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Some Remarks on a Class of Semi-Groups of Operators. I

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0. In [1, 2] some results are given about the class of all semi-groups of operators $(T_t, t \geqq 0)$ on a Banach space which have the property that for some $t>0$, $T_t - I$ is compact. Since there exist some important generalizations of the notion of compact operators, our purpose in this note is to obtain some results about semi-groups such that for some $t>0$, $T_t - I$ is in some class of operators which generalizes the class of compact operators, namely the class of α-contractions, densifying operators and Riesz operators.

1. First we recall some definitions for the reader's convenience. Let X be a Banach space and A be a bounded set in X. We define $\alpha(A)$, the Kuratowski number of the set A, as the infimum of all $\varepsilon>0$ such that there exists a decomposition of A into a finite number of subsets of diameter less than ε. It is easy to see that the following assertions hold:

1) if A, B are bounded sets in X; then $A \subseteq B$ implies $\alpha(A) \leqq \alpha(B)$;

2) if A is relatively compact; then $\alpha(A)=0$;

3) if $A+B=\{a+b, a\in A, b\in B\}$, then

$$\alpha(A+B) \leqq \alpha(A)+\alpha(B).$$

The notion of an α-contraction was introduced by G. Darbo.

Definition 1.1. A function $f\colon X \to X$ is called an α-contraction if there exists $k \in (0, 1)$ such that for every bounded set A we have $\alpha(fA) \leqq k\,\alpha(A)$ where $fA = \{y, y=f(x), x\in A\}$.

Definition 1.2. A function $f\colon X \to X$ is called densifying if for each bounded set A we have $\alpha(fA) < \alpha(A)$ if $\alpha(A)>0$.

Remark. It is easy to see that every operator of the form $T_1 + T_2$ where T_1 is a contraction map and T_2 is compact, is an α-contraction.

2. Let $\{T_t\}_{t\geqq 0}$ be a semi-group of class C_0 on X and define the set C as

$$\{t, t>0,\ T_t - I \text{ is an } \alpha\text{-contraction}\}.$$

Clearly these semi-groups include the semi-groups studied by Cuthbert. Our first result is a generalization of Theorem 1 in [1].

We suppose that the semi-group $\{T_t\}_{t\geqq 0}$ satisfies $\|T_t\| \leqq M e^{\omega t}$ for all $t \geqq 0$ with constants M, ω.

Theorem 2.1. *If $C \neq \emptyset$, then T_t is invertible for all t.*

Proof. Suppose that the assertion is false. Then $0 \in \sigma(T)$ for some t (and thus for all $t>0$). Since $T_t - I$ is α-contraction it is easy to see that the subspace $N(T_t)$

is finite-dimensional. We can show that $0 \in \sigma_p(T)$, the point spectrum of T_t. Indeed, we find a sequence of unit vectors $\{x_n\}$ such that $Tx_n \to 0$. But the set $\{x_n\}$ is relatively compact because we have, if $\alpha(\{x_n\}) > 0$,

$$\alpha(\{x_n\}) > \alpha((T_t - I(\{x_n\}))) = \alpha(\{x_n\})$$

since $\alpha(\{Tx_n\}) = 0$. From this it is clear that $0 \in \sigma_p(T_t)$.

We can now follow the arguments in [1] to obtain the assertion of the theorem.

Next we define the following set:

$$C_R = \{t > 0,\ T_t - I \text{ is of Riesz type}\}.$$

It is well known that every compact operator is also of Riesz type. Our aim in this section is to give some equivalent conditions on the semi-group T_t such that $C_R =]0, \infty[$.

Theorem 2.2. *The following conditions are equivalent for a strongly continuous semi-group $\{T_t\}$, $t \geqq 0$.*

1) $C_R =]0, \infty[$;

2) *S is of Riesz type;*

3) *$\lambda R_\lambda - I$ is of Riesz type for some $\lambda \geqq \omega$.*

Proof. If $\mu \neq 1$, then $\mu \in \sigma(T_t)$ implies that $\mu - 1$ is a Riesz point and thus $\mu \in P_{\sigma(T_t)}$ and

$$\sigma(T_t) = \{e^{xt}, x \in P_{\sigma(S)} \text{ for all } t > 0\}.$$

By Dieudonné's result [Ch. 11, § 4, Ex. 5], $\{\lambda, \lambda \in \sigma(T_t), |\lambda - 1| \geqq \varepsilon\}$ is finite for all $t > 0$.

Now it is easy to see that if $C_R \neq \emptyset$, then T_t is invertible for all t since the arguments in Theorem 1 of [1] or in our Theorem 2.1 work in this case. From this we obtain as in [1] that S is bounded and $T_t = e^{St}$. We show that S is of Riesz type. Since (see [1])

$$S = \left[\int_0^t T_s \, ds\right]^{-1} (T_t - I)$$

and $T_t - I$ is of Riesz type, we have that the image of S in the Calkin algebra $\mathscr{C} = \mathscr{L}(X)/\mathscr{K}$, $\mathscr{K}$ the ideal of all compact operators on X, is quasi-nilpotent and this shows that S is of Riesz type.

In the same way, since

$$T_t - I = S \int_0^t T_s \, ds$$

we obtain that $2) \to 1)$.

For the proof of $2) \to 3)$ we use again the characterization of Riesz operators with the Calkin algebra. For an element $T \in \mathscr{L}(X)$, $\hat{T}$ denotes its image in $\mathscr{C}$.

Since

$$\sigma(\lambda \hat{R}_\lambda - \hat{I}) = \{\sigma(\lambda(\lambda \hat{I} - \hat{S})^{-1} - 1) = \{\lambda(\lambda - \mu)^{-1} - 1, \mu \in \sigma(\hat{S})\} = \{0\}$$

the assertion is proved.

For the converse 3) → 2), we remark that

$$R_\lambda(T_t - I) = (\lambda R_\lambda - I)\int_0^t T_s\, ds$$

and thus

$$\hat{R}_\lambda(\hat{T}_t - \hat{I}) = (\lambda \hat{R}_\lambda - \hat{I})\int_0^t \hat{T}_s\, ds$$

which gives that $T_t - I$ is of Riesz type for all $t > 0$. This implies that S is of Riesz type.

We remark that, from the relation

$$R_\lambda - R_\mu = (\mu - \lambda) R_\lambda R_\mu$$

it is clear that if for some t_0, R_{t_0} is of Riesz type, then R_t is of Riesz type for all $t > 0$. This follows from the use of the Calkin algebra and the result of Basterfield [3],

$$z_t = \frac{z_{t_0}}{1 - (t_0 - t)\, z_{t_0}}$$

for all $z_{t_0} \in \sigma(R_{t_0})$, $z_t \in \sigma(R_t)$.

We remark that our result gives rise to the following problem: if R_t is of meromorphic type, or if $T_t - I$ is of meromorphic type, do the results remain valid?

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