Z. Wahrscheinlichkeitstheorie verw. Geb. 22, 158–160 (1972)

On Dvoretzky Coverings for the Circle

Benoit B. Mandelbrot

The problem of random cutouts treated in the preceding paper [3] – henceforth to be called the "R-problem" – is a more natural variant of a problem posed earlier by Dvoretzky: the "D-problem"; see [2, Chapter 9]. D cutouts are open arcs of the circle of unit circumference, whose positions t_n are random (t_n uniform from 0 to 1) but whose durations z_n are prescribed, with $z_n<1$. The problem is to determine whether covering of the circle by such cutouts is a.s., or has a probability less than 1. Let the z_n be ranked by nonincreasing value, and the number of cutouts satisfying $z_n \geqq z$ be designated by $-G(z)$. For various specific forms of $G(z)$, the probability of D covering has been obtained by Dvoretzky, J.P. Kahane, Erdös and Billard, but for other $G(z)$'s the problem had remained open. An example is $z_n=1/(n+1)$. I propose to show that the partial solution of the R-problem in [3] can help improve the existing very partial solution of the D-problem. In particular: if $z_n=1/(n+1)$, covering is shown to be a.s. We proceed by a sequence of small steps.

1. The probability of D covering is unaffected by adding and deleting cutouts in finite numbers, so that one can eliminate all z_n such that $z_n \geqq \frac{1}{2}$, and thus assume $-G(\frac{1}{2})=0$. Also, the circle $]0,1]$ is a.s. D covered iff the half circle $]\frac{1}{2},1]$ is a.s. D-covered.

2. *Comparison Rule:* Assume $G'(z)>G''(z)$ for all z. If D cutouts ruled by $G'(z)$ cover the circle a.s., then D cutouts ruled by $G''(z)$ also cover it a.s. If D cutouts ruled by $G''(z)$ do not cover the circle a.s., then D cutouts ruled by $G'(z)$ do not either cover the circle a.s.

3. To provide a connecting link between the R and the D problems, define a third variant, the "D_0 problem" as being the problem of cutouts on the real line R with random t_n (uniform from 0 to 1) and prescribed z_n. The number of D_0 cutouts with $z_n \geqq z$ will be denoted $-G_0(z)$. The half circle $]\frac{1}{2},1]$ is a.s. D-covered iff the set $]\frac{1}{2},1]$ of R is D_0 covered, with $G_0(z)=G(z)$. The R-problem on $]\frac{1}{2},1]$, when $M \leqq \frac{1}{2}$, can be split into two portions: Random selection of the values of z_n that correspond to cutouts such that $0<t_n<1$, and then random selection of the t_n, with uniform density between 0 and 1. This transforms the R problem ruled by the known function $F(z)$, into a random D_0 problem ruled by a r.f. $G(z)$, with independent increments and such that $EG(z)=F(z)$.

4. The comparison rule of step 2 continues to hold when either $G'(z)$, or $G''(z)$, or both, are random, and $G'(z)>G''(z)$ for all z holds a.s. If D cutouts ruled by $G'(z)$ cover the circle a.s., then D cutouts ruled by $G''(z)$ also cover it a.s. If D cutouts ruled by $G''(z)$ do not cover the circle a.s., then D cutouts ruled by $G'(z)$ do not either cover the circle a.s.

5. From step 4, it follows that one can solve a D_0 problem if one succeeds in bounding it on one side, or on both, by appropriate R problems. This is made possible by the law of the iterated logarithm, which implies that by modifying z_n for $z_n \geqq z_0$, where z_0 is a r.v. such that a.s. $z_0 > 0$ one can obtain a function $G_1(z)$ that satisfies a.s. the double inequality

$$|F(z)| - \sqrt{2|F(z)| \log\log|F(z)|} < |G_1(z)| < |F(z)| + \sqrt{2|F(z)| \log\log|F(z)|}.$$

6. *A Sufficient Condition for a.s. Covering in Dvoretzky's Problem.* Apply step 5 with $G'(z)$ a Poisson r.f. of expectation $F(x)$ and

$$G''(z) = G(z) = F(z) + \sqrt{2|F(z)| \log\log|F(z)|}.$$

If R cutouts ruled by $F(z)$ do not a.s. cover R, then D_0 cutouts ruled by $G'(z)$ do not a.s. cover $]\frac{1}{2}, 1]$, and further D cutouts ruled by $G(z)$ do not a.s. cover the circle.

This leads to the following test for Dvoretzky's problem. Solve for F the equation $G = F + \sqrt{2|F| \log\log|F|}$, and test the criterion $H(F) < \infty$ and $K(F) < \infty$ of Section 4 of [2]. It is sufficient that the criterion should apply to the slightly overevalued solution $F^* = G - \sqrt{2|G| \log\log|G|}$. In particular, when $|G| < 2/z$, and therefore

$$\int_s^1 \sqrt{2|G| \log\log|G|}\, dz < \infty \qquad \text{for all } s \geqq 0,$$

$H(G) < \infty$ is a sufficient condition for $H(F) < \infty$, and $K(G) < \infty$ a sufficient condition for $K(F) < \infty$. In summary,

Proposition. *If $|G| < 2/z$, then a sufficient ctiterion for the Pr of D covering to be less than 1 is that $H(G) < \infty$ and $K(G) < \infty$.*

7. *A Necessary Condition for a.s. Covering in Dvoretzky's Problem.* Now apply step 5 with $G''(z)$ a Poisson r.f. of expectation $F(z)$ and

$$G'(z) = G(z) = F(z) - \sqrt{2|F(z)| \log\log|F(z)|}.$$

If R cutouts ruled by $F(z)$ a.s. cover R, then D_0 cutouts ruled by $G''(z)$ a.s. cover $]\frac{1}{2}, 1]$ and D cutouts ruled by $G(z)$ a.s. cover the circle.

This leads to the following test for Dvoretzky's problem. Solve for F the equation $G = F - \sqrt{2|F| \log\log|F|}$, and test the criterion $H(F) = \infty$ and/or $K(F) = \infty$ of Section 4 of [2]. In particular,

Proposition. *If $|G| < 2/z$, then a sufficient criterion of a.s. D covering is that $H(G) = \infty$ and/or $K(G) = \infty$.*

8. *Application. Let $z_n = 1/(n+1)$, with $n > 1$. It follows that $G(z) = -1/z + 2$, with $G(\frac{1}{2}) = 0$. In this case, we have $H(G) = \infty$. This closes a case stressed by Dvoretzky and left open by Billard: Almost surely, D cutouts with $z_n = 1/(n+1)$ do cover the circle.*

Note Added in Proof. A full solution of the Dvoretzky problem has since been obtained by L. A. Shepp, in a paper titled "Covering the circle with randon arcs", to appear in Israel Journal of Mathematics. Also a partial solution different from mine has been obtained by S. Orey, in a paper titled "Random arcs on the circle" to appear in Journal d'Analyse.

References

1. Dvoretzky, A.: On covering a circle by randomly placed arcs. Proc. Nat. Acad. Sc. U.S.A. **42**, 199–203 (1956).
2. Kahane, J. P.: Some random series of functions. Lexington, Mass.: D. C. Heath and Co. 1968.
3. Mandelbrot, B.: Renewal sets and random cutouts. Z. Wahrscheinlichkeitstheorie verw. Geb. **22**, 145–157 (1972).

B. B. Mandelbrot
IBM Thomas J. Watson
Research Center
Yorktown Heights, N. Y. 10598
USA

(Received January 16, 1968/in revised form June 1, 1971)