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The Functional Law of the Iterated Logarithm for Dependent Random Variables

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1. Introduction

Let $X_1, X_2, \ldots$ be a sequence of random variables, $S_n = X_1 + \cdots + X_n$ $(S_0 = 0)$ and let $\chi_n(t)$ denote the random function in $0 \leqq t \leqq 1$ which is linear in every interval

$$[(k-1)/n, k/n] \ (1 \leqq k \leqq n) \quad \text{and} \quad \chi_n(k/n) = S_k \ (0 \leqq k \leqq n).$$

Put further $\varphi_n = n^{-\frac{1}{2}} \chi_n$, $\psi_n = (2n \log\log n)^{-\frac{1}{2}} \chi_n$. By a well-known theorem of Prohorov the sequence $X_1, X_2, \ldots$ obeys the functional central limit theorem (invariance principle) if and only if (i) the finite dimensional distributions of the process φ_n converge, as $n \to \infty$, to the corresponding finite dimensional distributions of the Wiener-process, (ii) $\lim_{h \to 0} \limsup_{n \to \infty} P\big(\sup_{|t-t'| \leqq h} |\varphi_n(t) - \varphi_n(t')| > \varepsilon\big) = 0$ for every $\varepsilon > 0$. The purpose of the present paper is to prove that under similar but slightly more restrictive conditions the sequence $X_1, X_2, \ldots$ obeys the functional (Strassen-type) law of the iterated logarithm, i.e. the sequence ψ_n is equicontinuous with probability one and the set of its norm-limit points (in the norm $C[0, 1]$) coincides with the set

$$K = \left\{x\colon\ x \text{ is absolutely continuous in } [0,1],\ x(0) = 0 \text{ and } \int_0^1 (\dot{x}(t))^2\, dt \leqq 1\right\}. \tag{1.1}$$

This is the case, e.g., if we assume that (i) the finite dimensional distributions of the process φ_n converge to the finite dimensional distributions of the Wiener-process in a certain rate, (ii) $P\big(\max_{1 \leqq \nu \leqq n} |S_{\nu+i} - S_i| \geqq C_1 \sqrt{n \log\log n}\big) \leqq C_2\, e^{-2 \log\log n}$, (iii) $\sup_n E|X_n|^{2+\delta} < \infty$ (Theorem 3). In certain situations (ii) can be omitted or can be replaced by an inequality of a simpler type (Theorems 1 and 2). Our results seem to be of wide applicability and we shall give some applications in a series of forthcoming papers.

2. Results

First we introduce some notations that we shall use throughout this paper.

Given a sequence $X_1, X_2, \ldots$ of random variables, let $S_n, \chi_n, \varphi_n, \psi_n$ denote the same as in the Introduction; let us also introduce the sums $S_n^{(m)} = X_{m+1} + \cdots + X_{m+n}$ $(m \geqq 0, n \geqq 1)$ and the piecewise constant random function $\tilde{\varphi}_n(t) = n^{-\frac{1}{2}} S_{[nt]}$ $(0 \leqq t \leqq 1)$. For every fixed $0 \leqq t_1 < t_1' \leqq \cdots \leqq t_r < t_r' \leqq 1$ $(r = 1, 2, \ldots)$ let $F_n^{(t_1, t_1', \ldots, t_r, t_r')}(x_1, x_2, \ldots, x_r)$ denote the distribution function of the random vector

$$\left(\frac{\tilde{\varphi}_n(t_1') - \tilde{\varphi}_n(t_1)}{\sqrt{t_1' - t_1}}, \ldots, \frac{\tilde{\varphi}_n(t_r') - \tilde{\varphi}_n(t_r)}{\sqrt{t_r' - t_r}}\right). \tag{2.1}$$

On the other hand, let $\Phi^{(t_1, t_1', \ldots, t_r, t_r')}(x_1, x_2, \ldots, x_r)$ denote the distribution function of the random vector

$$\left(\frac{\zeta(t_1')-\zeta(t_1)}{\sqrt{t_1'-t_1}}, \ldots, \frac{\zeta(t_r')-\zeta(t_r)}{\sqrt{t_r'-t_r}}\right) \tag{2.2}$$

where ζ is a standard Wiener-process. If $F(x_1, \ldots, x_r)$ is an r-dimensional distribution function and $B \subset R^r$ is a Borel-set, define $F(B) = \int_B dF(x_1, \ldots, x_r)$ i.e. let $F(B)$ denote the probability of the event that a random vector with distribution F belongs to B. Let us also introduce the characteristic functions of the random vectors appearing in (2.1) and (2.2):

$$f_n^{(t_1, \ldots, t_r')}(\lambda_1, \ldots, \lambda_r) = E\left(\exp\left\{\frac{i\lambda_1}{\sqrt{t_1'-t_1}}(\tilde{\varphi}_n(t_1') - \tilde{\varphi}_n(t_1)) + \cdots + \frac{i\lambda_r}{\sqrt{t_r'-t_r}}(\tilde{\varphi}_n(t_r') - \tilde{\varphi}_n(t_r))\right\}\right)$$

$$f_0^{(t_1, \ldots, t_r')}(\lambda_1, \ldots, \lambda_r) = E\left(\exp\left\{\frac{i\lambda_1}{\sqrt{t_1'-t_1}}(\zeta(t_1') - \zeta(t_1)) + \cdots + \frac{i\lambda_r}{\sqrt{t_r'-t_r}}(\zeta(t_r') - \zeta(t_r))\right\}\right).$$

Let Y be a metric space and $H \subset Y$. For every $c > 0$ let $H_{(c)}$ denote the neighbourhood of H of radius c, i.e. the set of those points of Y, which have a distance $< c$ from H. Let us further agree that if f is a real function in $[0, 1]$ and $m \geqq 2$ is an integer, then $\Pi_m f$ denotes the function which coincides with f at the points k/m $(0 \leqq k \leqq m)$ and is linear in every interval $[(k-1)/m, k/m]$ $(1 \leqq k \leqq m)$. For every r-dimensional vector $\lambda = (\lambda_1, \ldots, \lambda_r)$ let $\|\lambda\|$ stand for the number $(\lambda_1^2 + \cdots + \lambda_r^2)^{\frac{1}{2}}$. Finally, let the Lebesgue-measure of a Borel-set $B \subset R^r$ be denoted by $\mu(B)$.

Now we introduce some conditions concerning sequences of random variables. Given a sequence $X_1, X_2, \ldots$, we say that it satisfies

Condition **A**, if for every $n \geqq 1$, $t \geqq 0$ and every $a_1, a_2, \ldots$ we have

$$P(|a_1 X_1 + \cdots + a_n X_n| \geqq t\sqrt{a_1^2 + \cdots + a_n^2}) \leqq C_3 e^{-C_4 t^2} \tag{2.3}$$

where C_3, C_4 are positive constants depending only on the sequence $X_1, X_2, \ldots$;

Condition $\mathbf{B}_1$, if for every $0 \leqq t_1 < t_1' \leqq \cdots \leqq t_r < t_r' \leqq 1$ $(r = 1, 2, \ldots)$ and for every open rectangle $B \subset R^r$

$$\lim_{n \to \infty} F_n^{(t_1, \ldots, t_r')}(B) = \Phi^{(t_1, \ldots, t_r')}(B); \tag{2.4}$$

Condition $\mathbf{B}_2$, if for every $0 \leqq t_1 < t_1' \leqq \cdots \leqq t_r < t_r' \leqq 1$, $n \geqq 1$ and for every open rectangle $B \subset R^r$, furthermore for every open sphere $B \subset R^r$ centered in the origin we have

$$|F_n^{(t_1, \ldots, t_r')}(B) - \Phi^{(t_1, \ldots, t_r')}(B)| \leqq C_5(1 + \mu(B_{(1)}))\frac{1}{(nt)^\beta} \tag{2.5}$$

where $t = \min(t_1' - t_1, \ldots, t_r' - t_r)$, C_5 and β are positive constants which depend only on r and the sequence $X_1, X_2, \ldots$ (and are independent of n, B and $t_1, \ldots, t_r'$);

Condition $\mathbf{B}_3$, if for every $0 \leqq t_1 < t_1' \leqq \cdots \leqq t_r < t_r' \leqq 1$, $n \geqq 1$ and for every $\lambda = (\lambda_1, \ldots, \lambda_r)$ satisfying $\|\lambda\| \leqq C_6 (nt)^\gamma$ we have

$$|f_n^{(t_1, \ldots, t_r')}(\lambda_1, \ldots, \lambda_r) - f_0^{(t_1, \ldots, t_r')}(\lambda_1, \ldots, \lambda_r)| < C_7 \frac{1}{(nt)^\delta} \tag{2.6}$$

where $t = \min(t_1' - t_1, \ldots, t_r' - t_r)$, C_6, C_7, γ, δ are positive constants depending only on r and the sequence $X_1, X_2, \ldots$.

It is evident that condition $\mathbf{B}_2$ implies condition $\mathbf{B}_1$ but it is not clear what the connection is between conditions $\mathbf{B}_2$ and $\mathbf{B}_3$. We shall see later (in Lemma 3) that the situation is simple: condition $\mathbf{B}_3$ implies condition $\mathbf{B}_2$; the proof of this fact depends on a multidimensional analogue of Esseen's inequality obtained recently by von Bahr. As we already mentioned in the Introduction, condition $\mathbf{B}_1$ and the tightness of the sequence $\varphi_n(t)$ are necessary and sufficient conditions that the sequence $X_1, X_2, \ldots$ obey the functional central limit theorem.

We are now ready to formulate our first two theorems.

Theorem 1. *If the sequence X_1, X_2, ... is uniformly bounded and satisfies condition $\mathbf{B}_2$ (or condition $\mathbf{B}_3$), then it obeys the functional law of the iterated logarithm.*

Theorem 2. *If the sequence $X_1, X_2, \ldots$ satisfies conditions* **A** *and* $\mathbf{B}_2$ *(or* **A** *and* $\mathbf{B}_3$*), then it obeys the functional law of the iterated logarithm.*

Applications for some classes of dependent random variables, e.g. for multiplicative systems, lacunary orthogonal series, mixing sequences e.t.c. will be given elsewhere.

Condition **A** may be superfluous in Theorem 2 (at least under the assumption $\sup_n E|X_n|^{2+\delta} < \infty$) but we can not prove this.

3. Proofs

Lemma 1. *If a sequence $X_1, X_2, \ldots$ satisfies condition* **A**, *then there exist positive constants B, C_8, depending only on the sequence X_1, X_2, ... such that for every $m \geqq 0$, $n \geqq 1$ we have*

$$P\big(\max_{1 \leqq \nu \leqq n} |S_\nu^{(m)}| \geqq B \sqrt{n \log \log n}\big) \leqq C_8 \, e^{-2 \log \log n}. \tag{3.1}$$

Proof. Let $p > 0$ be an even number, then we obtain from condition **A** for every $a_1, a_2, \ldots$ and $n \geqq 1$:

$$\begin{aligned} E(|a_1 X_1 + \cdots + a_n X_n|^p) &\leqq \sum_{k=0}^{\infty} [(k+1)\sqrt{a_1^2 + \cdots + a_n^2}]^p \, P(|a_1 X_1 + \cdots + a_n X_n| \geqq k \sqrt{a_1^2 + \cdots + a_n^2}) \\ &\leqq (a_1^2 + \cdots + a_n^2)^{p/2} \sum_{k=0}^{\infty} (k+1)^p \, C_3 \, e^{-C_4 k^2} \\ &\leqq C_3 (a_1^2 + \cdots + a_n^2)^{p/2} \left[1 + \sum_{k=1}^{\infty} (2k)^p \, e^{-C_4 k^2}\right]. \end{aligned} \tag{3.2}$$

It is easy to show (comparing the sum with the integral

$$\int_0^\infty x^p e^{-C_4 x^2} dx = (2C_4)^{-\frac{p+1}{2}} \sqrt{\pi/2}\, 1\cdot 3\cdot \dots \cdot (p-1))$$

that

$$\sum_{k=1}^\infty k^p e^{-C_4 k^2} < (C_9 p)^{p/2}$$

where C_9 depends only on C_4. This relation, together with (3.2), gives

$$E(|a_1 X_1 + \dots + a_n X_n|^p) \leqq C_3 (a_1^2 + \dots + a_n^2)^{p/2} [1 + 2^p (C_9 p)^{p/2}] \leqq (C_{10} p)^{p/2} (a_1^2 + \dots + a_n^2)^{p/2}, \tag{3.3}$$

but this implies (cf. [4], pp. 513–514) that there exists a positive constant A^*, depending only on C_4 (i.e. only on the sequence $X_1, X_2, \dots$) such that the inequality

$$E(\exp\{t \max_{1\leqq \nu \leqq N} |a_1 X_1 + \dots + a_\nu X_\nu|\}) \leqq 2 \exp\{A^* t^2 (a_1^2 + \dots + a_N^2)\}$$

holds for $0 \leqq t < A^*$ and every $N \geqq 1$. Let us now put here $N = m+n$, $a_k = 0$ for $1 \leqq k \leqq m$, $a_k = 1$ for $m+1 \leqq k \leqq m+n$ and $t = t_0 = \sqrt{8}(2\sqrt{A^*})^{-1} (\log\log n/n)^{\frac{1}{2}}$. The value of t_0 satisfies $0 \leqq t_0 < A^*$ for $n \geqq n_0$ where n_0 depends only on A^*. Thus we obtain from the Markov-inequality for $n \geqq n_0$:

$$\begin{aligned} P\big(\max_{1\leqq \nu\leqq n} |S_\nu^{(m)}| \geqq \sqrt{8A^*} \sqrt{n \log\log n}\big) &\leqq \exp(-t_0 \sqrt{8A^*} \sqrt{n \log\log n}) \cdot E(\exp\{t_0 \max_{1\leqq \nu \leqq n} |S_\nu^{(m)}|\}) \\ &\leqq 2 \exp(-t_0 \sqrt{8A^*} \sqrt{n\log\log n} + A^* t_0^2 n) = 2\exp(-2\log\log n) \end{aligned}$$

which establishes (3.1) for $n \geqq n_0$ with $B = \sqrt{8A^*}$, $C_8 = 2$. On the other hand, by the appropriate choice of the constant $C_8 > 2$ we can guarantee that $C_8 e^{-2\log\log n_0} > 1$ and then (3.1) is valid (with the same value of B) also for $1 \leqq n \leqq n_0$ since the right side exceeds 1. This completes the proof of Lemma 1.

Lemma 2. *Let $A_1, A_2, \dots, A_n, \dots$ be events satisfying the following condition: there exist constants $C_{11} > 0$, $C_{12} > 0$, $\mu < 1$, $\rho > 0$, $\tau > 0$ such that*

$$P(A_n) \geqq C_{11} \frac{1}{n^\mu} \qquad (n \geqq 1), \tag{3.4}$$

$$|P(A_m A_n) - P(A_m) P(A_n)| \leqq C_{12} n^\rho e^{-\tau m} \quad (1 \leqq m < n). \tag{3.5}$$

Then

$$P(\limsup_{n\to\infty} A_n) = 1. \tag{3.6}$$

Proof. We use the method of P. Révész (cf. [6]). A well-known theorem of Erdős and Rényi (cf. [5], p. 391) states that if the events $A_1, A_2, \dots$ satisfy the conditions

$$\sum_{n=1}^\infty P(A_n) = +\infty, \tag{3.7}$$

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n}\sum_{l=1}^{n}(P(A_k A_l)-P(A_k)P(A_l))}{\left(\sum_{k=1}^{n} P(A_k)\right)^2}=0 \tag{3.8}$$

then (3.6) holds. Now (3.7) follows immediately from (3.4), therefore it suffices to prove that (3.4) and (3.5) together imply (3.8). Let us write

$$\sum_{k=1}^{n}\sum_{l=1}^{n}(P(A_k A_l)-P(A_k)P(A_l))=I_1+I_2+I_3+I_4,$$

where

$$I_1=\sum_{k=[\log^2 n]+1}^{n}\sum_{l=[\log^2 n]+1}^{n} d_{kl}, \qquad I_2=\sum_{k=1}^{[\log^2 n]}\sum_{l=1}^{[\log^2 n]} d_{kl},$$

$$I_3=\sum_{k=[\log^2 n]+1}^{n}\sum_{l=1}^{[\log^2 n]} d_{kl}, \qquad I_4=\sum_{k=1}^{[\log^2 n]}\sum_{l=[\log^2 n]+1}^{n} d_{kl},$$

$$d_{kl}=P(A_k A_l)-P(A_k)P(A_l).$$

Using (3.4) and (3.5), we obtain for sufficiently large n

$$|I_1|\leqq n^2\cdot C_{12}\, n^\rho\, e^{-\tau([\log^2 n]+1)}\leqq C_{12}\, n^{2+\rho}\, e^{-\tau\log^2 n}\leqq C_{12}$$

$$|I_2|\leqq \sum_{k=1}^{[\log^2 n]}\sum_{l=1}^{[\log^2 n]} 2\leqq 2\log^4 n$$

$$|I_3+I_4|\leqq \sum_{k=1}^{n}\sum_{l=1}^{[\log^2 n]} 2P(A_k)+\sum_{l=1}^{n}\sum_{k=1}^{[\log^2 n]} 2P(A_l)\leqq 4\log^2 n\sum_{r=1}^{n} P(A_r)$$

$$\sum_{k=1}^{n} P(A_k)\geqq C_{13}\, n^{1-\mu}$$

and thus the absolute value of the fraction in (3.8) can be majorized by

$$(C_{12}+2\log^4 n)\left(\sum_{k=1}^{n} P(A_k)\right)^{-2}+4\log^2 n\left(\sum_{k=1}^{n} P(A_k)\right)^{-1}=o(1).$$

Hence Lemma 2 is proved.

Proof of Theorem 2. The statement of the theorem will be proved under conditions **A** and $\mathbf{B}_2$, the remaining part is an immediate consequence of this result and Lemma 3. In our proof we shall use the method first applied by Chover for i.i.d. random variables in [2]; to avoid repetitions, we shall indicate only those steps where essential changes are necessary.

a) The a.s. equicontinuity of the sequence ψ_n is valid in the following form: Let ε be a positive number, choose an integer $q>2$ such that $\varepsilon^2\, 2^q>8B^2$ (B is the constant appearing in Lemma 1) and let us define

$$\delta_\varepsilon=2^{-q}. \tag{3.9}$$

Then we have with probability one for sufficiently large n

$$\sup_{|t-t'|\leqq \delta_\varepsilon} |\psi_n(t)-\psi_n(t')|<\varepsilon.$$

We omit the proof because our statement can be derived from inequality (3.1) essentially in the same way as the analogous statement (8) in [2] was deduced from relation (4) of the same paper.

b) To show that the set of all norm-limit points of the sequence ψ_n is $\subset K$ with probability one, it suffices to prove the next two statements:

1. For every $\varepsilon>0$, $m\geqq 2$ we have (as $n\to\infty$)

$$P\left(m\sum_{i=1}^{m}[\tilde{\varphi}_n(i/m)-\tilde{\varphi}_n((i-1)/m)]^2<r_0^2\right)=T_m(r_0^2)+O(n^{-\alpha})$$

where T_m denotes the d.f. of the χ_m^2 distribution, $r_0=(1+\varepsilon)\sqrt{2\log\log n}$, α and the constant implied by the O are positive numbers depending only on m, ε and the sequence $X_1, X_2, \ldots$.

2. We have

$$\sup_{0\leqq t\leqq 1} |\psi_n(t)-\tilde{\psi}_n(t)|\to 0 \tag{3.10}$$

with probability one for $n\to\infty$, where $\tilde{\psi}_n(t)=(2n\log\log n)^{-\frac{1}{2}}S_{[nt]}$.

Following Chover's method, we can deduce from statement 1 that $\Pi_m\,\tilde{\psi}_{n_k}\in K_{(2\varepsilon)}$ with probability one for sufficiently large k, where $n_k=[c^k]$, $c>1$. This relation and statement 2 imply $\Pi_m\,\psi_{n_k}\in K_{(3\varepsilon)}$ for sufficiently large k and hence—using Chover's argument again—we can obtain the desired result.

To prove statement 1, let us observe that the event

$$D_n=\left\{m\sum_{i=1}^{m}[\tilde{\varphi}_n(i/m)-\tilde{\varphi}_n((i-1)/m)]^2<r_0^2\right\}$$

can be written as follows:

$$D_n=\left\{\left(\frac{\tilde{\varphi}_n(1/m)-\tilde{\varphi}_n(0)}{\sqrt{1/m}},\ldots,\frac{\tilde{\varphi}_n(m/m)-\tilde{\varphi}_n((m-1)/m)}{\sqrt{1/m}}\right)\in G\right\}$$

where $G\subset R^m$ is an open sphere centered in the origin with radius r_0. Let us now apply condition $\mathbf{B}_2$ to obtain

$$\begin{aligned}P(D_n)&=P\left(\left(\frac{\zeta(1/m)-\zeta(0)}{\sqrt{1/m}},\ldots,\frac{\zeta(m/m)-\zeta((m-1)/m)}{\sqrt{1/m}}\right)\in G\right)+R \\ &=T_m(r_0^2)+R\end{aligned} \tag{3.11}$$

where

$$\begin{aligned}|R|&\leqq C_{14}(1+\mu(G_{(1)}))\frac{1}{(n/m)^{\alpha_1}}=C_{14}(1+C_{15}(r_0+1)^m)\frac{1}{(n/m)^{\alpha_1}} \\ &\leqq C_{16}(2r_0)^m\frac{1}{(n/m)^{\alpha_1}}\leqq C_{17}\frac{(\log\log n)^{m/2}}{n^{\alpha_1}}\leqq C_{17}\frac{1}{n^{\alpha_1/2}}\end{aligned} \tag{3.12}$$

for sufficiently large n; here the constants $C_{14}, \ldots, C_{17}, \alpha_1$ depend on m, ε and the sequence $X_1, X_2, \ldots$. Comparing (3.11) and (3.12), we get the desired result.

For the proof of statement 2 we remark that the moment condition $E|X_n|^{2+\delta} \leqq C_{18}$ $(n=1, 2, \ldots)$ implies $n^{-\frac{1}{2}}|X_n| \to 0$ and therefore $n^{-\frac{1}{2}} \max_{1\leqq k\leqq n} |X_k| \to 0$ with probability one; on the other hand, we have

$$|\psi_n(t) - \tilde{\psi}_n(t)| \leqq (2n \log\log n)^{-\frac{1}{2}} \max_{1\leqq k\leqq n} |X_k| \qquad (0 \leqq t \leqq 1).$$

Applying (3.3) with $p=4$, we obtain the result.

c) We turn now to the third—and most difficult—step of the proof of Theorem 3: we show that the set of all norm-limit points of the sequence ψ_n is $\supset K$ with probability one. The main problem arising here is a rather typical one: how to apply the Borel-Cantelli lemma for dependent events? In the present case the solution will be quite natural: it will be possible to show by means of condition $\mathbf{B}_2$ that the "critical" events are nearly independent (in some sense) and then we can apply Lemma 2.

Let $K^* \subset K$ be the set of functions defined by

$$K^* = \left\{x\colon\ x \in K \text{ and } \int_0^1 (\dot{x}(t))^2\, dt < 1\right\}.$$

It is obvious that K^* is dense in K (in the norm $C[0, 1]$) therefore it suffices to prove that the set of all norm-limit points of the sequence ψ_n is $\supset K^*$ with probability one. Let us fix $g \in K^*$, $\varepsilon > 0$ and choose a large integer m such that

$$\sup_{|t-t'|\leqq 1/m} |g(t) - g(t')| < \varepsilon, \qquad 1/m < \delta_\varepsilon \tag{3.13}$$

where δ_ε is defined by (3.9). Let us introduce the events

$$E_n = \{|(\tilde{\psi}_n(i/m) - \tilde{\psi}_n((i-1)/m)) - \Delta g_i| < \varepsilon/m \text{ for } i = 2, 3, \ldots, m\}$$
$$E_n^* = \{|(\psi_n(i/m) - \psi_n((i-1)/m)) - \Delta g_i| < 2\varepsilon/m \text{ for } i = 2, 3, \ldots, m\}$$
$$H_n = \{\sup_{0\leqq t\leqq 1} |\psi_n(t) - g(t)| < 6\varepsilon\} \qquad \Delta g_i = g(i/m) - g((i-1)/m).$$

Using the a.s. equicontinuity of the sequence ψ_n (in the form as it was formulated in a)), statement 2, and relation (3.13), we can easily see that

$$\limsup_{n\to\infty} E_n \subset \limsup_{n\to\infty} E_n^* \subset \limsup_{n\to\infty} H_n$$

(apart from a set of probability zero), therefore it suffices to show that

$$P(\limsup_{n\to\infty} E_n) = 1. \tag{3.14}$$

In the proof of (3.14) we intend to apply Lemma 2, so first we estimate $P(E_n)$ from below. A trite calculation shows that the event E_n can be written as follows:

$$E_n = \left\{\left(\frac{\tilde{\varphi}_n(2/m) - \tilde{\varphi}_n(1/m)}{\sqrt{1/m}}, \ldots, \frac{\tilde{\varphi}_n(m/m) - \tilde{\varphi}_n((m-1)/m)}{\sqrt{1/m}}\right) \in B\right\}$$

where $B \subset R^{m-1}$ is an open rectangle (actually a cube), namely

$$B=\{(x_1,\ldots,x_{m-1}):\ (\Delta g_{\nu+1}-\varepsilon/m)\sqrt{2m\log\log n}<x_\nu<(\Delta g_{\nu+1}+\varepsilon/m)\sqrt{2m\log\log n},\ \nu=1,2,\ldots,m-1\}.$$

Using condition $\mathbf{B}_2$, we obtain

$$\begin{aligned} P(E_n)&=P\left(\left(\frac{\zeta(2/m)-\zeta(1/m)}{\sqrt{1/m}},\ldots,\frac{\zeta(m/m)-\zeta((m-1)/m)}{\sqrt{1/m}}\right)\in B\right)+R_1\\ &=\prod_{i=2}^{m}\int_{(\Delta g_i-\varepsilon/m)\sqrt{2m\log\log n}}^{(\Delta g_i+\varepsilon/m)\sqrt{2m\log\log n}}\frac{1}{\sqrt{2\pi}}e^{-s^2/2}\,ds+R_1=V_1+R_1 \end{aligned} \tag{3.15}$$

where

$$\begin{aligned} |R_1|&\leqq C_{19}(1+\mu(B_{(1)}))\frac{1}{(n/m)^\beta}\\ &\leqq C_{19}\left[1+\left(2+\frac{2\varepsilon}{m}\sqrt{2m\log\log n}\right)^{m-1}\right]\frac{1}{(n/m)^\beta}\\ &\leqq C_{20}\frac{(\log\log n)^{\frac{m-1}{2}}}{n^\beta}\leqq C_{21}\frac{1}{n^{\beta/2}} \end{aligned} \tag{3.16}$$

for sufficiently large n; the constants C_{19}, C_{20}, C_{21}, β depend only on m, ε. (We used here the simple observation that if H, H' are open rectangles in R^k of the form

$$H=\{(x_1,\ldots,x_k):\ a_\nu<x_\nu<b_\nu,\ \nu=1,2,\ldots,k\}$$
$$H'=\{(x_1,\ldots,x_k):\ a_\nu-1<x_\nu<b_\nu+1,\ \nu=1,2,\ldots,k\}$$

then $H_{(1)}\subset H'$.) The expression V_1 has already appeared in [7], p. 214, where the estimate

$$V_1\geqq C_{22}\exp\left\{-\log\log n\int_0^1(\dot g(t))^2\,dt\right\}\cdot(\log\log n)^{-\frac{m-1}{2}} \tag{3.17}$$

was deduced; C_{22} is a constant independent of n. Now the assumption $g(t)\in K^*$ implies that $\int_0^1(\dot g(t))^2\,dt=1-\mu$ where $\mu>0$, thus we have for sufficiently large n

$$V_1\geqq C_{22}\exp\{-(1-\mu)\log\log n\}\cdot(\log\log n)^{-\frac{m-1}{2}}\geqq C_{23}(\log n)^{-(1-\mu/2)}$$

and therefore we obtain from (3.15) and (3.16)

$$P(E_n)\geqq C_{23}(\log n)^{-(1-\mu/2)}-C_{21}\,n^{-\beta/2}\geqq C_{24}(\log n)^{-(1-\mu/2)}.$$

Let us now put here $n=m^k$, then we obtain for sufficiently large k

$$P(E_{m^k})\geqq C_{25}\frac{1}{k^{1-\mu/2}} \tag{3.18}$$

where C_{25} does not depend on k; we can also achieve (changing the constant C_{25}) that (3.18) holds for every $k\geqq 1$.

Our next goal is to estimate the difference

$$|P(E_{m^k} E_{m^l}) - P(E_{m^k}) P(E_{m^l})|$$

for every $k<l$. A trite calculation shows that the events E_{m^k}, E_{m^l} and $E_{m^k} E_{m^l}$ can be written as follows:

$$E_{m^k} = \left\{\left(\frac{\tilde{\varphi}_{m^l}(2/m^{l-k+1}) - \tilde{\varphi}_{m^l}(1/m^{l-k+1})}{\sqrt{1/m^{l-k+1}}}, \ldots, \frac{\tilde{\varphi}_{m^l}(m/m^{l-k+1}) - \tilde{\varphi}_{m^l}((m-1)/m^{l-k+1})}{\sqrt{1/m^{l-k+1}}}\right) \in S\right\}$$

$$E_{m^l} = \left\{\left(\frac{\tilde{\varphi}_{m^l}(2/m) - \tilde{\varphi}_{m^l}(1/m)}{\sqrt{1/m}}, \ldots, \frac{\tilde{\varphi}_{m^l}(m/m) - \tilde{\varphi}_{m^l}((m-1)/m)}{\sqrt{1/m}}\right) \in S'\right\}$$

$$E_{m^k} E_{m^l} = \left\{\left(\frac{\tilde{\varphi}_{m^l}(2/m^{l-k+1}) - \tilde{\varphi}_{m^l}(1/m^{l-k+1})}{\sqrt{1/m^{l-k+1}}}, \ldots, \frac{\tilde{\varphi}_{m^l}(m/m^{l-k+1}) - \tilde{\varphi}_{m^l}((m-1)/m^{l-k+1})}{\sqrt{1/m^{l-k+1}}}, \frac{\tilde{\varphi}_{m^l}(2/m) - \tilde{\varphi}_{m^l}(1/m)}{\sqrt{1/m}}, \ldots, \frac{\tilde{\varphi}_{m^l}(m/m) - \tilde{\varphi}_{m^l}((m-1)/m)}{\sqrt{1/m}}\right) \in S''\right\}$$

where $S \subset R^{m-1}$, $S' \subset R^{m-1}$, $S'' \subset R^{2m-2}$ are open rectangles, namely

$$S = \{(x_1, \ldots, x_{m-1})\colon (\Delta g_{v+1} - \varepsilon/m)\sqrt{2m \log\log m^k} < x_v < (\Delta g_{v+1} + \varepsilon/m)\sqrt{2m \log\log m^k},\ v = 1, 2, \ldots, m-1\}$$

$$S' = \{(x_1, \ldots, x_{m-1})\colon (\Delta g_{v+1} - \varepsilon/m)\sqrt{2m \log\log m^l} < x_v < (\Delta g_{v+1} + \varepsilon/m)\sqrt{2m \log\log m^l},\ v = 1, 2, \ldots, m-1\}$$

$$S'' = \{(x_1, \ldots, x_{2m-2})\colon (x_1, \ldots, x_{m-1}) \in S, (x_m, \ldots, x_{2m-2}) \in S'\} = S \times S'.$$

Let further ζ be a standard Wiener-process (on an arbitrary probability space $(\Omega', \mathscr{A}', P')$) and let us consider the events

$$I_{k,l} = \left\{\left(\frac{\zeta(2/m^{l-k+1}) - \zeta(1/m^{l-k+1})}{\sqrt{1/m^{l-k+1}}}, \ldots, \frac{\zeta(m/m^{l-k+1}) - \zeta((m-1)/m^{l-k+1})}{\sqrt{1/m^{l-k+1}}}\right) \in S\right\}$$

$$J_l = \left\{\left(\frac{\zeta(2/m) - \zeta(1/m)}{\sqrt{1/m}}, \ldots, \frac{\zeta(m/m) - \zeta((m-1)/m)}{\sqrt{1/m}}\right) \in S'\right\}$$

$$K_{k,l} = \left\{\left(\frac{\zeta(2/m^{l-k+1}) - \zeta(1/m^{l-k+1})}{\sqrt{1/m^{l-k+1}}}, \ldots, \frac{\zeta(m/m^{l-k+1}) - \zeta((m-1)/m^{l-k+1})}{\sqrt{1/m^{l-k+1}}}, \frac{\zeta(2/m) - \zeta(1/m)}{\sqrt{1/m}}, \ldots, \frac{\zeta(m/m) - \zeta((m-1)/m)}{\sqrt{1/m}}\right) \in S''\right\}$$

(which are the formal analogues of the events E_{m^k}, E_{m^l}, $E_{m^k}E_{m^l}$ in the case when the process $\tilde{\varphi}_{m^l}(t)$ is replaced by $\zeta(t)$). We have evidently $K_{k,l}=I_{k,l}J_l$, furthermore $k<l$ implies $m/m^{l-k+1}\leqq 1/m$ and therefore $I_{k,l}$ and J_l are independent. We can now write

$$P(E_{m^k}E_{m^l})-P(E_{m^k})P(E_{m^l})$$
$$=\big(P(E_{m^k}E_{m^l})-P(K_{k,l})\big)+P(I_{k,l})\big(P(J_l)-P(E_{m^l})\big)+P(E_{m^l})\big(P(I_{k,l})-P(E_{m^k})\big)$$

and thus we obtain from condition $\mathbf{B}_2$ for $l>k\geqq 1$

$$\begin{aligned}
&|P(E_{m^k}E_{m^l})-P(E_{m^k})P(E_{m^l})|\\
&\quad\leqq |P(E_{m^k}E_{m^l})-P(K_{k,l})|+|P(E_{m^l})-P(J_l)|+|P(E_{m^k})-P(I_{k,l})|\\
&\quad\leqq C_{26}\big(1+\mu(S''_{(1)})\big)\frac{1}{(m^{k-1})^{\gamma_1}}+C_{27}\big(1+\mu(S'_{(1)})\big)\frac{1}{(m^{l-1})^{\gamma_2}}\\
&\qquad+C_{27}\big(1+\mu(S_{(1)})\big)\frac{1}{(m^{k-1})^{\gamma_2}}\\
&\quad\leqq C_{26}\left[1+\left(2+\frac{2\varepsilon}{m}\sqrt{2m\log\log m^k}\right)^{m-1}\left(2+\frac{2\varepsilon}{m}\sqrt{2m\log\log m^l}\right)^{m-1}\right]\\
&\qquad\cdot\frac{1}{(m^{k-1})^{\gamma_1}}+C_{27}\left[1+\left(2+\frac{2\varepsilon}{m}\sqrt{2m\log\log m^l}\right)^{m-1}\right]\frac{1}{(m^{l-1})^{\gamma_2}}\\
&\qquad+C_{27}\left[1+\left(2+\frac{2\varepsilon}{m}\sqrt{2m\log\log m^k}\right)^{m-1}\right]\frac{1}{(m^{k-1})^{\gamma_2}}\\
&\quad\leqq C_{28}\frac{(\log k)^{\frac{m-1}{2}}(\log l)^{\frac{m-1}{2}}}{m^{k\gamma_1}}+C_{29}\frac{(\log l)^{\frac{m-1}{2}}}{m^{l\gamma_2}}+C_{29}\frac{(\log k)^{\frac{m-1}{2}}}{m^{k\gamma_2}}\\
&\quad\leqq C_{30}\frac{(\log l)^{m-1}}{m^{k\gamma}}\leqq C_{30}\frac{l^{m-1}}{m^{k\gamma}}=C_{30}\,l^{m-1}e^{-\tau k}
\end{aligned}\tag{3.19}$$

where $\gamma_1,\gamma_2,C_{26},\ldots,C_{30}$ are constants which do not depend on k,l; $\gamma=\min(\gamma_1,\gamma_2)$, $\tau=\gamma\log m$.

Having the estimates (3.18) and (3.19), the proof of Theorem 2 can be finished very shortly: we have only to observe that by (3.18) and (3.19) the events E_{m^k} ($k=1,2,\ldots$) satisfy the conditions of Lemma 2, thus the lemma is applicable and we obtain

$$P(\limsup_{k\to\infty}E_{m^k})=1$$

which evidently implies (3.14).

Now we turn to the proof of the above mentioned fact that condition $\mathbf{B}_3$ implies condition $\mathbf{B}_2$.

Lemma 3. *If a sequence $X_1, X_2, \ldots$ satisfies condition $\mathbf{B}_3$ then it satisfies also condition $\mathbf{B}_2$.*

The statement of Lemma 3 is very typical: we assume the closeness of two characteristic functions and assert the closeness of the corresponding distributions. The proofs of such statements in one dimension are usually based on Esseen's inequality (see [3], p. 512) and for the proof of Lemma 3 we need a multidimensional analogue of this inequality. Such inequalities really exist, they have been obtained recently by von Bahr (see [1]). We present here one of them which is not the most general one but which is sufficient for our purposes.

Bahr's Formula. *Let F and G be k-dimensional distribution functions, let G have a continuous density function ψ and suppose that on the sphere $x_1^2+\cdots+x_k^2=R^2$, ψ satisfies the inequality $\psi(x_1, \ldots, x_k) \leqq \psi_1(R)$ where $\psi_1(R)$ is differentiable, $\psi_1(R)\cdot R^{k-1}\to 0$ if $R\to\infty$ and $\int_0^\infty |\psi_1'(R)|\, R^{k-1}\, dR = L < \infty$. Let $f(\lambda_1, \ldots, \lambda_k)$ and $g(\lambda_1, \ldots, \lambda_k)$ be the characteristic functions of F and G, put $h(\lambda_1, \ldots, \lambda_k)=f(\lambda_1, \ldots, \lambda_k)-g(\lambda_1, \ldots, \lambda_k)$ and let T be an arbitrary positive number. Then we have for every bounded convex Borel-set $B\subset R^k$:*

$$|F(B)-G(B)| \leqq E_1 \frac{L}{T} + E_2\, \mu(B_{(E_3/T)}) \int_{\|\lambda\|\leqq T} |h(\lambda_1, \ldots, \lambda_k)|\, d\lambda_1 \ldots d\lambda_k \tag{3.20}$$

where E_1, E_2, E_3 are positive constants depending only on k.

For further inequalities we refer to [1]. We also mention that inequality (3.20) is an easy consequence of formulas (5), (11), (12), (15) and Section 8 of [1].

Proof of Lemma 3. Let us fix $0\leqq t_1 < t_1' \leqq \cdots \leqq t_r < t_r' \leqq 1$ and apply Bahr's inequality with $k=r$, $F=F_n^{(t_1, \ldots, t_r')}$, $G=\Phi^{(t_1, \ldots, t_r')}$, $T=S(nt)^\nu$, where $F_n^{(t_1, \ldots, t_r')}$, $\Phi^{(t_1, \ldots, t_r')}$ are the distribution functions defined in Section 2, $t=\min(t_1'-t_1, \ldots, t_r'-t_r)$, S and ν are positive constants which we shall determine later. In this case we have evidently $\psi(x_1, \ldots, x_r)=(2\pi)^{-r/2} e^{-\frac{1}{2}(x_1^2+\cdots+x_r^2)}$ and therefore ψ_1 can be chosen as follows: $\psi_1(R)=(2\pi)^{-r/2} e^{-\frac{1}{2}R^2}$. This function satisfies all the additional requirements and we also see that in our case $L=(2\pi)^{-r/2}\int_0^\infty R^r e^{-\frac{1}{2}R^2}\, dR$ depends only on r. Let us now assume that

$$S(nt)^\nu \leqq C_6 (nt)^\gamma \tag{3.21}$$

where C_6, γ are the constants appearing in condition $\mathbf{B}_3$; then (2.6) holds for every $\lambda=(\lambda_1, \ldots, \lambda_r)$ such that $\|\lambda\|\leqq S(nt)^\nu$ and thus we obtain from (3.20) for every bounded convex Borel-set $B\subset R^r$:

$$\begin{aligned}
&|F_n^{(t_1, \ldots, t_r')}(B)-\Phi^{(t_1, \ldots, t_r')}(B)| \\
&\quad\leqq \frac{C_{31}}{S(nt)^\nu} + C_{32}\, \mu\big(B_{(\frac{C_{33}}{S(nt)^\nu})}\big)\cdot \int_{\|\lambda\|\leqq S(nt)^\nu} |f_n^{(t_1, \ldots, t_r')} - f_0^{(t_1, \ldots, t_r')}|\, d\lambda_1 \ldots d\lambda_r \\
&\quad\leqq \frac{C_{31}}{S(nt)^\nu} + C_{32}\, \mu\big(B_{(\frac{C_{33}}{S(nt)^\nu})}\big)\cdot \int_{\|\lambda\|\leqq S(nt)^\nu} \frac{C_7}{(nt)^\delta}\, d\lambda_1 \ldots d\lambda_r \\
&\quad= \frac{C_{31}}{S(nt)^\nu} + C_{32}\, \mu\big(B_{(\frac{C_{33}}{S(nt)^\nu})}\big)\cdot C_{34} \frac{S^r}{(nt)^{\delta-\nu r}}
\end{aligned} \tag{3.22}$$

where $C_{31}, \ldots, C_{34}$ (and, of course, C_7, δ) depend only on r. Let us now choose S and v as follows: $S=C_{33}$, $v=\min(\gamma/2, \delta/1+r)$. In this case relation (3.21) is obviously satisfied for $nt \geqq C_{35}$ where C_{35} is a positive constant depending on r. Furthermore, this number v satisfies $v>0$, $\delta - vr \geqq \delta - \frac{r}{r+1}\delta > 0$ thus $\eta = \min(v, \delta - vr)$ is a positive number depending on r. Finally, it is obvious that with this choice of S we have $C_{33} \leqq S(nt)^v$ for $nt \geqq 1$. Using these facts, we obtain from (3.22) for $nt \geqq \max(1, C_{35})$:

$$|F_n^{(t_1, \ldots, t'_r)}(B) - \Phi^{(t_1, \ldots, t'_r)}(B)| \leqq \frac{C_{36}}{(nt)^\eta} + \frac{C_{37}}{(nt)^\eta} \mu(B_{(1)}) \leqq C_{38}(1+\mu(B_{(1)})) \frac{1}{(nt)^\eta}.$$

On the other hand, it is possible to choose the number C_{38} so large that for $nt \leqq \max(1, C_{35})$ we have $C_{38} \cdot 1/(nt)^\eta > 2$ (this value of C_{38} depends only on C_{35} and η, that is only on r and the sequence $X_1, X_2, \ldots$), but then the inequality

$$|F_n^{(t_1, \ldots, t'_r)}(B) - \Phi^{(t_1, \ldots, t'_r)}(B)| \leqq C_{38}(1+\mu(B_{(1)})) \frac{1}{(nt)^\eta}$$

holds also for $nt \leqq \max(1, C_{35})$ since the left side is $\leqq 2$ and the right side is >2. This completes the proof of Lemma 3.

The next lemma will be used in the proof of Theorem 1.

Lemma 4. *Let $X_1, X_2, \ldots$ be a uniformly bounded sequence of random variables such that for every $m \geqq 0$, $n \geqq 1$, $0 \leqq t \leqq C_{39} \sqrt{\log n}$ we have $P(|S_n^{(m)}| \geqq t\sqrt{n}) \leqq C_{40} e^{-C_{41}t^2}$ where C_{39}, C_{40}, C_{41} are independent of m, n, t. Then the conclusion of Lemma 1 holds, i.e. there exist positive constants B', C'_8 such that for every $m \geqq 0$, $n \geqq 1$ we have*

$$P\Big(\max_{1 \leqq v \leqq n} |S_v^{(m)}| \geqq B' \sqrt{n \log\log n}\Big) \leqq C'_8 \, e^{-2\log\log n}.$$

This lemma is a simple variant of a lemma in Takahashi's paper [8] and it can be proved in the same manner.

Proof of Theorem 1. Let us apply condition $\mathbf{B}_2$ with $r=1$, $n=M+N$, $t_1 = M/(M+N)$, $t'_1 = 1$, $B=(-h, h)$, then we obtain

$$\begin{aligned} &|P(|S_N^{(M)}| < h\sqrt{N}) - P(|\zeta(1)| < h)| \\ &\quad \leqq C_{42}[1+(2h+2)]\, N^{-C_{43}} \leqq C_{42}\, N^{-C_{44}} \leqq C_{42}\, e^{-C_{45}h^2} \end{aligned}$$

provided that $h \leqq C_{46}\sqrt{\log N}$; this relation shows that the sequence $X_1, X_2, \ldots$ satisfies the conditions of Lemma 4. We can now observe that the equicontinuity statement in the proof of Theorem 2 was deduced only from relation (3.1), thus, in view of Lemma 4, it is valid also under the conditions of Theorem 1. Furthermore, in steps b), and c), we used only the equicontinuity property and condition $\mathbf{B}_2$, except the proof of (3.10) where we needed the relation $\sup_n E|X_n|^{2+\delta} < \infty$ for a suitable $\delta > 0$, and this was deduced from condition **A**. Our moment condition, however, is valid also under the conditions of Theorem 1, since in this case the sequence $X_1, X_2, \ldots$ is uniformly bounded.

4. A More General Theorem

It can be proved by calculations that a uniformly bounded sequence of independent random variables having mean 0 and variance 1 satisfies condition $\mathbf{B}_3$, therefore, in view of Theorem 1, the functional law of the iterated logarithm holds for such a sequence. It can also be checked that if $X_1, X_2, \ldots$ are independent random variables satisfying the conditions

$$EX_n=0, \qquad EX_n^2=1, \qquad P(|X_n|\geqq t)\leqq C_{47}\, e^{-C_{48}t^2}$$

where C_{47} and C_{48} are positive constants independent of n and t, then the conditions of Theorem 2 are satisfied, thus the functional law of the iterated logarithm holds also in this case. These remarks show that Theorems 1 and 2 imply certain results for independent random variables; these results, however, are not quite satisfactory, since their assumptions are too strong. Even the second, weaker condition $P(|X_n|\geqq t)\leqq C_{47}\, e^{-C_{48}t^2}$ implies the existence of the moments of all order of the variables X_n; moreover it implies the existence of the moment generating functions on the whole line. It does not follow either from Theorem 1 or from Theorem 2 that the functional law of the iterated logarithm holds if $X_1, X_2, \ldots$ are independent, identically distributed random variables with mean 0 and variance 1 (Strassen's result). The situation is not better even if we assume in addition that $E|X_1|^{2+\delta}<+\infty$. It is therefore worthwhile to formulate a common generalization of Theorem 1 and Theorem 2 which implies the latter result at least. The generalization is almost automatic from the proofs of Theorems 1 and 2.

Theorem 3. *If a sequence $X_1, X_2, \ldots$ of random variables satisfies condition $\mathbf{B}_2$, $E|X_n|^{2+\delta}\leqq C_{49}$ $(n=1, 2, \ldots)$ and*

$$P\Big(\max_{1\leqq v\leqq n} |S_v^{(m)}|\geqq B^* \sqrt{n\log\log n}\Big)\leqq C_{50}\, e^{-2\log\log n} \tag{4.1}$$

where C_{49}, C_{50}, B^*, δ are positive constants depending only on the sequence X_1, $X_2, \ldots$, then the functional law of the iterated logarithm is valid for this sequence.

Proof. See the proof of Theorem 1.

Let now $X_1, X_2, \ldots$ be a sequence of independent, identically distributed random variables such that $EX_1=0$, $EX_1^2=1$, $E|X_1|^{2+\delta}<+\infty$. Using the fact that the components of the random vector in (2.1) are independent, we obtain by standard calculations that the sequence $X_1, X_2, \ldots$ satisfies condition $\mathbf{B}_3$; on the other hand, the well-known inequality

$$P\Big(\max_{1\leqq v\leqq n} S_v^{(m)}\geqq c\Big)\leqq \tfrac{4}{3} P(S_n^{(m)}\geqq c-2\sqrt{n})$$

(which is valid for independent random variables having mean 0 and variance 1, cf. [5], p. 403) and the central limit theorem with remainder term $O(n^{-\delta/2})$ yield (4.1) with $B^*=3$. We omit the details.

References

1. Bahr, B. von: Multi-dimensional integral limit theorems. Ark. Mat. **7**, 71–88 (1967).
2. Chover, J.: On Strassen's version of the log log law. Z. Wahrscheinlichkeitstheorie verw. Gebiete **8**, 83–90 (1967).

3. Feller, W.: An introduction to probability theory and its applications. Vol. II. New York: Wiley 1966.
4. Gaposkin, V.F.: On the law of iterated logarithm for strongly multiplicative systems. In Russian. Teor. Verojatnost. i Primenen. **14**, 511–516 (1969).
5. Rényi, A.: Probability Theory. Budapest: Akadémiai Kiadó 1970.
6. Révész, P.: The law of the iterated logarithm for multiplicative systems. Indiana Univ. Math. J. **21**, 557–564 (1972).
7. Strassen, V.: An invariance principle for the law of the iterated logarithm. Z. Wahrscheinlichkeitstheorie verw. Gebiete **3**, 211–226 (1964).
8. Takahashi, S.: Notes on the law of the iterated logarithm. Studia Sci. Math. Hung. (In print.)

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