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Generalized Bounded Variation and Applications to Piecewise Monotonic Transformations*

Gerhard Keller
Universität Heidelberg, Institut für Angewandte Mathematik, Im Neuenheimer Feld 294, 6900 Heidelberg 1, Federal Republic of Germany

Summary. We prove the quasi-compactness of the Perron-Frobenius operator of piecewise monotonic transformations when the inverse of the derivative is Hölder-continuous or, more generally, of bounded p-variation.

Introduction

One of the most successfully used tools for the investigation of invariant measures for piecewise monotonic transformations T on $[0, 1]$ is the Perron-Frobenius-operator. If $0=a_0<a_1<\ldots<a_N=1$, if $T_i=T_{|(a_{i-1},a_i)}$ is strictly monotone and continuous ($i=1, \ldots, N$), and if m is a Borel-probability on $[0,1]$ with respect to which T is nonsingular, then the Perron-Frobenius-operator (PFO) of T and m is the linear, positive contraction

$$P\colon L^1_m \to L^1_m, \qquad Pf(x)=\sum_{i=1}^{N} (f\cdot g)(T_i^{-1}x)\cdot 1_{T(a_{i-1},a_i)}(x),$$

where $\frac{1}{g}=\frac{d}{dm}(m\circ T)$ is the Radon-Nikodym-derivative of T with respect to m.
P reflects very well the ergodic properties of the system (T, m), namely:

- $\mu=h\cdot m$ is a T-invariant probability
  if and only if
  $0\leqq h\in L^1_m$, $\int h\,dm=1$, and $Ph=h$.
- Mixing properties of T are closely related to spectral properties of P (cf. [7]).

A particularly favorable situation for the investigation of P occurs if

(1) $\|(g\circ T^{n-1})\cdot\ldots\cdot(g\circ T)\cdot g\|_\infty<1$ for some $n\in\mathbb{N}$ and

(2) g is of bounded variation.

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It has been shown in [7] that under these assumptions

(3) There is a $h: [0,1] \to \mathbb{R}_+$ of bounded variation such that $\mu = h \cdot m$ is a T-invariant probability on $[0,1]$.

(4) For some power T^k the measure μ splits up into finitely many ergodic components, on each of which T^k is weakly Bernoulli with exponential mixing rate. This is good enough to imply central limit theorems and almost sure invariance principles for stochastic processes $(f \circ T^{nk})_{n \in \mathbb{N}}$ with f of bounded variation.

Partial results in this direction can be found e.g. in [10] and [15]. M. Rychlik [12] has given a new, very elegant proof of (3) and (4), which applies also to a broad class of transformations with a countable number of monotonicity intervals. For further references see [7].

In [16] an attempt has been made to replace (2) in the case where m is Lebesgue-measure by "g is Hölder-continuous", but the result was unsatisfactory since some additional conditions had to be imposed, which in general cannot be checked effectively. Nevertheless a result in this direction is desirable because of two reasons:

a) Some problems related to the Lorenz-attractor can be reduced to problems concerning a piecewise monotonic transformation with Hölder-continuous derivative (cf. [16]). These problems could be solved setting m = Lebesgue-measure and $g = 1/|T'|$.

b) If $g = \lambda \cdot e^{\phi}$, $\lambda > 0$, $Ph = h$, and $\mu = h \cdot m$, then μ is called an equilibrium state for ϕ. For a particular class of transformations including the β-transformation ($x \to \beta x \bmod 1$, $\beta > 1$) and Markov-transformations it has been shown in [7, 8] that for each ϕ of bounded variation satisfying $\sum_{i=1}^{\infty} \mathrm{var}_i(\phi) < \infty$ ($\mathrm{var}_i(\phi) = \sup\{|\phi(x) - \phi(y)| \mid x, y \in I,\ I$ an interval on which T^i is monotone$\}$) there is a measure m and a real $\lambda > 0$ such that $g = \lambda \cdot e^{\phi}$ and g satisfies (1) and (2) above. For topological Markov-chains over a finite alphabet however (and hence for Markov-transformations), the same result was already known when ϕ is only Hölder-continuous (this implies $\sum \mathrm{var}_i(\phi) < \infty$, see [1]), although in this case ϕ is not necessarily of bounded variation.

The aim of this paper is to replace (2) above by

(2′) g is of universally bounded p-variation, i.e.

$$\mathrm{var}_p(g) = \sup_{0 \leq x_0 < \ldots < x_n \leq 1} \left(\sum_{i=1}^{n} |g(x_i) - g(x_{i-1})|^p \right)^{1/p} < \infty.$$

These are those functions which are called functions of bounded p-variation in [3]. At the end of Sect. 2 the word "universally" will be justified. The main result is Theorem 3.3 which asserts that under (1) and (2′) the transformation T has the properties briefly sketched in (3) and (4).

As each Hölder-continuous function with Hölder-exponent $0 < r \leq 1$ is of universally bounded $1/r$-variation, this solves the problems described under a) and b). As a matter of fact, problem a) can be solved under the weaker assumption that $1/|T'|$ is of universally bounded $1/r$-variation. I want to mention that Marek Rychlik orally announced me a solution of problem a) using basically the same idea.

In Sect. 1 we define a generalized concept of functions of bounded variation adapted to a quasi-compact, pseudo-metric space X equipped with a finite Borel-measure. This concept unifies Lipschitz-continuity, Hölder-continuity, Riemann-integrability, bounded variation, bounded p-variation, and gives many intermediate notions of bounded variation, some of which play an important role in Sect. 2 and 3. The main result is about compact embeddings of spaces of functions of generalized bounded variation into suitable L^p-spaces (Theorem 1.13). As in the theory of Sobolev-spaces, next to embedding theorems, trace theorems are the most fundamental ones. In Sect. 2 we prove such a theorem, when the underlying space is the unit-interval (not necessarily equipped with its Euclidean metric). This is the situation that occurs in Sect. 3, where results of Sects. 1 and 2 are used to show that PFO's satisfying (1) and (2) are quasi-compact as operators on some suitable space of functions of generalized bounded variation, which implies (3) and (4) as in [7] and [12].

1. Generalized Bounded Variation

Let (X,d) be a quasicompact topological space whose topology is defined by the pseudo-distance d. (This means that we do not require the Hausdorff-property and allow $d(x,y)=0$ for $x \neq y$, cf. XII.4 and Ex. XII.3.6 of [5].) $\mathscr{B}$ denotes the Borel-σ-algebra of (X,d) and m is a finite Borel-measure on $\mathscr{B}$. Open balls in X are denoted by $S_\varepsilon(x)=\{y\in X \mid d(x,y)<\varepsilon\}$. $F=\bigcap_{\substack{A \text{ closed} \\ m(X\setminus A)=0}} A$ is the support of m.

1.1. Definition. For an arbitrary function $h\colon X\to\mathbb{C}$ and $\varepsilon>0$ define $\operatorname{osc}(h,\varepsilon,.)\colon X\to[0,\infty]$ by

$$\operatorname{osc}(h,\varepsilon,x)=\begin{cases}\operatorname{ess\,sup}\{|h(y_1)-h(y_2)| \mid y_1,y_2\in S_\varepsilon(x)\} & \text{if } m(S_\varepsilon(x))>0\\ 0 & \text{if } m(S_\varepsilon(x))=0,\end{cases}$$

where the essential supremum is taken with respect to the product measure m^2 on X^2. As $\operatorname{osc}(h,\varepsilon,.)$ is lower semi-continuous and hence measurable, one can define for $1\leqq p\leqq\infty$:

$\operatorname{osc}_p(h,\varepsilon)=\|\operatorname{osc}(h,\varepsilon,.)\|_p$, where we admit the p-norm to take the value $+\infty$.

$\operatorname{osc}_p(h,\varepsilon)$ can be interpreted as an isotonic function (in the variable ε) from $(0,A]$ to $[0,\infty]$, where A is any positive constant. This motivates the next

1.2. Definition. Fix $A>0$ and denote by Φ the class of all isotonic maps $\phi(0,A]\to[0,\infty]$ with $\phi(x)\to 0$ $(x\to 0)$. Set

$$R_p=\{h\colon X\to\mathbb{C} \mid \operatorname{osc}_p(h,.)\in\Phi\},$$

and for $\phi \in \Phi$ set

$$R_{p,\phi} = \{h \in R_p \mid \mathrm{osc}_p(h,.) \leqq \phi\},$$
$$S_{p,\phi} = \bigcup_{n \in \mathbb{N}} R_{p,n\cdot\phi}.$$

If $\phi(x) = x^r$, we simply write $S_{p,r}$ instead of $S_{p,\phi}$.

The following lemma is trivial:

1.3. Lemma. a) $\phi, \psi \in \Phi$, $\phi \leqq \psi \Rightarrow R_{p,\phi} \subseteq R_{p,\psi}$, $S_{p,\phi} \subseteq S_{p,\psi}$.

b) $R_p = \bigcup_{\phi \in \Phi} R_{p,\phi} = \bigcup_{\phi \in \Phi} S_{p,\phi}$.

c) *If* $1 \leqq p \leqq q \leqq \infty$ *then* $S_{q,\phi} \subseteq S_{p,\phi}$ *for all* $\phi \in \Phi$.

d) *If M is one of the classes introduced in Definition* 1.2, *then* $h \in M \Rightarrow \mathrm{Re}\, h$, $\mathrm{Im}\, h \in M$.

The next lemma provides some elementary facts about the oscillation-functions:

1.4. Lemma. *For* $1 \leqq p \leqq \infty$ *holds:*

a) *If* $h_1 = h_2$ *m-a.e., then* $\mathrm{osc}(h_1, \varepsilon, .) = \mathrm{osc}(h_2, \varepsilon, .)$.

b) *Each* $h \in R_p$ *is bounded and* $\mathscr{B}_0$*-measurable, where* $\mathscr{B}_0$ *is the m-completion of* $\mathscr{B}$.

c) *If* $\{P_1, \ldots, P_N\}$ *is a measurable partition of* X *and if*

$$\operatorname{ess\,inf} h(P_n) \leqq f(x) \leqq \operatorname{ess\,sup} h(P_n) \quad \text{for all } x \in P_n \ (n = 1, \ldots, N),$$

then $\|f - h\|_p \leqq \mathrm{osc}_p(h, \varepsilon)$, *where* $\varepsilon = \sup\{\mathrm{diam}(P_n) \mid n = 1, \ldots, N\}$.

d) $\mathrm{osc}(h, \varepsilon, .)$ *is bounded on* X *for each* $h \in R_p$ *and* $\varepsilon > 0$.

e) *For each* $h \in R_p$ *there are elementary functions* $\underline{h}_n$, $\bar{h}_n$ *with*

$$\underline{h}_1 \leqq \ldots \leqq \underline{h}_n \leqq \ldots \leqq h \leqq \ldots \leqq \bar{h}_n \leqq \ldots \leqq \bar{h}_1,$$

such that $\|\bar{h}_n - \underline{h}_n\|_p \leqq \mathrm{osc}_p\left(h, \frac{1}{n}\right) \to 0$.

Proof. a) is obvious. We next prove c): For a.e. $x \in P_n$ holds:

$$|f(x) - h(x)| \leqq \operatorname{ess\,sup} h(P_n) - \operatorname{ess\,inf} h(P_n) \leqq \mathrm{osc}(h, \varepsilon, x),$$

hence $\|f - h\|_p \leqq \mathrm{osc}_p(h, \varepsilon)$ by definition.

d) is proved for small ε first: $h \in R_p$ implies that for sufficiently small $\varepsilon > 0$: $\mathrm{osc}(h, 4\varepsilon, .) \in L_m^p$. By the quasi-compactness of X one can choose $x_1, \ldots, x_n \in X$ with $X = \bigcup_{i=1}^{n} S_\varepsilon(x_i)$. As $\mathrm{osc}(h, 4\varepsilon, .) \in L_m^p$ and as for each

$$y \in S_{2\varepsilon}(x_i)\colon \mathrm{osc}(h, 4\varepsilon, y) \geqq \mathrm{osc}(h, 2\varepsilon, x_i),$$

it follows that $M = \max\{\mathrm{osc}(h, 2\varepsilon, x_i) \mid i = 1, \ldots, n\} < \infty$. Hence $\mathrm{osc}(h, \varepsilon, y) \leqq M$ for all y. In order to prove e) we choose for each $n \in \mathbb{N}$ a partition $\mathscr{P}_n = \{P_1(n), \ldots, P_{N(n)}(n)\}$ of X finitely generated from balls with diameter less than

$\frac{1}{n}$. We may assume that $\mathscr{P}_{n+1}$ is finer than $\mathscr{P}_n$. Define $\underline{h}_n, \bar{h}_n$ by $\underline{h}_n(x) = \operatorname{ess\,inf} h(P_i(n))$, $\bar{h}_n(x) = \operatorname{ess\,sup} h(P_i(n))$ if $x \in P_i(n)$. $\underline{h}_n, \bar{h}_n$ are $\mathscr{B}$-measurable elementary functions, $\underline{h}_n \leqq h \leqq \bar{h}_n$, and they are bounded for big n, as we know that d) holds for small ε at least. Furthermore $\|\bar{h}_n - \underline{h}_n\|_p \leqq \left\| \operatorname{osc}\left(h, \frac{1}{n}, .\right)\right\|_p = \operatorname{osc}_p\left(h, \frac{1}{n}\right) \to 0$ $(n \to \infty)$. Finally e) implies b), and because of b) assertion d) holds for arbitrary ε. □

The next lemma helps finding "smooth" versions of functions $h \in R_p$:

1.5. Lemma. a) *For each* $h \in R_p$ $(1 \leqq p \leqq \infty)$ *and* $\varepsilon > 0$ *holds:*

$$\operatorname{ess\,inf}\{h(y) \mid d(y, x) < \varepsilon\} \leqq h(x) \leqq \operatorname{ess\,sup}\{h(y) \mid d(y, x) < \varepsilon\} \quad m\text{-a.e.}$$

b) *For each* $h \in R_\infty$ *there is a* $h^*: X \to \mathbb{C}$ *with* $h^*_{|F} \in C(F)$ *and* $h = h^*$ *m-a.e. If* $h \in R_{\infty, \phi}$ *for continuous* ϕ, *then* $|h^*(x) - h^*(y)| \leqq \phi(d(x, y))$ *for all* $x, y \in F$.

Proof. a) If there were $x \in X$ and $\varepsilon > 0$ such that $h(y) > \operatorname{ess\,sup}\{h(z) \mid d(z, y) < 2\varepsilon\}$ for y in a subset of positive measure of $S_\varepsilon(x)$, then it would follow that $h(y) > \operatorname{ess\,sup}\{h(z) \mid d(x, z) < \varepsilon\}$ on a set of y's of positive measure in $S_\varepsilon(x)$ contradicting the definition of the essential supremum. As X can be covered by a finite number of such balls $S_\varepsilon(x)$ and as the same reasoning applies to the essential infimum, this proves a).

b) As $\operatorname{osc}(h, \varepsilon, .)$ is lower semi-continuous and bounded (by d of Lemma 1.4), $\sup_{x \in F} \operatorname{osc}(h, \varepsilon, x) = \operatorname{ess\,sup}_{x \in F} \operatorname{osc}(h, \varepsilon, x)$, such that $\operatorname{osc}(h, \varepsilon, x) \to 0$ $(\varepsilon \to 0)$ uniformly for all $x \in F$. For these x define now

$$h^*(x) = \lim_{\varepsilon \to 0} \operatorname{ess\,sup}\{h(y) \mid d(y, x) < \varepsilon\} = \lim_{\varepsilon \to 0} \operatorname{ess\,inf}\{h(y) \mid d(y, x) < \varepsilon\},$$

and set $h^*(x) = 0$ for $x \in X \setminus F$. As $h = h^*$ m-a.e. by part a), we only have to show that $h^*_{|F} \in C(F)$. But for $x, y \in F$ and arbitrary $\delta > 0$ we have

$$|h^*(x) - h^*(y)| \leqq \operatorname{osc}(h, d(x, y) + \delta, x) \leqq \|\operatorname{osc}(h, d(x, y) + \delta, .)\|_\infty \leqq \phi(d(x, y) + \delta)$$

for some $\phi \in \Phi$. Hence $h^*_{|F} \in C(F)$, and if ϕ is continuous,

$$|h^*(x) - h^*(y)| \leqq \phi(d(x, y)). \quad \square$$

1.6. Examples. a) For $1 \leqq p < \infty$ R_p is the class of all Riemann-integrable functions on X (mod m). This follows from e) of Lemma 1.4 (cf. [11], Chap. 7).

b) $R_{\infty|F} = C^*(F)$, the class of all continuous functions on F (mod m), by Lemma 1.5.

c) $S_{p, 1/p}$ will be called the class of functions of bounded p-variation. If $X = [0, 1]$ and m is the Lebesgue-measure, one can show that this class contains those functions of bounded p-variation considered in [3]. See Lemma 2.7.

d) $S_{\infty, r|F}$ is the class of Hölder-continuous functions on F with exponent r (mod m). This follows from Lemma 1.5.

Examples c) and d) suggest the following restriction of the class Φ:

1.7. Definition. $\Phi_1 = \{\phi \in \Phi \mid \phi(x) \geqq a \cdot x \ (0 < x \leqq A)$ for some $a > 0\}$.

Now we can prove the following density-result for S-classes:

1.8. Proposition. a) *$S_{p,\phi}$ is dense in $(L_m^p, \|.\|_p)$ for $1 \leqq p < \infty$ and $\phi \in \Phi_1$.*

b) *$S_{\infty,\phi|F}$ is dense in $(C^*(F), \|.\|_\infty)$ for all $\phi \in \Phi_1$.*

Proof. We first show b): $S_{\infty,\phi|F} \subseteq C^*(F)$ by Lemma 1.5. Furthermore, a) of Lemma 1.3 implies that for

$$\phi \in \Phi_1 : S_{\infty,\phi} = \bigcup_{n \in \mathbb{N}} R_{\infty, n \cdot \phi} \supseteq \bigcup_{n \in \mathbb{N}} R_{\infty,(\varepsilon \to n\varepsilon)} = S_{\infty,1},$$

and $S_{\infty,1|F}$ is dense in $C^*(F)$, as it is the space of Lipschitz-continuous functions on F (see Ex. 1.6.d).

We now show a): By b) of Lemma 1.4, $S_{p,\phi} \subseteq L_m^p$. By c) of Lemma 1.3, $S_{\infty,\phi} \subseteq S_{p,\phi}$, and the denseness of $C^*(X)$ in L_m^p together with part b) implies the denseness of $S_{p,\phi}$ in L_m^p. □

In order to make the S-spaces into Banach-spaces we pass to m-equivalence classes of functions and introduce a norm on them:

1.9. Definition. For $p \geqq 1$ and $\phi \in \Phi$ we define:

a) $BV_{p,\phi}$ is the space of m-equivalence classes of functions in $S_{p,\phi}$.

b) For $h: X \to \mathbb{C}$ set $\operatorname{var}_{p,\phi}(h) = \sup_{0 < \varepsilon \leqq A} \dfrac{\operatorname{osc}_p(h, \varepsilon)}{\phi(\varepsilon)}$.

c) For $h \in BV_{p,\phi}$ set $\|h\|_{p,\phi} = \operatorname{var}_{p,\phi}(h) + \|h\|_p$. ($\|.\|_{p,\phi}$ is well defined because of a) of Lemma 1.4.)

If $\phi(\varepsilon) = \varepsilon^r$ we simply write $BV_{p,r}$, $\|.\|_{p,r}$, $\operatorname{var}_{p,r}$. Observe that the definition depends on the constant A!

The proof of the following lemma is straightforward:

1.10. Lemma. *$BV_{p,\phi}$ is a linear space, and $\|.\|_{p,\phi}$ is a norm on it $(1 \leqq p \leqq \infty$, $\phi \in \Phi)$.*

In order to show that $(BV_{p,\phi}, \|.\|_{p,\phi})$ is a Banach-space with a compact embedding into L_m^p we need two preparatory lemmas:

1.11. Lemma. *If $h \in BV_{p,\phi}$ and if $\mathscr{P} = \{P_1, \ldots, P_N\}$ is a measurable partition of X, then $\|h - E_m[h|\mathscr{P}]\|_p \leqq \operatorname{var}_{p,\phi}(h) \cdot \phi(d)$, provided $d = \sup_{P \in \mathscr{P}} \operatorname{diam}(P) \leqq A$. Here $E_m[h|\mathscr{P}] = \sum_{i=1}^{N} m(P_i)^{-1} \cdot \int_{P_i} h\,dm \cdot 1_{P_i}$.*

Proof. By c) of Lemma 1.4 we have $\|h - E_m[h|\mathscr{P}]\|_p \leqq \operatorname{osc}_p(h, \varepsilon)$, where $\varepsilon = \sup_{P \in \mathscr{P}} \operatorname{diam}(P)$. As $\operatorname{osc}_p(h, \varepsilon) \leqq \operatorname{var}_{p,\phi}(h) \cdot \phi(\varepsilon)$, this proves the lemma. □

1.12. Lemma. *Let $(h_n)_{n \in \mathbb{N}}$ be a sequence in $BV_{p,\phi}$ converging in $\|.\|_p$-norm to some element $h \in L_m^p$. Then*

$$\operatorname{var}_{p,\phi}(h) \leqq \liminf_{n \to \infty} \operatorname{var}_{p,\phi}(h_n).$$

Proof. Passing, if necessary, to a subsequence we can assume that $\liminf_{n\to\infty} \mathrm{var}_{p,\phi}(h_n) = \lim_{n\to\infty} \mathrm{var}_{p,\phi}(h_n)$. Passing to a subsequence again we also can assume that $h_n(x) \to h(x)$ for all $x \in X \setminus N$, where N is some set of measure 0. Fix $x \in X$, $\varepsilon > 0$. There is a subset $N_x \subseteq X^2$ with $m^2(N_x) = 0$, such that for all $(y, z) \in X^2 \setminus N_x$ with $d(x, y)$, $d(x, z) < \varepsilon$ holds:

$$|h(y) - h(z)| = \lim_{n\to\infty} |h_n(y) - h_n(z)| \leqq \liminf_{n\to\infty} \mathrm{osc}(h_n, \varepsilon, x).$$

Hence $\mathrm{osc}(h, \varepsilon, x) \leqq \liminf_{n\to\infty} \mathrm{osc}(h_n, \varepsilon, x)$. By Fatou's lemma we get:

$$\int \mathrm{osc}(h, \varepsilon, x)^p \, dm(x) \leqq \liminf_{n\to\infty} \int \mathrm{osc}(h_n, \varepsilon, x)^p \, dm(x)$$

for $1 \leqq p < \infty$, while (for $p = \infty$) obviously

$$\operatorname*{ess\,sup}_{x\in X} \mathrm{osc}(h, \varepsilon, x) \leqq \liminf_{n\to\infty} \operatorname*{ess\,sup}_{x\in X} \mathrm{osc}(h_n, \varepsilon, x).$$

Hence, for $1 \leqq p \leqq \infty$,

$$\mathrm{osc}_p(h, \varepsilon) \leqq \liminf_{n\to\infty} \mathrm{osc}_p(h_n, \varepsilon) \leqq \phi(\varepsilon) \cdot \liminf_{n\to\infty} \mathrm{var}_{p,\phi}(h_n)$$

for all $\varepsilon \leqq A$ implying $\mathrm{var}_{p,\phi}(h) \leqq \liminf_{n\to\infty} \mathrm{var}_{p,\phi}(h_n)$. □

The main result of this section is:

1.13. Theorem. *For $1 \leqq p \leqq \infty$ and $\phi \in \Phi$ we have:*

a) *$E = \{f \in BV_{p,\phi} \mid \|f\|_{p,\phi} \leqq c\}$ is a compact subset of L^p_m for each $c > 0$.*

b) *$(BV_{p,\phi}, \|\cdot\|_{p,\phi})$ is a Banach-space.*

c) *For $\phi \in \Phi_1$, $BV_{p,\phi}$ is dense in L^p_m (in case $1 \leqq p < \infty$) or in $C^*(F)$ (in case $p = \infty$) respectively.*

Proof. a) Let f_n be in E $(n \geqq 1)$. From Lemma 1.11 and Theorem IV.8.18 in [6] it follows that there is a subsequence (g_n) of (f_n) and an element $f \in L^p_m$ with $\lim_{n\to\infty} \|f - g_n\|_p = 0$. Hence Lemma 1.12 implies that

$$\begin{aligned}\|f\|_{p,\phi} = \|f\|_p + \mathrm{var}_{p,\phi}(f) &\leqq \lim_{n\to\infty} \|g_n\|_p + \liminf_{n\to\infty} \mathrm{var}_{p,\phi}(g_n) \\ &= \liminf_{n\to\infty} \|g_n\|_{p,\phi} \leqq c, \quad \text{i.e.: } f \in E.\end{aligned}$$

b) follows immediately from I.1.6 in [13] now, and c) from Proposition 1.8. □

The following lemma will be used in the next section:

1.14. Lemma. *For fixed f and p, $\mathrm{osc}_p(f, \varepsilon)$ is continuous from below and isotone as a function of ε.*

Proof. $\mathrm{osc}(f, \varepsilon, x)$ is continuous from below and isotone as a function of ε for fixed x. For $1 \leqq p < \infty$ the assertion then follows from the monotone convergence theorem, while for $p = \infty$ it is enough to observe the fact that "$g_n \uparrow g$ pointwise" implies "$\|g_n\|_\infty \to \|g\|_\infty$" if $g_n, g > 0$. □

2. The One-Dimensional Case with $p=1$ (Trace Theorem, Products and Transformations)

Working with variation-norms one often needs theorems of the following type: Let Y be a "nice" subset of X, $f\in BV$. Then

$$\operatorname{var}(f\cdot 1_Y)\leqq C_1\cdot \operatorname{var}(f_{|Y})+C_2\cdot \int_Y |f|\,dm,$$

where C_1, C_2 are constants depending on Y only. (This is a combination of an extension- and a trace theorem, cf. [14].) In general such theorems are hard to establish. The constants C_1 and C_2 will depend on the dimension and shape of the boundary of Y, and there are many combinations of p and ϕ for which $\operatorname{var}_{p,\phi}$ does not satisfy such a relation at all. Therefore we restrict our interest here to the one-dimensional case needed in Sect. 3, i.e. X is the unit interval, m is an atom-free Borel-measure on X, and d is the pseudo-distance given by $d(x,y)=m\{z\,|\,x\leqq z\leqq y \text{ or } y\leqq z\leqq x\}$. As the d-topology is coarser than the usual topology on $[0,1]=X$, (X,d) is quasicompact, and m can be restricted to the σ-algebra $\mathscr{B}$, which – in accordance with Sect. 1 – denotes the Borel-σ-algebra of the d-topology. Throughout this section all topological and measure-theoretical statements will refer to d and $\mathscr{B}$.

2.1. Theorem. *Let $Y\subseteq X$ be an interval, $m(Y)\geqq 4A$. For each f: $Y\to\mathbb{C}$ and each $0<\tilde{\varepsilon}\leqq A$ we have*

$$\operatorname{osc}_1(f\cdot 1_Y,\tilde{\varepsilon})\leqq\left(2+\frac{8A}{m(Y)-2A}\right)\cdot\int_Y \operatorname{osc}(f_{|Y},\tilde{\varepsilon},x)\,dm(x)+\frac{4\tilde{\varepsilon}}{m(Y)}\cdot\int_Y |f(x)|\,dm(x).$$

Proof. Fix $0<\tilde{\varepsilon}\leqq A$ and $0<\varepsilon<\tilde{\varepsilon}$. There is a $n\in\mathbb{N}$ such that

$$2(n-1)\varepsilon<m(Y)\leqq 2n\varepsilon. \tag{5}$$

We introduce the following notations: Let f be a function from $Y\to\mathbb{C}$. Observe the different meanings of

$$\operatorname{osc}(f_{|Y},\varepsilon,x)=\operatorname{ess\,sup}\{|f(y_1)-f(y_2)|\,|\,y_1,y_2\in S_\varepsilon(x)\cap Y\}$$

and

$$\operatorname{osc}(f\cdot 1_Y,\varepsilon,x)=\operatorname{ess\,sup}\{|\tilde{f}(y_1)-\tilde{f}(y_2)|\,|\,y_1,y_2\in S_\varepsilon(x)\},$$

where $\tilde{f}(y)=f(y)$ $(y\in Y)$ and $\tilde{f}(y)=0$ $(y\in X\setminus Y)$. Now suppose that a_1 and a_2 are the left and the right endpoint of Y. Set

$$I_i=S_\varepsilon(a_i)\quad(i=1,2),\qquad I_0=Y\setminus(I_1\cup I_2),$$
$$h_i(x)=\operatorname{osc}(f_{|Y},\varepsilon,x)\cdot 1_{I_i}(x)\qquad(i=0,1,2),$$
$$s_i(x)=\operatorname*{ess\,sup}_{y\in S_\varepsilon(x)\cap Y}|f(y)|\cdot 1_{I_i}(x)\qquad(i=1,2).$$

Then

$$\operatorname{osc}(f\cdot 1_Y,\varepsilon,x)=h_0(x)+\sum_{i=1,2}\max\{h_i(x),s_i(x)\}. \tag{6}$$

Set

$$x(t)=\begin{cases}\sup\{y\in X\,|\,d(a_1,y)=t\} & \text{if } t\geqq 0\\ \inf\{y\in X\,|\,d(a_1,y)=-t\} & \text{if } t<0\end{cases}$$

and consider the intervals $J_k=Y\cap S_\varepsilon(x(2k\varepsilon))$ $(k=1,\dots,n-1)$. As the J_k are pairwise disjoint, there is a $k_0\in\{1,\dots,n-1\}$ with

$$\int_{J_{k_0}}\operatorname{osc}(f_{|Y},\tilde{\varepsilon},x)\,dm(x)\leqq\frac{1}{n-1}\cdot\int_Y\operatorname{osc}(f_{|Y},\tilde{\varepsilon},x)\,dm(x). \tag{7}$$

For $\xi\in[-\varepsilon,\varepsilon]$ define now $z_k(\xi)=x(\xi+2k\varepsilon)$ $(k=0,\dots,k_0)$ and $z_k(\xi)=x(\xi+\delta+2(k-1)\varepsilon)$ $(k=k_0+1,\dots,n)$, where $\delta=m(Y)-2(n-1)\varepsilon>0$ according to (5). Furthermore set $F_i(\xi)=\operatorname{ess\,sup}\{|f(y)-f_0|\,|\,y\in U_i(\xi)\cap Y\}$ where $U_1(\xi)=S_{\varepsilon+\xi}(a_1)$, $U_2(\xi)=S_{\varepsilon-\xi}(a_2)$, and $f_0=m(Y)^{-1}\cdot\int_Y f\,dm$. Then

$$\begin{aligned} &s_1(x)\leqq F_1(\xi)+|f_0| \quad \text{for } x=x(\xi)\in I_1,\\ &s_2(x)\leqq F_2(\xi)+|f_0| \quad \text{for } x=x(m(Y)+\xi)\in I_2,\end{aligned} \tag{8}$$

and

$$\begin{aligned}&\max\{h_1(x(\xi)),F_1(\xi)\}+\max\{h_2(x(m(Y)+\xi)),F_2(\xi)\}\\ &\leqq\frac{m(Y)}{m(Y)-2\varepsilon}\sum_{k=0}^{n}\operatorname{osc}(f_{|Y},\tilde{\varepsilon},z_k(\xi)).\end{aligned} \tag{9}$$

In order to show the latter inequality one has to consider four cases:

i) $h_1(x(\xi))\geqq F_1(\xi)$, $h_2(x(m(Y)+\xi))\geqq F_2(\xi)$. This case is trivial.

ii) $h_1(x(\xi))<F_1(\xi)$, $h_2(x(m(Y)+\xi))<F_2(\xi)$. In this case there is a $k\in\{1,\dots,n-1\}$ such that

$$\operatorname*{ess\,inf}_{y\in S_{\tilde{\varepsilon}}(z_k(\xi))}f(y)\leqq f_0\leqq\operatorname*{ess\,sup}_{y\in S_{\tilde{\varepsilon}}(z_k(\xi))}f(y),$$

which proves the inequality in this case.

iii) $h_1(x(\xi))\geqq F_1(\xi)$, $h_2(x(m(Y)+\xi))<F_2(\xi)$. This case is more delicate and is responsable for the factor $\dfrac{m(Y)}{m(Y)-2\varepsilon}$. If there is a $k\in\{1,\dots,n\}$ as in ii), the inequality again is true (even without the factor). Otherwise, setting $u=x(\xi+\varepsilon)$ we can suppose w.l.o.g. that $c=\operatorname*{ess\,inf}_{y\geqq u}(f(y)-f_0)>0$. Then

$$F_2(\xi)\leqq\sum_{k=1}^{n}\operatorname{osc}(f_{|Y},\tilde{\varepsilon},z_k(\xi))+c$$

and

$$\begin{aligned}m([u,a_2])\cdot c&\leqq\int_u^{a_2}(f(y)-f_0)\,dm(y)=-\int_{a_1}^{u}(f(y)-f_0)\,dm(y)\\ &\leqq 2\varepsilon\cdot F_1(\xi)\leqq 2\varepsilon\cdot h_1(x(\xi)),\end{aligned}$$

i.e.

$$c\leqq h_1(x(\xi))\cdot\frac{2\varepsilon}{m(Y)-2\varepsilon}$$

such that

$$h_1(x(\xi))+F_2(\xi)\leqq \sum_{k=0}^{n} \operatorname{osc}(f_{|Y}, \tilde{\varepsilon}, z_k(\xi))+h_1(x(\xi))\cdot\frac{2\varepsilon}{m(Y)-2\varepsilon}$$
$$\leqq\frac{m(Y)}{m(Y)-2\varepsilon}\cdot\sum_{k=0}^{n}\operatorname{osc}(f_{|Y}, \tilde{\varepsilon}, z_k(\xi)).$$

iv) $h_1(x(\xi))<F_1(\xi)$, $h_2(x(m(Y)+\xi))\geqq F_2(\xi)$. This case is analogous to iii). From (6), (8), and (9) it follows that

$$\begin{aligned}
&\operatorname{osc}_1(f\cdot 1_Y, \varepsilon)\\
&\quad=\int \operatorname{osc}(f\cdot 1_Y, \varepsilon, x)\,dm(x)\\
&\quad\leqq\int h_0(x)\,dm(x)+|f_0|\cdot 4\varepsilon+\frac{m(Y)}{m(Y)-2\varepsilon}\cdot\int_{-\varepsilon}^{\varepsilon}\left(\sum_{k=0}^{n}\operatorname{osc}(f_{|Y}, \tilde{\varepsilon}, z_k(\xi))\right)d\xi\\
&\quad\leqq\int h_0(x)\,dm(x)+|f_0|\cdot 4\varepsilon+\frac{m(Y)}{m(Y)-2\varepsilon}\cdot\int_{x(-\varepsilon)}^{x(m(Y)+\varepsilon)}\operatorname{osc}(f_{|Y}, \tilde{\varepsilon}, x)\,dm(x)\\
&\qquad\qquad+\frac{m(Y)}{m(Y)-2\varepsilon}\cdot\int_{J_{k_0}}\operatorname{osc}(f_{|Y}, \tilde{\varepsilon}, x)\,dm(x)
\end{aligned}$$

according to the choice of the $z_k(\xi)$,

$$\leqq\frac{m(Y)}{m(Y)-2\varepsilon}\cdot\left(2+\frac{1}{n-1}\right)\cdot\int_Y\operatorname{osc}(f_{|Y}, \tilde{\varepsilon}, x)\,dm(x)+4\varepsilon\cdot|f_0|$$

by (7) and since $\sup_{x\in S_\varepsilon(a_i)\setminus Y}\operatorname{osc}(f_{|Y}, \tilde{\varepsilon}, x)\leqq\inf_{x\in S_\varepsilon(a_i)\cap Y}\operatorname{osc}(f_{|Y}, \tilde{\varepsilon}, x)\quad(i=1,2)$,

$$\leqq\left(2+\frac{8\varepsilon}{m(Y)-2\varepsilon}\right)\int_Y\operatorname{osc}(f_{|Y}, \tilde{\varepsilon}, x)\,dm(x)+4\varepsilon\cdot|f_0|$$

by (5) and as $m(Y)\geqq 4A\geqq 4\varepsilon$.

As $\operatorname{osc}_1(f\cdot 1_Y, \varepsilon)$ is continuous from below in the variable ε (see Lemma 1.14), this implies that for each $0<\tilde{\varepsilon}\leqq A$:

$$\operatorname{osc}_1(f\cdot 1_Y, \tilde{\varepsilon})\leqq\left(2+\frac{8A}{m(Y)-2A}\right)\cdot\int_Y\operatorname{osc}(f_{|Y}, \tilde{\varepsilon}, x)\,dm(x)$$
$$+4\tilde{\varepsilon}\cdot m(Y)^{-1}\cdot\int_Y|f|\,dm.\quad\square$$

The proof of the following lemma is similar but easier. We skip it:

2.2. Lemma. *For bounded* $f\colon Y\to\mathbb{C}$ *($Y\subseteq X$ an interval) and* $0<\varepsilon\leqq A$ *we have*

$$\|f\|_\infty\leqq\varepsilon^{-1}\cdot\left(\int_Y\operatorname{osc}(f_{|Y}, \varepsilon, x)\,dm(x)\right)+m(Y)^{-1}\cdot\int_Y|f|\,dm.$$

(This estimate is quite rough but sufficient for our purposes.)

2.3. Lemma. *Let $Y, Z \subseteq X$, $T\colon Y \to Z$ be bijective, and $f\colon Y \to \mathbb{C}$. Set*

$$\tilde{\varepsilon}(z) = \sup\{d(T^{-1}y, T^{-1}z) \mid y \in S_\varepsilon(z) \cap Z\}.$$

Then $\operatorname{osc}(f \circ T^{-1}{}_{|Z}, \varepsilon, z) \leqq \operatorname{osc}(f_{|Y}, \tilde{\varepsilon}(z), T^{-1}z)$ *for each* $z \in Z$.

The proof is straightforward.

2.4. Lemma. *Let $Y, Z \subseteq X$ be intervals, $T\colon Y \to Z$ an order-isomorphism or -antiisomorphism, non-singular with respect to m, and call $T' = \frac{d}{dm}(m \circ T)$ the Radon-Nikodym derivative of T with respect to m. Suppose that $T'(y) \geqq \alpha > 0$ for* a.e. $y \in Y$. *Then for each $f\colon Y \to \mathbb{C}$*

$$\int_Z \operatorname{osc}\left(\frac{f}{T'} \circ T^{-1}{}_{|Z}, \varepsilon, z\right) dm(z) \leqq \int_Y \operatorname{osc}(f_{|Y}, \alpha^{-1}\varepsilon, y)\, dm(y)$$
$$+5 \cdot \int_Z \operatorname{osc}\left(\frac{1}{T'} \circ T^{-1}{}_{|Z}, \varepsilon, z\right) dm(z) \left(m(Y)^{-1} \cdot \int_Y |f|\, dm + \frac{1}{A} \cdot \int_Y \operatorname{osc}(f_{|Y}, A, y)\, dm(y)\right).$$

Proof. Using a) of Lemma 1.5 one easily shows that for a.e. $z \in Z$

$$\operatorname{osc}\left(\frac{f}{T'} \circ T^{-1}{}_{|Z}, \varepsilon, z\right) \leqq |f(T^{-1}z)| \cdot \operatorname{osc}\left(\frac{1}{T'} \circ T^{-1}{}_{|Z}, \varepsilon, z\right)$$
$$+ \frac{1}{T'(T^{-1}z)} \cdot \operatorname{osc}(f \circ T^{-1}{}_{|Z}, \varepsilon, z)$$
$$+2 \cdot \operatorname{osc}\left(\frac{1}{T'} \circ T^{-1}{}_{|Z}, \varepsilon, z\right) \cdot \operatorname{osc}(f \circ T^{-1}{}_{|Z}, \varepsilon, z). \tag{10}$$

We first treat the integral over the second term in this sum:

$$\int_Z \operatorname{osc}(f \circ T^{-1}{}_{|Z}, \varepsilon, z) \cdot (T^{-1})'(z)\, dm(z) \tag{11}$$
$$\leqq \int_Z \operatorname{osc}(f_{|Y}, \tilde{\varepsilon}(z), T^{-1}z) \cdot (T^{-1})'(z)\, dm(z) \quad \text{(by 2.3)}$$
$$\leqq \int_Y \operatorname{osc}(f_{|Y}, \alpha^{-1}\varepsilon, y)\, dm(y)$$

by integral transformation and observing that $T' \geqq \alpha > 0$ implies $\tilde{\varepsilon} \leqq \alpha^{-1}\varepsilon$.

The integral over the first and the third term is estimated as follows:

$$\int_Z \operatorname{osc}\left(\frac{1}{T'} \circ T^{-1}{}_{|Z}, \varepsilon, z\right) \cdot (|f(T^{-1}z)| + 2 \cdot \operatorname{osc}(f \circ T^{-1}{}_{|Z}, \varepsilon, z))\, dm(z) \tag{12}$$
$$\leqq 5 \cdot \|f_{|Y}\|_\infty \cdot \int_Z \operatorname{osc}\left(\frac{1}{T'} \circ T^{-1}{}_{|Z}, \varepsilon, z\right) dm(z)$$
$$\leqq 5 \cdot \int_Z \operatorname{osc}\left(\frac{1}{T'} \circ T^{-1}{}_{|Z}, \varepsilon, z\right)$$
$$\cdot \left(m(Y)^{-1} \cdot \int_Y |f|\, dm + \frac{1}{A} \cdot \int_Y \operatorname{osc}(f_{|Y}, A, y)\, dm(y)\right) dm(z) \quad \text{(by 2.2)}$$

(10), (11), and (12) together prove the lemma. □

If m is not the Lebesgue-measure on $[0, 1]$ and d is not the Euclidean metric, it is hard to recognize functions in $BV_{p,\phi}$. As we shall deal with such situations in Sect. 3, it is important to characterize subclasses of $BV_{p,\phi}$ at least.

2.6. Definition. For a function f: $[0, 1]\to\mathbb{C}$ define the universal p-variation by

$$\operatorname{var}_p(f)=\sup_{0\leqq a_0<a_1<\ldots<a_n\leqq 1}\left(\sum_{i=1}^{n}|f(a_i)-f(a_{i-1})|^p\right)^{1/p}$$

and denote the space of functions of universally bounded p-variation by $UBV_p=\{f\colon[0,1]\to\mathbb{C}\,|\operatorname{var}_p(f)<\infty\}$.

These are exactly the functions of bounded p-variation in the sense of [3]. The following lemma justifies the notation:

2.7. Lemma. *$UBV_p\subseteq\bigcap_m BV_{p,1/p}$ for all $1\leqq p<\infty$, where the intersection ranges over all spaces $BV_{p,1/p}$ which stem from any atom-free finite Borel-measure m on $[0, 1]$ and its associated pseudo-distance d. In particular, if m is a probability-measure, then $\operatorname{var}_{p,1/p}(f)\leqq 2^{1/p}\cdot\operatorname{var}_p(f)$.*

Proof. As each $f\in UBV_p$ is bounded, $\int|f|\,dm<\infty$ for any finite measure m. With a similar argument as in the proof of Theorem 2.1 one shows that $\int\operatorname{osc}(f,\varepsilon,x)^p\,dm(x)\leqq 2\varepsilon\cdot(\operatorname{var}_p(f))^p$. We leave the details to the reader. □

Problem. Can the inclusion in Lemma 2.7 be replaced by equality?

3. Perron-Frobenius Operators for Piecewise Monotonic Transformations Acting on BV_1-Classes

As in [7] we now assume the following situation: Let $\{I_1, \ldots, I_N\}$ be a finite partition of $X=[0, 1]$ into intervals, and let T: $X\to X$ be a transformation which is monotone and continuous on each I_i. (We call such a transformation piecewise monotonic.) Assume that

(13) there is a Borel-probability m on X with respect to which T is non-singular. Call $g=\left(\frac{d}{dm}(m\circ T)\right)^{-1}=\frac{1}{T'}$, and suppose $g(x)\leqq\alpha^{-1}<\infty$ m-a.e.

Define

(14) P: $mb(X)\to mb(X)$ $(=\{f\colon X\to\mathbb{C}\,|\,f$ measurable and bounded$\})$ by

$$Pf(x)=\sum_{i=1}^{N}(f\cdot g)\circ T_i^{-1}\cdot 1_{TI_i},\qquad\text{where } T_i=T_{|I_i}.$$

Then $m(P1_B)=\sum_{i=1}^{N}\int_{T(B\cap I_i)}\frac{d}{dm}(m\circ T_i^{-1})\,dm=\sum_{i=1}^{N}\int_{B\cap I_i}dm=m(B)$ for each $B\in\mathscr{B}$, such that

(15) $m(Pf)=m(f)$ for all $f\in mb(X)$, and P extends to a positive linear contraction on L^1_m.

From these assumptions it follows that m is atom-free (cf. Lemma 2 in [7]), and with $d(x,y)=m(\{z|x\leqq z\leqq y \text{ or } y\leqq z\leqq x\})$ we are in the situation of Sect. 2. Our goal is to find a spectral decomposition for P as in [7]. To this end we need:

3.1. Lemma. $\Big($*Remember that* $\operatorname{var}_{p,\phi}(f)=\sup\limits_{0<\varepsilon\leqq A}\dfrac{\operatorname{osc}_p(f,\varepsilon)}{\phi(\varepsilon)}$ *depends on the constant* A!$\Big)$

If $g\in UBV_p$ $(1\leqq p<\infty)$, *then for each* $\delta>0$ *there are constants* $A=A(\delta)>0$ *and* $K=K(\delta)>0$ *such that for* $f\colon X\to\mathbb{C}$

$$\operatorname{var}_{1,1/p}(Pf)\leqq\frac{2+\delta}{\alpha^{1/p}}\operatorname{var}_{1,1/p}(f)+K\cdot\|f\|_1.$$

Proof. Set $M=\frac{\delta}{20}\cdot\alpha^{-1/p}\cdot\left(\frac{\delta}{16+2\delta}\right)^{1-1/p}$. Refining, if necessary, the partition $\{I_1,\dots,I_N\}$ we can assume that

$$\operatorname{var}_p(g_{|I_i})<M\qquad(i=1,\dots,N). \tag{16}$$

This is possible because each g of universally bounded p-variation has one-sided limits in each point and its set of discontinuities is at most countable. If d_j denotes the height of the j-th discontinuity, then $\sum_{j=1}^{\infty}d_j^p<\infty$. Hence there are only finitely many j's with $d_j\geqq M$, and these d_j can be taken as new partition-points. Refining the partition further one finally obtains (16). We remark that the refinement can be done in such a way that if

$$\Gamma_-=\min\{m(TI_i)|i\in\{1,\dots,N\},\ m(I_i)>0\}$$

and

$$\Gamma_+=\max\{m(TI_i)|i\in\{1,\dots,N\}\},$$

then

$$\Gamma_+\leqq 2\cdot\Gamma_-. \tag{17}$$

Now set

$$A=A(\delta)=\Gamma_-\cdot\frac{\delta}{16+2\delta}. \tag{18}$$

Then we have for $0<\varepsilon\leqq A$:

$$\begin{aligned}\operatorname{osc}_1(Pf,\varepsilon)&\leqq\sum_{\substack{i=1\\ m(I_i)>0}}^{N}\operatorname{osc}_1((f\cdot g)\circ T_i^{-1}\cdot 1_{TI_i},\varepsilon)\leqq\sum_{\substack{i=1\\ m(I_i)>0}}^{N}\left(\left(2+\frac{8A}{m(TI_i)-2A}\right)\right.\\ &\quad\left.\cdot\int_{TI_i}\operatorname{osc}\left(\frac{f}{T_i'}\circ T_i^{-1}{}_{|TI_i},\varepsilon,x\right)dm(x)+4\varepsilon m(TI_i)^{-1}\cdot\int_{TI_i}\left|\frac{f}{T_i'}\right|\circ T_i^{-1}\,dm\right)\quad\text{(by 2.1)}.\end{aligned} \tag{19}$$

Furthermore

$$\int_{TI_i} (|f| \circ T_i^{-1}) \cdot \left(\frac{1}{T_i'} \circ T_i^{-1}\right) dm = \int_{I_i} |f|\, dm, \tag{20}$$

$$2 + \frac{8A}{m(TI_i) - 2A} \leqq 2 + \frac{\delta}{2} \quad \text{(by (18))}, \tag{21}$$

$$\begin{aligned} &\int_{TI_i} \operatorname{osc}\left(\frac{f}{T_i'} \circ T_i^{-1}{}_{|TI_i}, \varepsilon, x\right) dm(x) \\ &\leqq \int_{I_i} \operatorname{osc}(f_{|I^i}, \alpha^{-1}\varepsilon, y)\, dm(y) + 5 \cdot \int_{TI_i} \operatorname{osc}\left(\frac{1}{T_i'} \circ T_i^{-1}{}_{|TI_i}, \varepsilon, z\right) dm(z) \\ &\quad \cdot \left(m(I_i)^{-1} \int_{I_i} |f|\, dm + \frac{1}{A} \int_{I_i} \operatorname{osc}(f_{|I_i}, A, y)\, dm(y)\right) \quad \text{(by 2.4)}, \end{aligned} \tag{22}$$

and

$$\begin{aligned} &\int_{TI_i} \operatorname{osc}\left(\frac{1}{T_i'} \circ T_i^{-1}{}_{|TI_i}, \varepsilon, z\right) dm(z) \\ &\leqq \left(\int_{TI_i} \operatorname{osc}\left(\frac{1}{T_i'} \circ T_i^{-1}{}_{|TI_i}, \varepsilon, z\right)^p dm(z)\right)^{1/p} \cdot (m(TI_i))^{(1-1/p)} \\ &\leqq (2\varepsilon)^{1/p} \cdot \operatorname{var}_p\left(\frac{1}{T_i'} \circ T_i^{-1}{}_{|TI_i}\right) \cdot \Gamma_+^{(1-1/p)} \quad \text{(by 2.7)} \\ &= (2\varepsilon)^{1/p} \cdot \Gamma_+^{(1-1/p)} \cdot \operatorname{var}_p(g_{|I_i}), \quad \text{as the universal } p\text{-variation is invariant under order-iso- and -antiisomorphisms,} \\ &< 2 \cdot \varepsilon^{1/p} \cdot \Gamma_-^{(1-1/p)} \cdot M \quad \text{(by (16) and (17))}. \end{aligned} \tag{23}$$

If we set $\gamma = \min\{m(I_i) \mid i \in \{1, \ldots, N\}, m(I_i) > 0\}$, then (19)–(23) imply

$$\begin{aligned} \operatorname{osc}_1(Pf, \varepsilon) \leqq \left(2 + \frac{\delta}{2}\right) &\left(\int_X \operatorname{osc}(f, \alpha^{-1}\varepsilon, y)\, dm(y) + 10 \cdot \varepsilon^{1/p} \cdot \Gamma_-^{1-1/p} \cdot M \right. \\ &\left. \cdot \left(\frac{1}{\gamma} \cdot \int_X |f|\, dm + \frac{1}{A} \cdot \int_X \operatorname{osc}(f, A, y)\, dm(y)\right)\right) \\ &+ 4\varepsilon \cdot \Gamma_-^{-1} \cdot \int_X |f|\, dm, \end{aligned}$$

such that

$$\begin{aligned} \frac{\operatorname{osc}_1(Pf, \varepsilon)}{\varepsilon^{1/p}} &\leqq \left(\left(2 + \frac{\delta}{2}\right) \cdot \alpha^{-1/p} + 10 \cdot \Gamma_-^{1-1/p} \cdot M \cdot A^{1/p-1}\right) \cdot \operatorname{var}_{1,1/p}(f) \\ &\quad + \left(\left(2 + \frac{\delta}{2}\right) \cdot 10 \cdot \Gamma_-^{1-1/p} \cdot M \cdot \frac{1}{\gamma} + 4 \cdot A^{1-1/p} \cdot \Gamma_-^{-1}\right) \cdot \int_X |f|\, dm \\ &\leqq (2 + \delta) \cdot \alpha^{-1/p} \cdot \operatorname{var}_{1,1/p}(f) + K \cdot \int_X |f|\, dm \end{aligned}$$

by definition of M and A, which proves the lemma. □

The following theorem is an easy consequence of this lemma:

3.2. Theorem. *Let* T, P, *and* m *be as described in* (13)–(15). *If* $g \in UBV_p$ $(1 \leqq p < \infty)$ *and if there is a* $n \in \mathbb{N}$ *with* $\|g_n\|_\infty < 1$ $\left(\text{where } g_n(x) = \left(\frac{d}{dm}(m \circ T^n)\right)^{-1}(x) = g(T^{n-1}x) \cdot \ldots \cdot g(Tx) \cdot g(x)\right)$, *then there are* $k \in \mathbb{N}$, $0 < \beta < 1$, *and* $C > 0$ *such that for each* $f \in BV_{1,1/p}$

$$\|P^k f\|_{1,1/p} \leqq \beta \cdot \|f\|_{1,1/p} + C \cdot \|f\|_1.$$

Proof. It is easy to see that T^j is piecewise monotonic and $g_j \in UBV_p$ for all $j \in \mathbb{N}$. (UBV_p is closed under product and order-(anti)isomorphism.) Hence there is a $k \in \mathbb{N}$ with $g_k \in UBV_p$ and $\|g_k\|_\infty \leqq (\frac{1}{4})^p$. Applying Lemma 3.1 with $\delta = 1$ to P^k gives

$$\begin{aligned}\|P^k f\|_{1,1/p} &= \operatorname{var}_{1,1/p}(P^k f) + \|P^k f\|_1 \\ &\leqq \tfrac{3}{4} \operatorname{var}_{1,1/p}(f) + (K+1) \cdot \|f\|_1 \leqq \tfrac{3}{4} \|f\|_{1,1/p} + (K+1) \cdot \|f\|_1. \quad \square\end{aligned}$$

As P is a L^1_m-contraction, the last theorem and Theorem 1.13 allow to apply an ergodic theorem of Ionescu-Tulcea and Marinescu [9] stating in an abstract Banach space setting the results 3.3.1–3.3.3 of the following theorem (they are formulated here for the special situation under consideration). From this 3.3.4–3.3.8 can be derived as in [7].

3.3. Theorem. *Under the assumptions of the preceding theorem holds:*

3.3.1. $P\colon L^1_m \to L^1_m$ *has a finite number of eigenvalues* $\lambda_1, \ldots, \lambda_r$ *of modulus* 1.

3.3.2. $E_i = \{f \in L^1_m \mid Pf = \lambda_i f\} \subseteq BV_{1,1/p}$ *and* $\dim(E_i) < \infty$ $(i = 1, \ldots, r)$.

3.3.3. $P = \sum_{i=1}^{r} \lambda_i \Psi_i + Q$, *where the* Ψ_i *are projections onto the eigenspaces* E_i, $\|\Psi_i\|_1 \leqq 1$, *and* Q *is a linear operator on* L^1_m *with* $Q(BV_{1,1/p}) \subseteq BV_{1,1/p}$, $\sup_{n \in \mathbb{N}} \|Q^n\|_1 < \infty$, *and* $\|Q^n\|_{1,1/p} = O(q^n)$ *for some* $0 < q < 1$. *Furthermore* $\Psi_i \Psi_j = 0$ $(i \neq j)$ *and* $\Psi_i Q = Q \Psi_i = 0$ *for all* i. *(This means that* P *is quasi-compact as operator on* $(BV_{1,1/p}, \|\cdot\|_{1,1/p})$.*)*

3.3.4. 1 *is an eigenvalue of* P, *and assuming* $\lambda_1 = 1$ *and* $h = \Psi_1(1)$, $\mu = h \cdot m$ *is the greatest* T*-invariant probability on* X *absolutely continuous with respect to* m, *i.e. if* $\tilde{\mu}$ *is* T*-invariant and* $\tilde{\mu} \ll m$, *then* $\tilde{\mu} \ll \mu$.

3.3.5. There is a partition $\{C_{k,l} \mid k = 1, \ldots, r,\ l = 1, \ldots, L_k\}$ *of* X *such that* $TC_{k,l} = C_{k,(l+1) \bmod L_k}$ *and* $T^{L_k}_{|C_{k,l}}$ *is weakly mixing for all* k *and* l.

3.3.6. If (T, μ) *is weakly mixing and* Π *is an arbitrary finite partition of* X *into intervals, then there are constants* $K > 0$ *and* $0 < \rho < 1$ *such that* $\sum_R \sum_S |\mu(R \cap S) - \mu(R) \cdot \mu(S)| \leqq K \cdot \rho^l$ *where the summation extends over all* $R \in \bigvee_{i=0}^{n-1} T^{-i}\Pi$, $S \in \bigvee_{i=n+l}^{n+l+k-1} T^{-i}\Pi$, *and* $k, l, n \in \mathbb{N}$ *are arbitrary. This means that* Π *is a weak*

Bernoulli-partition for (T, μ) with exponentially decreasing mixing-coefficients. In particular, the natural extension of (T, μ) is isomorphic to a Bernoulli-shift (cf. [2]). (The proof of this fact needs some minor modifications compared to [7], as g is no longer in UBV_1 but in UBV_p.)

3.3.7. If (T, μ) is weakly mixing, $f \in UBV_s$ for some $1 \leqq s < \infty$, f real-valued, $\int f d\mu = 0$, and if $S(t)$ is defined as $S(t) = \sum_{0 \leqq i < t} f \circ T^i$, then the series $\sigma^2 = \int f^2 d\mu + 2 \cdot \sum_{k=1}^{\infty} \int f \cdot (f \circ T^k) d\mu$ converges absolutely, $\int S(t)^2 d\mu = t \cdot \sigma^2 + O(1)$, and if $\sigma^2 \neq 0$ the following holds:

a) $\sup_{z \in \mathbb{R}} \left| \mu \left\{ \frac{1}{\sigma \sqrt{t}} S(t) \leqq z \right\} - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-x^2/2} dx \right| = O(t^{-\theta})$ *for some* $\theta > 0$.

b) *Without changing its distribution one can redefine the process $(S(t))_{t \geqq 0}$ on a richer probability space together with a standard Brownian motion $(B(t))_{t \geqq 0}$ such that $|\sigma^{-1} \cdot S(t) - B(t)| = O(t^{1/2 - \lambda})$ μ-a.e. for some $\lambda > 0$.*

3.3.8. μ is an equilibrium state for $\log g$ *on X, i.e.*

$$h(\mu) + \int \log g \, d\mu = \sup \{ h(\nu) + \int \log g \, d\nu \mid \nu \text{ is a } T\text{-invariant probability on } X \},$$

where $h(\nu)$ is the entropy of (T, ν).

3.3.9. In [4] it has been shown that 3.3.6 *is sufficient for limit theorems for U-statistics and v. Mises' functionals based on data from a stationary process $X_n(\omega) = f(T^n \omega)$ $(\omega \in X)$, where again $f \in UBV_s$ for some $1 \leqq s < \infty$.*

Recently, M. Rychlik [13] has shown for the case $p = 1$ how one can derive the spectral decomposition 3.3.3 without referring to the theorem of Ionescu-Tulcea and Marinescu. Using a slightly refined version of the inequality of Theorem 3.2 he gives a direct proof that the operator $P: BV \to BV$ is quasicompact (that is what 3.3.1–3.3.3 mainly say) and a very short proof of 3.3.6. Furthermore, his technique allows him to treat transformations with countably many intervals of monotonicity in a very elegant way.

As in [7] and [8] one can show that the hypothesis of Theorem 3.2 (and hence of 3.3) are satisfied in the following situations:

3.4. Theorem. *Let $\phi \in UBV_p$, $(\sup \phi - \inf \phi) < h_{top}(T)$. Then there is a probability m on X and a real $\lambda > 0$ such that m and $g = \lambda e^{\phi}$ satisfy* (13)–(15) *and the hypothesis of* 3.3.

Proof. It easy to see that $\phi \in UBV_p$ implies $g \in UBV_p$. The existence of the measure m and of a natural number n with $\|g_n\|_\infty < 1$ can be proved as in [8].

3.5. Theorem. *Suppose $T_{|I_i}$ is differentiable for each i, $\frac{1}{T'} \in UBV_p$, and $|(T^n)'| \geqq \alpha > 1$ for some $n \in \mathbb{N}$. Then the hypothesis of* 3.3 *are satisfied for m = Lebesgue-measure.*

Let $\mathscr{P}=\{I_1, \ldots, I_N\}$ be a partition of $[0,1]$ into intervals on which T is monotone and continuous, and call $J_n(x)$ the element of $\bigvee_{i=0}^{n-1} T^{-i}\mathscr{P}$ containing x.

3.6. Definition. T is completely covering, if for each $x\in[0,1]$ there is a $k\in\mathbb{N}$ and an infinite subset $B\subseteq\mathbb{N}$ such that for all $n\in B$:

$$\bigcup_{j=1}^{k} T^{n+j} J_n(x)=[0,1].$$

This is a kind of weak specification property that has been introduced in [8], § 3. Also the following examples can be found there:

3.7. Examples. a) Irreducible Markov-transformations are completely covering. (That are transformations with $T(I_i)\cap I_j \neq \emptyset \Rightarrow I_j\subseteq T(I_i)$ and $\bigcup_{n=0}^{\infty} T^n(I_i)=[0,1]$ for all I_i.)

b) β-transformations $(x\to\beta x \bmod 1,\ \beta>1)$ are completely covering.

c) $T(x)=\beta x+\alpha \bmod 1$ $(\beta>1)$ is completely covering, if $1\notin \text{closure}\{T^k(0)\,|\,k\in\mathbb{N}\}$ or $0\notin \text{closure}\{T^k(1)\,|\,k\in\mathbb{N}\}$, particularly if 0 or 1 is periodic under T. b) is a special case of c), of course.

3.8. Theorem. *If T is completely covering and*

$$\operatorname{var}_n(\phi)=\sup\left\{|\phi(x)-\phi(y)|\,\middle|\, x,y\in J\in\bigvee_{i=0}^{n-1} T^{-i}\mathscr{P}\right\}=O(q^n)$$

for some $0<q<1$, then there is a probability-measure m satisfying (13)–(15), *and for this m the hypothesis of* 3.3 *are satisfied.*

Proof. From $\operatorname{var}_n(\phi)=O(q^n)$ it follows that $g\in UBV_p$ for $p>-\dfrac{\ln N}{\ln q}$, where N is the number of monotonicity-intervals of T. The existence of a measure m satisfying (13)–(15) is proved as in §1 of [8], and $\|g_n\|_\infty<1$ for some $n\in\mathbb{N}$ is proved as Theorem 3 of [8].

Remark. The proofs from [8] carry over to 3.4 and 3.8 since in [8] the fact that ϕ is of bounded variation has been used only in order to show that one-sided limits of ϕ exist in every point and that ϕ has at most countably many discontinuities, and all this is true also for $\phi\in UBV_p$.

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