+1}+1}^{\lambda_{k+2}} n^{-1}P\{\sup_{\lambda_k<j\leqq n}|{}^{(r)}S_j-B_j|>xB_{[\lambda^{-2}n]}\}\tag{2.4}$$

the last inequality following from $\sum_{\lambda_k+1}^{\lambda_{k+1}} n^{-1}\leqq(\lambda_{k+1}-\lambda_k)\lambda_k^{-1}\sim(\lambda-1)$. Applying Lemma 5.1 with ε of that Lemma replaced by $x/2(2r+4)$ shows that the last series is

$$\leqq c\lim_m\sum_{n>m} n^{-1}P\{|{}^{(r)}S_n-B_n|>\tfrac{1}{2}xB_{[\lambda^{-2}n]}\}$$
$$+c\lim_m\sum_{n>m} n^{-1}P\{|X_n^{(r+1)}|>x/2(2r+4)B_{[\lambda^{-2}n]}\}.$$

The convergence of the last two series follows from (1.5) and Lemma 5.2 below. (The argument $\lambda^{-2}n$ of B can be replaced by n since B is regularly varying). Since $x>0$ is arbitrary, (1.3) holds.

Next, (1.3) implies (1.4) as follows. By the linear nature of $S_n(t)$ (and ${}^{(r)}S_k\equiv 0$ for $0\leqq k\leqq r$) we have (taking the + sign as usual)

$$\sup_{0\leqq t\leqq 1}|{}^{(r)}S_n(t)-t|=\sup_{r<k\leqq n}\left|\frac{{}^{(r)}S_k}{B_n}-\frac{k}{n}\right|+\sup_{0\leqq k\leqq r}\left|\frac{k}{n}\right|$$

where ${}^{(r)}S_n(t)=0$ for $0\leqq t\leqq r/n$. Thus it suffices to show from (1.3) that $\sup_{k_0\leqq k\leqq n}\left|\frac{{}^{(r)}S_k}{B_n}-\frac{k}{n}\right|$ can be made arbitrarily small if $n>k_0$ and k_0 is large enough. It is elementary to show that

$$\sup_{k_0\leqq k\leqq n}\left|\frac{{}^{(r)}S_k}{C_n}-\frac{k}{n}\right|=\sup_{k_0\leqq k\leqq n}\left|\frac{{}^{(r)}S_k}{B_n}-\frac{k}{n}\right|+o(1) \text{ a.s.}$$

if C_n is another sequence for which ${}^{(r)}S_n/C_n\to 1$ a.s. Thus we can assume B_n is the special (nondecreasing) sequence satisfying $B_n=nA(B_n)$. Then

$$\begin{aligned}\sup_{k_0\leqq k\leqq n}\left|\frac{{}^{(r)}S_k}{B_n}-\frac{k}{n}\right| &= \sup_{k_0\leqq k\leqq n}\left|\frac{{}^{(r)}S_k}{B_k}-\frac{kB_n}{nB_k}\right|\frac{B_k}{B_n}\\ &\leqq \sup_{k_0\leqq k\leqq n}\left|\frac{{}^{(r)}S_k}{B_k}-1\right|+\sup_{k_0\leqq k\leqq n}B_n^{-1}n^{-1}|nB_k-kB_n|\end{aligned}$$

and $\sup_{k_0\leqq k\leqq n}\left|\frac{{}^{(r)}S_k}{B_k}-1\right|$ can be made small for k_0 large and $n>k_0$ since ${}^{(r)}S_n/B_n\to 1$ a.s. Also for $n\geqq k$

$$\begin{aligned}B_n^{-1}n^{-1}|nB_k-kB_n| &= B_n^{-1}k|A(B_k)-A(B_n)|\\ &= B_n^{-1}k\left|\int_{B_k}^{B_n}[1-F(u)-F(-u)]\,du\right|\leqq kH(B_k)\end{aligned}$$

which is small for $k\geqq k_0$ since $kH(B_k)\to 0$. Thus (1.4) holds, so (1.4) and (1.3) are equivalent since (1.4) clearly implies (1.3).

Finally we show the equivalence of (1.6) and (1.7). First let (1.7) hold. This implies $S_n/B_n \xrightarrow{P} +1$ (as usual we take the + sign) where $B_n\sim n\,v(B_n)$. It also implies $E\sup_{n>r}|X_n^{(r+1)}|/B_n<+\infty$; because, for this it suffices that $\int_1^\infty P\{\sup|X_n^{(r+1)}|/B_n>x\}\,dx$ be finite, and for this (by Lemma 5.7) it suffices that $\sum n^r\int_{x_0}^\infty H^{r+1}(xB_n)\,dx$ be finite for some $x_0>0$. The last series is bounded by a multiple of (noting that B_n is regularly varying with index 1)

$$\begin{aligned}\sum\int_{B_{j-1}}^{B_j}H^{r+1}(x)\,dx\sum_{n=1}^{j}n^rB_n^{-1} &\ll \sum_j\int_{B_{j-1}}^{B_j}H^{r+1}(x)\,dx\,(j-1)^{r+1}B_j^{-1}\\ &\ll \int_{x_0}^\infty x^{-1}[B^{-1}(x)]^{r+1}H^{r+1}(x)\,dx \ll \int_{x_0}^\infty x^r|v(x)|^{-r-1}H^{r+1}(x)\,dx,\end{aligned}$$

since $B^{-1}(x)\sim x\,v(x)$ (see Lemma 5.4), and so is convergent by (1.7). Similarly, $\Sigma\lambda_j^{r+1}\int_{x_0}^\infty H^{r+1}(xB_{\lambda_j})\,dx$ is finite.

Next, using the maximal inequality of Lemma 5.1 with $\varepsilon=\frac{1}{2}x/(2r+4)\geqq\varepsilon_0=\frac{1}{2}/(2r+4)$ $(x\geqq 1)$ we obtain for x_0, N_0, j_0 large enough

$$E\sup_{n>n_0}|^{(r)}S_n|/B_n\leqq E\sup_{j>j_0}\sup_{\lambda_j<n\leqq\lambda_{j+1}}|^{(r)}S_n-B_n|/B_n+1$$

$$\leqq x_0+\sum_j\int_{x_0}^{\infty}P\{\sup_{\lambda_j<n\leqq\lambda_{j+1}}|^{(r)}S_n-B_n|>xB_{\lambda_j}\}\,dx$$

$$\ll\sum_j\int_{x_0}^{\infty}P\{|^{(r)}S_{\lambda_{j+1}}-B_{\lambda_{j+1}}|>\tfrac{1}{2}xB_{\lambda_j}\}\,dx$$

$$+\sum_j\int_{x_0}^{\infty}P\{|X_{\lambda_{j+1}}^{(r+1)}|>\tfrac{1}{2}xB_{\lambda_j}/(2r+4)\}\,dx.$$

By Lemma 5.6 (of course $h^{-r-1}(x)\to 1$ as $x\to+\infty$) the second series is bounded by a multiple of $\Sigma\lambda_j^{r+1}\int_{x_0}^{\infty}H^{r+1}(xB_{\lambda_j})\,dx$, which is finite. Remembering that B_n is regularly varying with index 1, the first series is bounded by a multiple of

$$\sum_j\sum_{\lambda_j<n\leqq\lambda_{j+1}}n^{-1}\int_{x_0}^{\infty}P\{|^{(r)}S_{\lambda_j}-B_{\lambda_j}|>xB_{\lambda_j}\}\,dx$$

$$\leqq\sum_j\sum_{\lambda_j<n\leqq\lambda_{j+1}}n^{-1}\int_{x_0}^{\infty}P\{\sup_{\lambda_j\leqq k\leqq n}|^{(r)}S_k-B_k|>xB_{\lambda_j}\}\,dx$$

$$\ll\sum_n n^{-1}\int_{x_0}^{\infty}P\{|^{(r)}S_n-B_n|>(5r+3)xB_n\}\,dx+c$$

the last step following by a change of variable. (We use c to denote a finite constant resulting from another application of Lemma 5.1).

Now truncate at xB_n and use the inequality following (2.2) (with x replacing ε) to replace $^{(r)}S_n$ with $S_n^n=\sum_{i=1}^{n}X_i I(|X_i|\leqq xB_n)$. We can also re-center at $nv(xB_n)=ES_n^n$, since $B_n-nv(B_n)=o(B_n)$ and for $x\geqq 1$ and n large

$$n|v(B_n)-v(xB_n)|=n\left|\int_{B_n}^{xB_n}u\,d[1-F(u)-F(-u)]\right|\leqq n\,x\,B_n\,H(B_n)\leqq\tfrac{1}{2}xB_n$$

since $nH(B_n)\to 0$. Thus it will suffice to prove the convergence of

$$\Sigma n^{-1}\int_{x_0}^{\infty}P\{|S_n^n-n\,v(xB_n)|>2(2r+1)xB_n\}\,dx.$$

But by Lemma 5.5 with $M=2xB_n$, $t=2(2r+1)xB_n$, $s_n^2=nV(xB_n)$, this series is

$$\ll\Sigma n^{-1}\int_{x_0}^{\infty}n^{2r+1}V^{2r+1}(xB_n)\,dx/x^{4r+2}B_n^{4r+2}$$

$$\ll\int_{x_0}^{\infty}x^{-2r-2}V^{2r+1}(x)\,v^{-2r-1}(x)\,dx.$$

This is (2.3) and it converges by (1.7).

Now let (1.6) hold. Then $\sup_{n>r}|^{(r)}S_n-B_n|/B_n<+\infty$ a.s., so (anticipating the result of Theorem 3, which is proved independently) we have $(S_n-nv(B_n))/B_n\xrightarrow{P}0$.

Since we assume in (1.6) that $B_n \sim n|\nu(B_n)|$, each sequence of integers contains a subsequence for which $S_{n_i}/B_{n_i} \xrightarrow{P} \pm 1$. By Theorem 2 of [17] this means that S_n is relatively stable, $S_n/B_n \xrightarrow{P} +1$ say, and that $B_{n+1} \sim B_n$. Then by (2.1)

$$|X_n^{(r+1)}|/B_n = \sup_{r<j\leq n} |({}^{(r)}S_j - B_j) - ({}^{(r)}S_{j-1} - B_{j-1})|/B_n + 0(1)$$
$$\leq 2 \sup_{r<j\leq n} |{}^{(r)}S_j - B_j|/B_j + 0(1)$$

giving $E \sup_{n>r} |X_n^{(r+1)}|/B_n < +\infty$. By Lemma 5.7 then, $\Sigma n^r \int_{x_0}^{\infty} H^{r+1}(xB_n)\,dx$ converges, and by the usual manipulations this implies (1.7). The proof is complete.

3. Proof of Theorem 2

Suppose $(S_n - M_n^{(1)})/B_n \to 1$ a.s. Then $(S_{n-1} - M_{n-1}^{(1)})/B_n \xrightarrow{P} 1$, because $M_n^{(1)}$ and $M_{n-1}^{(1)}$ can only differ if $X_n > M_{n-1}^{(1)}$, and the probability of this is $n^{-1} \to 0$. These mean that $B_n \sim B_{n-1}$, so actually $(S_{n-1} - M_{n-1}^{(1)})/B_n \to 1$ a.s. Then (c.f. (2.1)) $M_n^{(2)}/B_n \to 0$ a.s. Now $M_n^{(2)}/B_n \xrightarrow{P} 0$ implies $M_n^{(1)}/B_n \xrightarrow{P} 0$ by proceeding as in Lemma 5.3. Also, then, $M_n^{(2)}/B_n \to 0$ a.s. implies $\Sigma n[1 - F(\varepsilon B_n)]^2 < +\infty$. In addition, $S_n/B_n \xrightarrow{P} 1$, so B_n may be taken to satisfy $B_n = n\,\nu(B_n)$ where $\nu(x) > 0$ for x large enough. From the convergence of $\Sigma n[1 - F(\varepsilon B_n)]^2$ follows that of $\int_1^{\infty} \nu^{-2}(x)\,x[1 - F(x)]^2\,dx$. Clearly

$$\nu(x) = \int_{-x}^{x} u\,dF(u) \leq \int_0^x u\,dF(u), \quad \text{so} \quad \int_1^{\infty} \left[\int_0^x u\,dF(u)\right]^{-2} x[1 - F(x)]^2\,dx$$

converges, from which we deduce by applying Theorem 1 to X_i^+ that

$$\left(\sum_{i=1}^{n} X_i^+ - \sup_{1\leq i\leq n} X_i^+\right) \Big/ C_n \to 1 \text{ a.s.} \quad \text{where} \quad C_n = n \int_0^{C_n} x\,dF(x) \geq B_n.$$

Thus $|S_n - M_n^{(1)}|/C_n$ is bounded almost surely and since $(M_n^{(1)} - \sup_{1\leq i\leq n} X_i^+)/C_n \to 0$ a.s. (easily checked), $\sum_{i=1}^{n} X_i^-/C_n$ is bounded almost surely. This means $\Sigma F(-\varepsilon C_n)$ is bounded for some $\varepsilon > 0$, hence for all $\varepsilon > 0$, since C_n is regularly varying with index 1 [17 Lemma 1]. The convergence of $\Sigma F(-C_n)$ then implies that of $\int_1^{\infty} \left[\int_0^x u\,dF(u)\right]^{-1} x|dF(-x)|$, which means further that

$$\int_0^x u|dF(-u)| \Big/ \int_0^x u\,dF(u) \to 0, \quad \text{so} \quad B_n = \int_{-B_n}^{B_n} x\,dF(x) \sim n \int_0^{B_n} x\,dF(x).$$

Conversely, convergence of $\int_1^{\infty} \left[\int_0^x u\,dF(u)\right]^{-2} x(1 - F(x))^2\,dx$ means by Theorem 1 that $(\Sigma X_i^+ - \sup X_i^+)/B_n \to 1$ a.s., where $B_n = n \int_0^{B_n} x\,dF(x)$ and B_n is regularly varying with index 1. Convergence of

$$\int_1^\infty \left[\int_0^x u\,dF(u)\right]^{-1} x|dF(-x)|$$

then implies $\Sigma F(-B_n)<+\infty$, hence $\Sigma F(-\varepsilon B_n)<+\infty$ for $\varepsilon>0$, from which $(\Sigma X_i^- - n\int_0^{B_n} x|dF(-x)|)/B_n \to 0$ a.s. follows by standard methods. Now

$$\int_0^x u|dF(-u)| \Big/ \int_0^x u\,dF(u) \to 0, \quad \text{so} \quad n\int_0^{B_n} x\,dF(-x)=O(B_n),$$

hence $\Sigma X_i^-/B_n\to 0$ a.s., leading to $(S_n-\sup X_i^+)/B_n\to 1$ a.s., hence $(S_n - M_n^{(1)})/B_n\to 1$ a.s.

4. Proof of Theorem 3 and the Corollary

We require the following representations of the distributions of ${}^{(r)}S_n$ and $X_n^{(r)}$, which are implicit in e.g. Arov and Bobrov [1]. If $y>0$ define $X_i(y)\overset{D}{=}(X_i\,|\,|X_i|\leqq y)$, and let $S_n(y)=\sum_{i=1}^n X_i(y)$. Using the continuity of F, we have

$$P\{{}^{(r)}S_n<x\}=\int_0^\infty P\{S_{n-r}(y)<x\}\,dg_n^r(y), \tag{4.1}$$

$$P\{X_n^{(r+1)}<x\}=\int_0^\infty P\{\sup_{1\leqq i\leqq n-r} X_i(y)<x\}\,dg_n^r(y), \tag{4.2}$$

where $dg_n^r(y)=r\binom{n}{r}h^{n-r}(y)[1-h(y)]^{r-1}dh(y)$, and $h(y)=1-H(y)=P(|X_1|\leqq y)$.

We now restrict ourselves to $r\geqq 1$, and proceed by showing that

$$\limsup |{}^{(r)}S_n - A_n|/B_n<+\infty \text{ a.s.}$$

implies $X_n^{(r+1)}/B_n \xrightarrow{P} 0$. If this were not so, we could find $\varepsilon,\delta>0$ and an infinite sequence N for which $P\{|X_n^{(r+1)}|>2\varepsilon B_n\}\geqq 3\delta(r-1)!$ when $n\in N$. Then for such n,

$$\begin{aligned} 3\delta(r-1)! &\leqq P\{|X^{(r+1)}|>2\varepsilon B_n\}=\int_0^\infty P\{\sup_{1\leqq i\leqq n-r}|X_i(y)|>2\varepsilon B_n\}\,dg_n^r(y)\\ &\leqq \int^\infty P\{\sup_{1\leqq i\leqq n-r}|X_i(y)-m(y)|>\varepsilon B_n\}\,dg_n^r(y)\\ &\leqq 2\int_0^\infty P\{\sup_{1\leqq i\leqq n-r}|X_i^s(y)|>\varepsilon B_n\}\,dg_n^r(y) \end{aligned} \tag{4.3}$$

by [16, p. 259], where $X_i^s(y)$ are symmetrised versions of $X_i(y)$ and $m(y)$ is a median of $X_1(y)$. We used the fact, easily verified, that $|m(y)-m|\leqq c$ for some c, where m is a median of X_1, then assumed N contains only integers large enough for $(|m|+c)/B_n\leqq\varepsilon$.

Make the transformation $z=nH(y)$ in (4.3), let $y_n(z)=H^{-1}(zn^{-1})$ where H^{-1} denotes the left-continuous inverse of H, and note that

$$P\{\sup_{1\leq i\leq n-r}|X_i^s(y_n(z))|\geqq \varepsilon B_n\}$$

is, for each n, a nonincreasing function of z. Thus by Helly's theorem a further subsequence of N can be taken, if necessary, so that

$$P\{\sup_{1\leq i\leq n-r}|X_i^s(y_n(z))|>\varepsilon B_n\}\to f(z),$$

a nonincreasing function of z. Note that $e^{(n-r)\log(1-z/n)}$ is bounded by $e^{-z/(1-\eta)}$ in $0\leqq z\leqq \eta n$, $\eta<1$, and by $e^{n\log(1-\eta)}$ in $\eta n<z<n$, so it is easy to deduce from (4.3) that

$$2\int_0^\infty f(z)z^{r-1}e^{-z}\,dz=2\lim_{n\in N}\int_0^\infty P\{\sup_{1\leq i\leq n-r}|X_i^s(y_n(z))|>\varepsilon B_n\}\,e^{n\log(1-z/n)}z^{r-1}\,dz$$

$$\geqq 3\delta(r-1)! \tag{4.4}$$

This means $f(z_0)>\delta$ for some $z_0>0$, so by further restricting N if necessary,

$$P\{\sup_{1\leq i\leq n}|X_i^s(y_n(z_0))|>\varepsilon B_n\}\geqq\delta,\qquad n\in N.$$

Applying now a result due to Rosalsky and Teicher [28], for any integer $m>1$ there are integers $0=v_0<v_1<\ldots<v_m<n-r$, depending on N, m, and z_0, such that for $0<z\leqq z_0$,

$$\min_{1\leq k\leq m}P\{\sup_{i\in I_k}|X_i^s(y_n(z))|>\varepsilon B_n\}\geqq\min_{1\leq k\leq m}P\{\sup_{i\in I_k}|X_i^s(y_n(z_0))|>\varepsilon B_n\}\geqq\delta/2m$$

where $I_k=(v_{k-1},v_k]$, $1\leqq k\leqq m$.

Note that, again by symmetrisation, ([16, p. 259),

$$2P\{|^{(r)}S_n-A_n|>m\varepsilon B_n\}=2\int_0^\infty P\{|S_{n-r}(y)-A_n|>m\varepsilon B_n\}\,dg_n^r(y)$$

$$\geqq\int_0^\infty P\{|S_{n-r}^s(y)|>2m\varepsilon B_n\}\,dg_n^r(y)=2\int_0^\infty P\{S_{n-r}^s(y)>2m\varepsilon B_n\}\,dg_n^r(y)$$

$$\geqq 2\int_0^\infty P\{\min_{1\leq k\leq m}\sum_{i\in I_k}X_i^s(y)>2\varepsilon B_n\}\,P\left\{\sum_{i=v_m+1}^n X_i^s(y)\geqq 0\right\}dg_n^r(y)$$

$$\geqq\int_0^\infty\prod_{k=1}^m P\{\sum_{i\in I_k}X_i^s(y)>2\varepsilon B_n\}\,dg_n^r(y)$$

$$\geqq\frac{1}{2}\int_0^\infty\prod_{k=1}^m P\left\{\sup_{i\in I_k}\left|\sum_{j=v_{k-1}+1}^i X_j^s(y)\right|>2\varepsilon B_n\right\}dg_n^r(y)$$

$$\geqq\frac{1}{2}\int_0^\infty\prod_{k=1}^m P\{\sup_{i\in I_k}|X_i^s(y)|>\varepsilon B_n\}\,dg_n^r(y)$$

$$\geqq\frac{1}{2}\,\frac{(n-1)\ldots(n-r+1)}{n^{r-1}(r-1)!}\int_0^{z_0}\Pi P\{\sup_{i\in I_k}|X_i^s(y_n(z))|>\varepsilon B_n\}e^{n\log(1-z/n)}z^{r-1}\,dz,$$

where in the last inequality the variable was changed to $z=nH(y)$, as in the steps leading to (4.4).

By Fatou's Lemma, we now deduce

$$\liminf_{n\in N} P\{|{}^{(r)}S_n - A_n| > \tfrac{1}{2} m\varepsilon B_n\} \geqq \frac{1}{(r-1)!}\left(\frac{\delta}{2m}\right)^m \int_0^{z_0} e^{-z} z^{r-1}\, dz > 0$$

which, since m is arbitrary, means $\limsup_{n\in N} |{}^{(r)}S_n - A_n|/B_n = +\infty$ a.s., a contradiction.

This shows that $X_n^{(r+1)}/B_n \xrightarrow{P} 0$, giving further $nP(|X_1| > \varepsilon B_n) \to 0$ for $\varepsilon > 0$ (Lemma 5.3), and so, as in [28] or [12], $nB_n^{-2} V(\varepsilon B_n) \to 0$. Hence $(S_n - \alpha_n)/B_n \xrightarrow{P} 0$ for some α_n.

To prove the Corollary, we have by Theorem 3 that $\limsup |{}^{(r)}S_n|/B_n < +\infty$ a.s. implies $(S_n - \alpha_n)/B_n \xrightarrow{P} 0$, equivalently (Lemma 5.3), $({}^{(r)}S_n - \alpha_n)/B_n \xrightarrow{P} 0$. Since α_n may be chosen as $nv(B_n)$, we must have $n|v(B_n)|/B_n$ bounded away from 0 and $+\infty$; if not, ${}^{(r)}S_n/B_n \xrightarrow{P} 0$ or $\pm\infty$ for a subsequence and ${}^{(r)}S_n/B_n \to 0$ or $\pm\infty$ a.s. for a further subsequence, contradicting (1.9). Thus every sequence of integers contains a subsequence through which $nv(B_n)/B_n \to -C$, $0 < |C| < +\infty$, i.e. $S_n/B_n \xrightarrow{P} C$ through this subsequence. By Theorem 2 of [17] this means S_n is relatively stable so ${}^{(r)}S_n/C_n \xrightarrow{P} \pm 1$ for some C_n. If C_n/B_n contained a subsequence converging to 0 or $+\infty$ we could take a further subsequence through which ${}^{(r)}S_n/C_n \to \pm 1$ a.s. yet

$$\frac{{}^{(r)}S_n}{B_n} = \frac{{}^{(r)}S_n}{C_n}\frac{C_n}{B_n} \sim \pm\frac{C_n}{B_n} \to 0 \text{ or } \pm\infty \text{ a.s.}$$

as $n \to +\infty$ through this sequence. Again this contradicts (1.9) so C_n/B_n is bounded away from 0 and $+\infty$. This implies $\limsup |{}^{(r)}S_n|/C_n < +\infty$ a.s., hence $\limsup |X_n^{(r+1)}|/C_n < +\infty$ a.s. by (2.1), so, as in Lemma 5.4, (5.2) holds for some $\varepsilon > 0$; but then it holds for every $\varepsilon > 0$ since C_n is regularly varying with index 1. Thus (1.7) and hence (1.3) hold.

5. Some Lemmas

Lemma 5.1. *Suppose* α_n *and* $B_n > 0$, $B_n \uparrow +\infty$, *are constants for which* $(S_n - \alpha_n)/B_n \xrightarrow{P} 0$. Given $\varepsilon_0 > 0$, $\delta > 0$ *there is a constant* n_0 *depending only on* ε_0, δ, *for which*

$$\begin{aligned}(1-\delta) P\{&\sup_{0\leqq j\leqq n} ({}^{(r)}S_j - \alpha_j) > xB_n\}\\ &\leqq P\{S_n^n - \alpha_n > (x - (r+3)\varepsilon) B_n\} + P\{|X_n^{(r+1)}| > \varepsilon B_n\}\\ &\leqq P\{{}^{(r)}S_n - \alpha_n > (x - (2r+4)\varepsilon) B_n\} + 2P\{|X_n^{(r+1)}| > \varepsilon B_n\}\end{aligned}$$

for every x and $\varepsilon \geqq \varepsilon_0$ *whenever* $n \geqq n_0$, *where* $S_k^n = \sum_{i=1}^{k} X_i I(|X_i| \leqq \varepsilon B_n)$.

Proof. Since $(S_n - \alpha_n)/B_n \xrightarrow{P} 0$, given ε_0, $\delta > 0$ we can find constants n_0, k_0, $n_0 > k_0$, for which $n \geqq n_0$ implies

$$\sup_{k_0 \leqq k \leqq n} |\alpha_n - \alpha_k - \alpha_{n-k}| \leqq \varepsilon_0 B_n \tag{5.1}$$

and

$$\sup_{k_0 \leqq k \leqq n} P\{|S_{n-k} - \alpha_{n-k}| > \varepsilon_0 B_n\} \leqq \tfrac{1}{2}\delta; \tag{5.2}$$

cf. [19]. (Let $S_0 = \alpha_0 = 0$). We can actually take $k_0 = 0$ in (5.1) and (5.2) because

$$\max_{0 \leqq k \leqq k_0} |\alpha_n - \alpha_{n-k}| = \max_{n-k_0 \leqq k \leqq n} |\alpha_n - \alpha_k| \leqq \max_{n-k_0 \leqq k \leqq n} |\alpha_n - \alpha_k - \alpha_{n-k}| + 0(1) \leqq \varepsilon_0 B_n + o(B_n)$$

by (5.1), if $n - k_0 > k_0$, which holds if $n_0 \geqq 2k_0$ and $n \geqq n_0$. Thus also

$$\max_{0 \leqq k \leqq k_0} |\alpha_n - \alpha_{n-k} - \alpha_k| = o(B_n),$$

so (5.1) holds with $k_0 = 0$; similarly for (5.2). Thus if $\varepsilon \geqq \varepsilon_0$

$$\begin{aligned}
&\inf_{0 \leqq k \leqq n} P\{|S_n^n - S_k^n - \alpha_n + \alpha_k| \leqq 2\varepsilon B_n\} \\
&\quad \geqq \inf_{0 \leqq k \leqq n} P\{|S_n^n - S_k^n - \alpha_{n-k}| \leqq \varepsilon B_n\} = \inf_{0 \leqq k \leqq n} P\left\{\left|\sum_{i=1}^{n-k} X_i I(|X_i| \leqq \varepsilon B_n) - \alpha_{n-k}\right| \leqq \varepsilon B_n\right\} \\
&\quad \geqq \inf_{0 \leqq k \leqq n} [P\{|S_{n-k} - \alpha_{n-k}| \leqq \varepsilon B_n\} - (n-k) H(\varepsilon B_n)] \geqq 1 - \delta
\end{aligned}$$

because we can also make $nH(\varepsilon B_n) \leqq nH(\varepsilon_0 B_n) \leqq \frac{1}{2}\delta$ if $n \geqq n_0$ and $\varepsilon \geqq \varepsilon_0$. Now by (2.2) and the fact that $\sup_{r<k \leqq n} |X_k^{(r+1)}| \leqq |X_n^{(r+1)}|$,

$$\begin{aligned}
P\{\sup_{0 \leqq k \leqq n} ({}^{(r)}S_k - \alpha_k) > xB_n\} &\leqq P\{\sup_{0 \leqq k \leqq n} (S_k^n - \alpha_k) > (x - (r+1)\varepsilon) B_n\} \\
&\quad + P\{\sup_{0 \leqq k \leqq n} |{}^{(r)}S_k - S_k^n| > (r+1)\varepsilon B_n\} \\
&\leqq P\{\sup_{0 \leqq k \leqq n} (S_k^n - \alpha_k) > (x - (r+1)\varepsilon) B_n\} + P\{|X_n^{(r+1)}| > \varepsilon B_n\}.
\end{aligned}$$

Finally by independence (interpret $\sup_{0 \leqq j < 0}$ as $-\infty$)

$$\begin{aligned}
&P\{\sup_{0 \leqq k \leqq n} (S_k^n - \alpha_k) > (x - (r+1)\varepsilon) B_n\} \\
&\quad \leqq \sum_{k=0}^{n} P\{\sup_{0 \leqq j < k} (S_j^n - \alpha_j) \leqq (x - (r+1)\varepsilon) B_n,\ S_k^n - \alpha_k > (x - (r+1)\varepsilon) B_n\} \\
&\quad \leqq (1-\delta)^{-1} \sum_{k=0}^{n} P\{\sup_{0 \leqq j < k} (S_j^n - \alpha_j) \leqq (x - (r+1)\varepsilon) B_n, \\
&\qquad S_k^n - \alpha_k > (x - (r+1)\varepsilon) B_n,\ |S_n^n - S_k^n - \alpha_n + \alpha_k| \leqq 2\varepsilon B_n\} \\
&\quad \leqq (1-\delta)^{-1} P\{S_n^n - \alpha_n > (x - (r+3)\varepsilon) B_n\} \\
&\quad \leqq (1-\delta)^{-1} [P\{{}^{(r)}S_n - \alpha_n > (x - (2r+4)\varepsilon) B_n\} + P\{|X_n^{(r+1)}| > \varepsilon B_n\}]
\end{aligned}$$

using (2.2) again. Lemma 5.1 follows from these estimates.

Lemma 5.2. *For any constants α_n, $B_n>0$, $B_n\uparrow+\infty$, $x>0$:*

$$P\{|^{(r)}S_n-\alpha_n|>xB_n\}\geqq\tfrac{1}{8}P\{|X_n^{(r+1)}|>5xB_n\}$$

if $n\geqq n_0(x)$.

Proof. This is Lemma 1 of [20].

Lemma 5.3. *If $B_n>0$, $B_n\uparrow+\infty$, then $X_n^{(r+1)}/B_n\xrightarrow{P}0$ for $r\geqq 0$ if and only if $X_n^{(1)}/B_n\xrightarrow{P}0$.*

Proof. Clearly $X_n^{(1)}/B_n\xrightarrow{P}0$ implies $X_n^{(r+1)}/B_n\xrightarrow{P}0$, so let $X_n^{(r+1)}/B_n\xrightarrow{P}0$. This implies $nH(\varepsilon B_n)\to 0$ for every $\varepsilon>0$, which can be shown as follows: suppose $n_i H(\varepsilon B_{n_i})\to c>0$ for some $\varepsilon>0$ and a sequence $n_i\to+\infty$. Choose $\delta>0$, $\delta<1$, $\delta/(1-\delta)<c$, and define a sequence C_n by

$$[1-H(\varepsilon C_n)]^n=1-\delta.$$

Then

$$n_i H(\varepsilon C_{n_i})\sim n_i\log[1-H(\varepsilon C_{n_i})]\to-\log(1-\delta).$$

We can assume $B_{n_i}<C_{n_i}$ for this subsequence; if not, we could take another subsequence (also denoted n_i) for which $B_{n_i}\geqq C_{n_i}$, and then $n_i H(\varepsilon B_{n_i})\leqq n_i H(\varepsilon C_{n_i})$ shows that

$$c\leqq-\log(1-\delta)\leqq\delta/(1-\delta)<c,$$

a contradiction. Thus $B_{n_i}<C_{n_i}$ and so $X_{n_i}^{(r+1)}/C_{n_i}\xrightarrow{P}0$. But by the inequality of Lemma 5.6

$$P\{|X_{n_i}^{(r+1)}|>\varepsilon C_{n_i}\}\geqq\binom{n_i}{r+1}H^{r+1}(\varepsilon C_{n_i})[1-H(\varepsilon C_{n_i})]^{n_i-r-1}$$

$$\to-(1-\delta)\log(1-\delta)/(r+1)!>0$$

giving a contradiction. Thus $nH(\varepsilon B_n)\to 0$ for every $\varepsilon>0$ and so $X_n^{(1)}/B_n\xrightarrow{P}0$.

Lemma 5.4. *Suppose $B_n>0$, $B_n\uparrow+\infty$ and $r\geqq 0$. Then ${}^{(r)}S_n/B_n\xrightarrow{P}\pm 1$ if and only if $S_n/B_n\xrightarrow{P}\pm 1$. If one of these holds, the following are equivalent to each other and to* (1.7):

$$X_n^{(r+1)}/B_n\to 0 \text{ a.s.}; \tag{5.3}$$

$$\sum_{n\geqq 1} n^r H^{r+1}(\varepsilon B_n)<+\infty \quad \text{for every } \varepsilon>0; \tag{5.4}$$

$$\sum_{j\geqq 1}\lambda_j^{r+1}H^{r+1}(\varepsilon B_{\lambda_j})<+\infty, \quad \text{for every } \varepsilon>0, \quad \text{where } \lambda_j=[\lambda^j],\ \lambda>1; \tag{5.5}$$

$$\sum_{n\geqq 1} n^{-1}P\{|X_n^{(r+1)}|>\varepsilon B_n\}<+\infty \quad \text{for every } \varepsilon>0; \tag{5.6}$$

$$\sum_{j\geqq 1} P\{|X_{\lambda_j}^{(r+1)}|>\varepsilon B_{\lambda_j}\}<+\infty \quad \text{for every } \varepsilon>0. \tag{5.7}$$

Proof. Suppose ${}^{(r)}S_n/B_n \xrightarrow{P} 1$. From Lemma 5.2 we deduce $X_n^{(r+1)}/B_n \xrightarrow{P} 0$, so $X_n^{(1)}/B_n \xrightarrow{P} 0$ by Lemma 5.3, and this means $S_n/B_n \xrightarrow{P} 1$. Conversely if $S_n/B_n \xrightarrow{P} 1$ then by [8, p. 140], $nH(\varepsilon B_n) \xrightarrow{P} 0$ for every $\varepsilon > 0$, so $X_n^{(1)}/B_n \to 0$ and this means ${}^{(r)}S_n/B_n \xrightarrow{P} 1$.

Suppose now that $S_n/B_n \xrightarrow{P} 1$. We show that then (5.4) and (1.7) are equivalent. By [17, Lemma 2], $B_n \sim B(n)$ where B is a positive nondecreasing function satisfying $B(x) = xA(B(x))$. By the monotonicity of H and B, then, (5.4) is equivalent to the convergence of

$$\int_{x_0}^{\infty} x^r H^{r+1}(B(x))\, dx. \tag{5.8}$$

To change variable in this, note that

$$B'(x) = A(B(x))\{1 - xG(B(x))\}^{-1} = [1 + o(1)]\, v(B(x))$$

where $xG(B(x)) \to 0$ since $nH(\varepsilon B_n) \to 0$ for $\varepsilon > 0$. Thus $B'(x) > 0$ for large x, so $B(x)$ is ultimately strictly increasing, its inverse $B^{-1}(x)$ exists for large x, diverges to $+\infty$ as $x \to +\infty$, and satisfies $B'(B^{-1}(x)) \sim v(x)$. From $B(x) = xA(B(x))$ it follows that $B^{-1}(x) \sim x/v(x)$, so convergence of (5.4) is equivalent to convergence of

$$\int_{x_0}^{\infty} [B^{-1}(x)]^r H^{r+1}(x)\, [B'(B^{-1}(x)]^{-1}\, dx,$$

or to (1.7). Thus (5.4) and (1.7) are equivalent.

For the remainder of the lemma: the equivalence of (5.3) and (5.4) is Lemma 3 of [22], while the equivalence of (5.4)–(5.7) follows easily from

$$P\{|X_n^{(r+1)}| > \varepsilon B_n\} = \sum_{j=r+1}^{n} \binom{n}{j} H^j(\varepsilon B_n)\,[1 - H(\varepsilon B_n)]^{n-j} \sim n^{r+1} H^{r+1}(\varepsilon B_n)/(r+1)!$$

when $nH(\varepsilon B_n) \to 0$ ([22, Lemma 2]).

Lemma 5.5. *If Y_i are independent r.v.'s, $|Y_i - EY_i| \leqq M$ and $s_n^2 = \sum_{i=1}^{n} \operatorname{Var}(Y_i)$, then for $t > 0$,*

$$P\left\{\left|\sum_{i=1}^{n} (Y_i - EY_i)\right| > t\right\} \leqq 2e^{t/M}(s_n^2/tM)^{t/M}.$$

Proof. From inequality (8b) of Bennett [2],

$$\begin{aligned} P\left\{\left|\sum_{i=1}^{n} (Y_i - EY_i)\right| > t\right\} &\leqq 2\exp\{-t[(1 + s_n^2/Mt)\log(1 + tM/s_n^2) - 1]/M\} \\ &\leqq 2\exp\{-t[\log(1 + tM/s_n^2) - 1]/M\} \\ &\leqq 2e^{t/M}(1 + tM/s_n^2)^{-t/M} \leqq 2e^{t/M}(s_n^2/tM)^{t/M}. \end{aligned}$$

Lemma 5.6. *If $n > r = 0, 1, 2, \ldots$ and $x > 0$,*

$$\binom{n}{r+1} H^{r+1}(x)\, h^{n-r-1}(x) \leqq P\{|X_n^{(r+1)}| > x\} \leqq \binom{n}{r+1} H^{r+1}(x)\, h^{-r-1}(x).$$

Proof.

$$\begin{aligned}
&P\{|X_n^{(r+1)}|>x\}\\
&=P\{(r+1) \quad \text{or more of } |X_i|>x\}=\sum_{j=r+1}^{n}\binom{n}{j}H^j(x)h^{n-j}(x)\\
&=n(n-1)\dots(n-r)H^{r+1}(x)h^{-r-1}(x)\sum_{j=0}^{n-r-1}\binom{n-r-1}{j}\frac{H^j(x)h^{n-j}(x)}{(j+r+1)\dots(j+1)}\\
&\leqq\binom{n}{r+1}H^{r+1}(x)h^{-r-1}(x).
\end{aligned}$$

For the other inequality, simply take the term for $j=r+1$ from the sum.

Lemma 5.7. *Suppose $r=0,1,2,\dots$, $\delta>0$, and $B_n\uparrow+\infty$. Then for some $c(\delta)$, $C(\delta)$ and $N(\delta)$,*

$$c\sum_{n\geqq 1}n^r H^{r+1}(xB_n)\leqq P\{\sup_{n>N}|X_n^{(r+1)}|/B_n>x\}\leqq C\sum_{n\geqq 1}n^r H^{r+1}(xB_n)$$

uniformly in $x\geqq\delta$, provided (for the lefthand inequality) the series converges when δ replaces x.

Proof. For the right hand inequality, using Lemma 5.5, and the same argument used in deriving (2.4),

$$\begin{aligned}
&P\{\sup_{n>\lambda_{j_0}}|X_n^{(r+1)}|/B_n>x\}\\
&\leqq P\{\sup_{j\geqq j_0}\sup_{\lambda_j<n\leqq\lambda_{j+1}}|X_n^{(r+1)}|>xB_{\lambda_j}\}\leqq\sum_{j\geqq j_0}P\{|X_{\lambda_{j+1}}^{(r+1)}|>xB_{\lambda_j}\}\\
&\leqq\sum_{j\geqq j_0}\binom{\lambda_{j+1}}{r+1}H^{r+1}(xB_{\lambda_j})h^{-r-1}(xB_{\lambda_1})\\
&\leqq\sum_{j\geqq j_0}\lambda_{j+1}^{r+1}H^{r+1}(xB_{\lambda_j})h^{-r-1}(\delta B_{\lambda_1})/(r+1)!\\
&\leqq\sum_{n\geqq 1}n^r H^{r+1}(xB_n)\lambda^2 h^{-r-1}(\delta B_{\lambda_1})/\{(r+1)!(\lambda-1)(\lambda-1-2\lambda^{-j_0})\}
\end{aligned}$$

assuming $x\geqq\delta$, j_0 is such that $\lambda-1-2\lambda^{-j_0}>0$, and $h(\delta B_{\lambda_1})>0$. Defining the constants appropriately gives the inequality.

For the left hand inequality choose N so large that

$$\{h^{-r-1}(\delta B_{\lambda_1})/(r+1)!\}\sum_{n>N}n^r H^{r+1}(\delta B_n)\leqq\tfrac{1}{2}.$$

Then for j_0 large enough,

$$\begin{aligned}
&P\{\sup_{n>N}|X_n^{(r+1)}|/B_n>x\}\geqq P\{\sup_{j\leqq j_0}\sup_{\lambda_{j-1}<n\leqq\lambda_j}|X_n^{(r+1)}|>xB_{\lambda_j}\}\\
&\geqq P(\bigcup_j\{(r+1)\text{ or more of }|X_{\lambda_{j-1}+1}|,\dots|X_{\lambda_j}|,\text{ are }>xB_{\lambda_j}\})=P(\bigcup_j E_j),\text{ say.}
\end{aligned}$$

By stationarity and Lemma 5.6, if $j>\log N$

$$P(E_j)=P\{|X^{(r+1)}_{\lambda_j-\lambda_{j-1}}|>xB_{\lambda_j}\}\leqq\binom{\lambda_j-\lambda_{j-1}}{r+1}H^{r+1}(xB_{\lambda_j})h^{-r-1}(xB_{\lambda_j})$$
$$\leqq\lambda_j^{r+1}H^{r+1}(xB_{\lambda_j})h^{-r-1}(\delta B_{\lambda_1})/(r+1)!$$

if $x\geqq\delta$. By the way we chose N we thus have $\sum_{j>\log N}P(E_j)\leqq\frac{1}{2}$.

Now when $\Sigma n^r H^{r+1}(\delta B_n)<+\infty$, $nH(\delta B_n)\to 0$ so $h^n(\delta B_n)\to 1$. Again using Lemma 5.6 this gives the lower bound

$$P(E_j)\geqq 2c'\lambda_j^{r+1}H^{r+1}(xB_{\lambda_j})$$

for some c' depending on δ but not on x if $x\geqq\delta$. Finally we use Bonferroni's inequality and the independence of the E_j to deduce that

$$P(\bigcup_j E_j)\geqq\sum_j P(E_j)-\Sigma P(E_j)P(E_k)\geqq\tfrac{1}{2}\sum_j P(E_j)$$
$$\geqq c'\sum_j\lambda_j^{r+1}H^{r+1}(xB_{\lambda_j})\geqq c\sum_n n^r H^{r+1}(xB_n)$$

where c, c' depend on δ but not on x if $x\geqq\delta$. This proves the Lemma.

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