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On the Transformation of Martingales with a Two Dimensional Parameter Set by Convex Functions

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Introduction

It is a well-known fact that the transformation of a martingale with real parameters by a convex function leads to a submartingale. P.A. Meyer remarked in his review paper on processes with a two dimensional parameter set [4] that besides the result which states that the square of a martingale is a weak submartingale[1] he has not found in the literature any other result on the transformation of martingales with two dimensional parameter sets by convex functions.

Recently X. Guyon showed in [3] that for the two dimensional parameter set case the transformation of a strong martingale adapted to the Brownian fields by a convex function u gives a weak submartingale provided that u belongs to the class $C^4(R)$ and furthermore its fourth derivative $u^{(4)}$ is non-negative. However since Guyon derives the above mentioned result from his very sophisticated Ito-formula (see [2], Chap. 6), it should be understood that the validity of his result depends on the huge set of conditions of his Ito-formula for stochastic integrals with respect to martingales with a two dimensional parameter set.

The aim of this note is twofold:

a) We reduce the problem of the transformation of martingales with a two dimensional parameter set by convex functions to that for the one-dimensional case and thus give an elementary proof of the following result:

The transformation of a "strong" martingale by a convex function u is a weak submartingale provided that u belong to the class $C^2(R)$ and furthermore its derivative $u^{(2)}$ is convex (Theorem 1).

b) We give an example which shows that the sufficient conditions of our main Theorem 1 are also "nearly" necessary.

[1] This fact is essential for the proof of the existence of the Doob-Meyer decomposition for square-integrable martingales with a two dimensional parameter set

The Main Theorem

We mostly follow the notation and terminology of Cairoli and Walsh's paper [1] or those of the proceedings [4]. The two-dimensional parameter set will always be the subset $\{z|z<z_0\}$ of R^2_+ equipped with the usual partial ordering of the plane. If a probability system $(\Omega, \mathscr{F}_z, z<z_0, \mathscr{F}, P)$ is given, then it is always assumed in this note that the family $\{\mathscr{F}_z, z<z_0\}$ of sub-σ-fields of $\mathscr{F}$ satisfies the four standard regularity conditions:

(F1) $\mathscr{F}_0$ contains all null sets of $\mathscr{F}$;

(F2) If $z<z'<z_0$ then $\mathscr{F}_z \subset \mathscr{F}_{z'}$;

(F3) For each $z \ll z_0$, $\mathscr{F}_z = \bigcap_{z' \gg z} \mathscr{F}_{z'}$;

(F4) For each $z<z_0$, $\mathscr{F}_{st_0}$ and $\mathscr{F}_{s_0t}$ are conditionally independent given $\mathscr{F}_z$ where $z=(s,t)$ and $z_0=(s_0,t_0)$.

We recall that a process with a two-dimensional parameter set $M=\{M_z, z<z_0\}$ is said to be a weak submartingale (resp. martingale, strong martingale) if it is adapted to $\{\mathscr{F}_z, z<z_0\}$ and integrable (i.e. $E(|M_z|)<+\infty, \forall z<z_0$) and if

$$E\{M(z,z']|\mathscr{F}_z\} \geqq 0 \quad \text{for all } z=(s,t) \ll z'=(s',t')<z_0 \tag{1}$$

where $M(z,z']$ denotes the quantity

$$M_{s't'}+M_{st}-M_{st'}-M_{s't}$$

resp.

$$E\{M_{z'}-M_z|\mathscr{F}_z\}=0, \tag{1'}$$

$$E\{M(z,z']|\mathscr{F}_{st_0} \vee \mathscr{F}_{s_0t}\}=0. \tag{1''}$$

Theorem 1. *Let $M=\{M_z, z<z_0\}$ be a right-continuous, square-integrable martingale which is null on the axes of R^2_+, and let f be a function of the class $C^2(R)$ such that f'' is convex and non-negative.*

Suppose that the corresponding increasing process $\langle M\rangle$ has the property:

$$M^2-\langle M\rangle=\{M_z^2-\langle M\rangle_z, z<z_0\} \quad \textit{is a martingale.}$$

Then the process $f(M)=\{f(M_z), z<z_0\}$ is a weak submartingale provided that one of the two following conditions is satisfied:

(a) *For all $t\in[0,t_0]$, $M_{.t}=\{M_{st}, 0\leqq s\leqq s_0\}$ is a continuous process, and there is a constant K such that*

$$\operatorname{Sup}_{z<z_0}|M_z| \leqq K \quad \text{a.e.};$$

(b) *$\{\mathscr{F}_z, z<z_0\}$ is the Brownian filtration, and for all $t\in[0,t_0]$,*

$$E\left\{\int_0^{s_0} |f'(M_{st})|^2 d_s\langle M\rangle_{st}\right\}<\infty,$$

$$E\{\sup_{0\leqq s\leqq s_0} f''(M_{st})\cdot\langle M\rangle_{s_0t}\}<\infty.$$

Remarks. Before proving Theorem 1 let us give some remarks which justify in some sense the hypotheses stated here.

1) By symmetry we can permute the indices t and s in the conditions (a) or (b) of the theorem.

2) If M is supposed to be a continuous strong martingale, then our additional condition on the increasing process ($M^2-\langle M\rangle$ is a martingale) becomes superfluous (see [1], Theorem 1.9).

3) The inequality $\operatorname{Sup}_{z<z_0}|M_z|\leqq k$ a.e. and its equivalent form $|M_{z_0}|\leqq k$ imply the two inequalities in (b), and on the other hand if the filtration is brownian, then the continuity of the process $M_{.t}$ is a consequence of the Wong-Zakai representation theorem. In other words if the filtration is brownian, then the hypotheses in (a) of the Theorem 1 are less good than those in (b).

4) On the one hand by the Schwarz inequality

$$E(\operatorname*{Sup}_{0\leqq s\leqq s_0} f''(M_{st})\cdot\langle M\rangle_{s_0t})\leqq E(\operatorname*{Sup}_{0\leqq s\leqq s_0} f''(M_{st})^2)^{1/2}\cdot E(\langle M\rangle^2_{s_0t})^{1/2},$$

and on the other hand it follows from the convexity of f'', the Doob inequality and the Burkholder inequality for the martingale $M_{s_0\cdot}$ that

$$E(\operatorname*{Sup}_{0\leqq s\leqq s_0} f''(M_{st})^2)\leqq 4E(f''(M_{s_0t})^2)\leqq 4E(f''(M_{z_0})^2),$$
$$E(\langle M\rangle^2_{s_0t})\leqq E(\langle M\rangle^2_{z_0})\leqq \text{const.}\,E(M^4_{z_0}).$$

We conclude that the last inequality in (b) is satisfied if

$$E(f''(M_{z_0})^2)<\infty \quad\text{and}\quad E(M^4_{z_0})<\infty.$$

It is noted that in the Ito-formula for the two dimensional case used by Guyon it is supposed that $E(|M_{z_0}|^6)<\infty$ (see [3]).

Lemma 2. *Under the same hypotheses as in Theorem 1, let* $z_1=(s_1,t_1)\ll z_2=(s_2,t_2)<z_0$ *and let* $s_1=s_n^0<s_n^1<\ldots<s_n^n=s_2$ *be a subdivision of the interval* $[s_1,s_2]$ *into n equal intervals. Put*

$$f_n''(M_{st})=f''(M_{s_n^k t}) \quad \text{if } s_n^k\leqq s<s_n^{k+1}.$$

Then for all $t\in[0,t_0]$,

$$\lim_{n\to\infty} E\left\{\int_{s_1}^{s_2}|f_n''(M_{st})-f''(M_{st})|\,d_s\langle M\rangle_{st}\right\}=0. \tag{2}$$

Proof. Let $Q=[s_1,s_2]\times\Omega$, $\mathscr{B}=\mathscr{B}(s_1,s_2]\otimes\mathscr{F}$ and let μ^t be the Doléans measure generated by the increasing process $d_s\langle M\rangle_{st}$.

It follows from the last condition of (b) that the following function φ belongs to $\mathscr{L}^1(Q,\mathscr{B},\mu^t)$,

$$\varphi:=\operatorname*{Sup}_{s_1\leqq s\leqq s_2} f''(M_{st}).$$

Since f'' is continuous and since $M_{.t}$ is a continuous process, we get

$$\text{a.e.-}\mu^t \lim_{n\to\infty} f_n''(M_{st})=f''(M_{st})\leqq\varphi$$

for all $s \in [s_1, s_2]$. (2) is now an immediate consequence of the Lebesgue convergence theorem. qed.

Proof of Theorem 1. By the Ito formula for martingales with a one dimensional parameter set, we have

$$f(M_{st}) - f(0) = \int_0^s f'(M_{vt}) d_v M_{vt} + \tfrac{1}{2} \int_0^s f''(M_{vt}) d_v \langle M \rangle_{vt} \tag{3}$$

(Note that $M_{0t} = 0$ a.e.)

It follows from the condition (b) that the 1st term in the right-hand side member of (3) is a martingale and that $f(M_{st})$ must belong to $\mathscr{L}^1$. Thus

$$\begin{aligned} &E\{f(M_{s_2 t_i}) - f(M_{s_1 t_i}) | \mathscr{F}_{s_1 t_i}\} \\ &= \tfrac{1}{2} E\left\{\int_{s_1}^{s_2} f''(M_{vt_i}) d_v \langle M \rangle_{vt_i} | \mathscr{F}_{s_1 t_i}\right\}, \quad (i = 1, 2). \end{aligned}$$

Since $f'' \geqq 0$, $\mathscr{F}_{s_1 t_1} = \mathscr{F}_{z_1} \subset \mathscr{F}_{s_1 t_2}$, and $d_v \langle M \rangle_{vt_2} \geqq d_v \langle M \rangle_{vt_1}$ we have

$$\begin{aligned} &E\{f(M)(z_1, z_2] | \mathscr{F}_{z_1}\} \\ &\geqq \tfrac{1}{2} E\left\{\int_{s_1}^{s_2} (f''(M_{vt_2}) - f''(M_{vt_1})) d_v \langle M \rangle_{vt_1} | \mathscr{F}_{z_1}\right\}. \end{aligned} \tag{4}$$

On the one hand it follows from Lemma 2 that

$$\begin{aligned} \tfrac{1}{2} I(n) := \tfrac{1}{2} E\left\{\int_{s_1}^{s_2} (f_n''(M_{vt_2}) - f_n''(M_{vt_1})) d_v \langle M \rangle_{vt_1} | \mathscr{F}_{z_1}\right\} \\ \to \text{the right hand side term of (4)} \\ \text{in } \mathscr{L}^1 \text{ when } n \to \infty. \end{aligned}$$

On the other hand,

$$\begin{aligned} I_{(n)} &= \sum_{k=0}^{n-1} E\{(f''(M_{s_n^k t_2}) - f''(M_{s_n^k t_1}))(\langle M \rangle_{s_n^{k+1} t_1} - \langle M \rangle_{s_n^k t_1}) | \mathscr{F}_{z_1}\} \\ &:= \sum_{k=0}^{n-1} E\{I_k | \mathscr{F}_{z_1}\}, \end{aligned} \tag{5}$$

but for each $0 \leqq k \leqq n-1$

$$\{f''(M_{s_n^k t}), \mathscr{F}_{s_0 t}, 0 \leqq t \leqq t_0\}$$

is a non-negative submartingale. It follows that

$$E\{I_k | \mathscr{F}_{s_0 t_1}\} = (E\{f''(M_{s_n^k t_2}) | \mathscr{F}_{s_0 t_1}\} - f''(M_{s_n^k t_1})) \cdot (\langle M \rangle_{s_n^{k+1} t_1} - \langle M \rangle_{s_n^k t_1}) \geqq 0 \quad \text{a.e.}$$

Therefore

$$E\{I_k | \mathscr{F}_{z_1}\} = E\{E\{I_k | \mathscr{F}_{s_0 t_1}\} | \mathscr{F}_{z_1}\} \geqq 0. \tag{6}$$

Thus, it follows from (4)–(6) that

$E\{f(M)(z_1, z_2] | \mathscr{F}_{z_1}\} \geqq 0.$ qed.

Corollary 3. *Under the same hypotheses as in Theorem 1 we suppose furthermore that the process,* $f(M)=\{f(M_z)|z<z_0\}$ *is right-continuous in* $\mathcal{L}^1$ *and that* f *is bounded from below. Then there exists an increasing process* A *such that* $f(M)-A$ *is a weak martingale.*

In particular if f is non-negative and is either such that $\sqrt{f}$ is convex or such that

$$E\{f(M_{z_0})(\text{Log}^+ f(M_{z_0}))^2\} \quad \text{is finite,}$$

then there exists an increasing process A such that

$$f(M)-A \quad \text{is a weak martingale.}$$

Proof. Since f is bounded from below,

$$\inf_{x\in R} f(x) = -c > -\infty.$$

Let $\varphi=f+c$, then φ and M also verify all the hypotheses of Corollary 3 as f and M do. Since $\varphi \geqq 0$, it follows from a known result (see for instance Theorem 3.1, [4]) that there exists an increasing process A such that

$$\varphi(M)-A \quad \text{is a weak martingale}$$

and so is $f(M)-A$.

Suppose that φ is a non-negative convex function on R. Then it follows from the Doob-Cairoli inequality [1] that

$$E\{\sup_{z<z_0} \varphi(M_z)^2\} \leqq 16 E\{\varphi(M_{z_0})^2\} \tag{7}$$

and

$$E\{\sup_{z<z_0} \varphi(M_z)\} \leqq \text{const.}\, E\{\varphi(M_{z_0})(\text{Log}^+ \varphi(M_{z_0}))^2\} + \text{const.} \tag{8}$$

If we put either $\varphi=\sqrt{f}$ or $\varphi=f$, the inequalities (7) or (8) imply that

$$E(\sup_{z<z_0} f(M_z)) < \infty.$$

The right continuity of $z \mapsto f(M_z)$ in $\mathcal{L}^1$ follows at once from the Lebesgue dominated convergence theorem. q.e.d.

Remarks. 1) In [3] Guyon gave a counter-example showing that there exists a strong convex function f (i.e. $f^{(i)} \geqq 0$, $i=2,3,4$) and a martingale M with respect to the Brownian filtration such that $f(M)$ is not a weak submartingale. In fact his proof shows essentially that there exists a martingale M such that $M^2-\langle M\rangle$ is not a martingale and that $f(M)$ is not a weak submartingale. We conclude that the fact "$M^2-\langle M\rangle$ should be a martingale" is in some sense the weakest condition required for M in Theorem 1.

2) We give in the following a counter-example showing that there exists a positive convex function f such that

$$f(w) \quad \text{is not a weak submartingale,}$$

where $w=\{w_z, z\in R_+^2\}$ is the Brownian sheet. Hence the condition "f'' is convex" in the Theorem 1 is in some sense the weakest required for the given function f.

We take $f(x)=2+x^2+\sin x>0$.

We have $f''(x)=2-\sin x>0$ and $f^{(IV)}(x)=\sin x$, hence f'' is not convex. Let $z=(s,t)\ll z'=(s',t')$, $\varphi(x)=2+x^2$ and put

$$A=(z,z'],\quad A_1=((0,t),(s,t')],\quad A_2=((s,0),(s',t)],$$
$$z_1=(s,t'),\quad z_2=(s',t).$$

Then

$$E\{\varphi(w)(z,z']\mid\mathscr{F}_z\}=m(A)$$

where $m(A)$ is the Lebesgue measure of the Borel set A. Consider $\xi>z$ and put

$$R_\xi=(0,\xi],\quad Q_\xi=R_\xi\setminus R_z.$$

Then

$$w_\xi=w((0,\xi])=w(R_z)+w(Q_\xi).$$

Since $w(R_z)$ and $w(Q_\xi)$ are two independent Gaussian random variables, it follows that

$$E\{\sin(w(Q_\xi))\}=0$$

and

$$E\{\sin(w_\xi)\mid\mathscr{F}_z\}=b_\xi\cdot\sin(w(R_z))$$

where

$$b_\xi=E\{\cos(w(Q_\xi))\}.$$

It can be shown that

$$b_\xi=e^{-\frac{1}{2}m(Q_\xi)}.$$

It turns out that

$$\begin{aligned}E\{f(w)(z,z']\mid\mathscr{F}_z\}&=m(A)+(1+b_{z'}-b_{z_1}-b_{z_2})\sin(w_z)\\&=\underset{I}{m(A)}+(\underset{II}{1+e^{-\frac{1}{2}m(A\cup A_1\cup A_2)}-e^{-\frac{1}{2}m(A_1)}-e^{-\frac{1}{2}m(A_2)}})\cdot\sin(w_z)\\&:=I+II\cdot\sin(w_z).\end{aligned}$$

It is easy to see that if

$$z=(1{,}000,\ 1{,}000)\ \text{and}$$
$$z'=(1{,}000.1,\ 1{,}000.1),$$

then $II>I>0$. Hence the variable $E\{f(w)(z,z']\mid\mathscr{F}_z\}$ could have negative values on a set of positive probability.

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