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The Square of Shot Noise

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1. Introduction

We consider the stochastic process of shot noise. This is usually taken as a model for the fluctuating part of the plate current in a vacuum-tube (diode, triode) due to the random emission of electrons from the cathode.

For a formal definition of the process $s(t)$ we put

$$s_\lambda(t) = \int_{-\infty}^{\infty} f(t-s)\, dN_\lambda(s).$$

Here $N_\lambda(s)$ stands for the Poisson process with a fixed rate λ and f is a (non random) function giving the "current pulse" due to a single electron. We assume that f is a function decreasing at infinity faster than any power, although less stringent conditions would suffice. In practice f has compact support but otherwise is quite arbitrary. For a variety of examples see [8] and its references.

For each λ, $s_\lambda(t)$ turns out to be a stationary process of a well known kind, see for instance [2] or [3].

We are interested in identifying the function $f(t)$ from observations relating to the shot noise process. In the case when $s_1(t)$ itself is available it is not hard to see that f is (essentially) determined by the moments of $s_1(t)$. Here we consider the case when only the modulus, but not the signature of the process is available.

Imagine that we have the means to speed up the Poisson process by choosing different values of the rate λ, $\lambda \in \Lambda$. This will give us a family of stochastic processes

$$X_\lambda(t) = s_\lambda^2(t) \qquad \lambda \in \Lambda.$$

We take Λ to be an arbitrary infinite set. The Poisson process being a nice point process one can easily write down expressions for the moments

$$E\big(X_\lambda(t_1) \ldots X_\lambda(t_n)\big)$$

of the process $X_\lambda(t)$. We are now in a position to describe the main result of this paper.

Theorem. *The moments of the processes $X_\lambda(t)$, $\lambda \in \Lambda$, are enough to determine the function f up to a translation and a fixed signature.*[1]

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This work is an expansion of research carried out while the author was with IBM Watson Research Center in 1972.

[1] A complete proof is presented under the technical condition $\int f \neq 0$.

2. Preliminaries

We start recalling some well known facts which are going to be used below. The Poisson process with rate λ is a particular case of a point process with nice "product density functions", see [2, 3, 7]. If we set

$$N_\lambda(s_1, s_2] = N_\lambda(s_2) - N_\lambda(s_1),$$

we get a stationary additive interval function such that if $\Delta_1, \ldots, \Delta_k$ are disjoint intervals and $s_j \in \Delta_j$ we have

$$\lim_{|\Delta_j| \to 0} |\Delta_1|^{-1} \ldots |\Delta_k|^{-1} \operatorname{Prob}\{N(\Delta_1)=1, \ldots, N(\Delta_k)=1\} = \lambda^k$$

uniformly in $s_1, \ldots, s_k$.

Using Theorem 3.1 in [3] we can write down for arbitrary intervals Δ_i $(i=1, \ldots, n)$

$$E(N_\lambda(\Delta_1) \ldots N_\lambda(\Delta_n)) = \sum_{l=1}^{n} \lambda^l \int \ldots \int \Big[\prod_{j\in\nu_1} \chi_{\Delta_j}(\tau_1)\Big] \ldots \Big[\prod_{j\in\nu_l} \chi_{\Delta_j}(\tau_l)\Big]\, d\tau_1 \ldots d\tau_l.$$

Here the sum extends over all partitions $(\nu_1, \ldots, \nu_l)$ of the set $(1, 2, \ldots, n)$ and χ_Δ denotes the characteristic function of the interval Δ.

It is clear that we have

$$E(s_\lambda(t_1) \ldots s_\lambda(t_n)) = \sum_{l=1}^{n} \lambda^l \int \ldots \int \Big[\prod_{j\in\nu_1} f(t_j - \tau_1)\Big] \ldots \Big[\prod_{j\in\nu_l} f(t_j - \tau_l)\Big]\, d\tau_1 \ldots d\tau_l. \tag{1}$$

This formula is valid if f has compact support, and an application of Fubini's theorem shows that it is true almost everywhere $(t_1, \ldots, t_n)$ as soon as $f \in L'(R)$.

Particular cases of (1) include:

$$E(s_\lambda(t_1)) = \lambda \int f(t-\tau_1)\, d\tau_1 = \lambda \int f(\tau)\, d\tau,$$

$$E(s_\lambda(t_1)\, s_\lambda(t_2)) = \lambda^2 \left(\int f(\tau)\, d\tau\right)^2 + \lambda \int f(t_1-\tau) f(t_2-\tau)\, d\tau,$$

$$\begin{aligned} E(s_\lambda(t_1)\, s_\lambda(t_2)\, s_\lambda(t_3)) &= \lambda^3 \left(\int f(\tau)\, d\tau\right)^3 \\ &\quad + \lambda^2 \left(\int f(\tau)\, d\tau\right) \left[\int [f(t_1-\tau) f(t_2-\tau) + f(t_2-\tau) f(t_3-\tau) + f(t_3-\tau) f(t_1-\tau)]\, d\tau\right] \\ &\quad + \lambda \int f(t_1-\tau) f(t_2-\tau) f(t_3-\tau)\, d\tau. \end{aligned}$$

The results used above can be appropriately modified to deal with point processes not having nice "product densities". For such a general case the reader can consult [6], especially Theorem 4 and Corollary 2 to Theorem 3.

A look at (1), or a direct computation of the cumulants instead of the moments of $s_\lambda(t)$, shows that when $s_1(t)$ is available one knows all the integrals

$$\int f(t_1-\tau) f(t_2-\tau) \ldots f(t_n-\tau)\, d\tau$$

a.e. in $(t_1, \ldots, t_n)$. This allows for the determination of a function $f \in L'(R)$ unique up to a translation, see [1] and [4].

Expression (1) will be used below to compute the moments of $X_\lambda(t) = s_\lambda^2(t)$.

Notice that

$$E(X_\lambda(t)) = \lambda^2 \left(\int f(\tau)\, d\tau\right)^2 + \lambda \left(\int f^2(\tau)\, d\tau\right),$$

and thus knowing $E(X_\lambda(t))$ for enough many values of λ allows one to decide if $\int f(\tau)\,d\tau$ vanishes or not. It is convenient to look at these two cases separately.

3. Proof of the Theorem

Case 1. $\int f(\tau)\,d\tau \neq 0$. Assume that $\int f(\tau)\,d\tau > 0$. From (1) one gets that the coefficient of λ^3 in $E(X_\lambda(t_1)\,X_\lambda(t_2))$ is

$$\left(\int f(\tau)\,d\tau\right)^2 \left(2\int f^2(\tau)\,d\tau + 4\int f(t_1-\tau)\,f(t_2-\tau)\,d\tau\right). \tag{2}$$

The reader will easily see the relation between this expression and the six partitions

(12)(3)(4), (34)(1)(2), (13)(2)(4), (14)(2)(3), (23)(1)(4), (24)(1)(3)

of the set (1, 2, 3, 4). The first two partitions give the contribution

$$\left(\int f(\tau)\,d\tau\right)^2 \int f^2(\tau)\,d\tau \tag{3}$$

while the last four ones give

$$\left(\int f(\tau)\,d\tau\right)^2 \int f(t_1-\tau)\,f(t_2-\tau)\,d\tau. \tag{4}$$

Now (3) is already known from $E(X_\lambda(t))$, thus so is (4), and therefore the integral

$$\int f(t_1-\tau)\,f(t_2-\tau)\,d\tau$$

becomes known once we have

$$E(X_\lambda(t_1)\,X_\lambda(t_2)\,\dots\,X_\lambda(t_n)) \tag{5}$$

for $n \leqq 2$.

We will proceed now by induction to show that the knowledge of expression (5) for $2 \leqq n \leqq N$, $\lambda \in \Lambda$, and arbitrary $(t_1 \dots t_n)$, implies the knowledge of the integral

$$\int f(t_1-\tau)\dots f(t_N-\tau)\,d\tau. \tag{6}$$

The coefficient of λ^{N+1} in expression (5) with $n=N$ is in correspondence with the partitions of the set $(1, 2, \dots, 2N)$ into $N+1$ blocks. Among these we can restrict our attention, using the inductive hypothesis, to those partitions into a block of N elements and N blocks of a single element each. There are $\binom{2N}{N}$ such partitions. Examples of contributions from this partitions are

$$\left(\int f(\tau)\,d\tau\right)^N \int f(t_1-\tau)\dots f(t_N-\tau)\,d\tau$$

from the partition

$$(1\,3\,5\,\dots\,2N-1)\,(2)\,(4)\,\dots\,(2N)$$

and

$$\left(\int f(\tau)\,d\tau\right)^N \int f^2(t_1-\tau)\,f(t_2-\tau)\dots f(t_{N-1}-\tau)\,d\tau$$

from the partition

$$(1\,2\,3\,5\,\dots\,2N-2)\,(2N-1)\,(4)\,(6)\,\dots\,(2N).$$

Each one of the $\binom{2N}{N}$ partitions under consideration gives a contribution of the kind

$$(\int f(\tau)\,d\tau)^N \int \prod_{i\in A_2} f^2(t_i-\tau) \prod_{i\in A_1} f(t_i-\tau)\,d\tau \tag{7}$$

with A_1 and A_2 arbitrary disjoint subsets of $(1\ 2\ \dots\ N)$ such that

$$2|A_2|+|A_1|=N. \tag{8}$$

The examples given above correspond to the cases

$$A_1=(1\ 2\ \dots\ N), \qquad A_2=\{\phi\}$$

and

$$A_1=(2\ \dots\ N-1), \qquad A_2=(1)$$

respectively.

The total contribution can be split in blocks according to the value of $|A_2|= 0, 1, 2, \dots [N/2]$.

One can further subdivide each block by grouping together the elements with the same $A_1 \cup A_2$. In this fashion the contribution from the k-th blocks is

$$(\int f(\tau)\,d\tau)^N \sum_{A_1\cup A_2} \sum I(A_1, A_2) \tag{9}$$

with the inner summation running over all pairs of disjoint subsets A_1, A_2 in $(1\ 2\ \dots\ N)$, $|A_2|=k$, $|A_1|=N-2k$, and having a specified union $A_1\cup A_2$. $I(A_1, A_2)$ is shorthand for the second factor in (7).

To conclude the proof we need to show that if $|A_2|=k\neq 0$, each inner sum in (9) can be obtained from (5) with $2\leqq n\leqq N$, $\lambda\in A$.

Indeed it is enough to look at (5) with $n=N-k=|A_1\cup A_2|<N$. Pick t_i, $i\in A_1\cup A_2$ and look at expression (5) for these values of t_i. We are dealing with a polynomial in λ of degree $2(N-k)=2(|A_1|+|A_2|)$. The coefficient of $\lambda^{|A_1|+1}$ is the sum of

$$(\int f(\tau)\,d\tau)^{|A_1|} \sum I(A_1, A_2)$$

plus products of integrals each involving less than N factors and thus known by inductive hypothesis.

A fortiori (6) is known and we can invoke once more the result in [1] to conclude that f is determined up to a translation, under the assumption $\int f(\tau)\,d\tau>0$.

Had we started from $\int f(\tau)\,d\tau<0$ we would have another solution unique up to translation. A look at the proof above shows that this new class should be the class of translates of the negative of a function in the previous class.

This concludes the proof under the assumption $\int f(\tau)\,d\tau\neq 0$.

Case 2. $\int f(\tau)\,d\tau=0$.

Now it is clear that any partition containing a block of a single element gives a vanishing contribution, in particular all the integrals considered above are of this type.

As a matter of fact

$$E(X_\lambda(t_1)\dots X_\lambda(t_n)) \tag{10}$$

is a polynomial of degree at most n.

Using the results contained in the previous section we get, for instance

$$E(X_\lambda(t))=\lambda\int f^2(\tau)\,d\tau$$

$$E(X_\lambda(t_1)\,X_\lambda(t_2))=\lambda^2\left[\left(\int f^2(\tau)\,d\tau\right)^2+2\left(\int f(t_1-\tau)f(t_2-\tau)\,d\tau\right)^2\right]$$
$$+\lambda\int f^2(t_1-\tau)f^2(t_2-\tau)\,d\tau.$$

It is clear that for each n the new contribution to the coefficient of λ^n in (10) is given by

$$\sum_\pi R(t_{\pi 1}-t_{\pi 2})\dots R(t_{\pi n}-t_{\pi 1})$$

where $R(t)=\int f(t+\tau)f(\tau)\,d\tau$ and the summation extends to the group of permutations of n elements.

Now this implies, see [5], that the function $R(t)$ itself can be obtained from the knowledge of the moments of the processes X_λ, $\lambda\in\Lambda$. This amounts to the knowledge of the Fourier transform $\hat f(\lambda)$ of f up to an arbitrary phase factor $e^{i\theta(\lambda)}$, θ real valued and measurable.

On the other hand, the coefficient of λ in $E(X_\lambda(t_1)\dots X_\lambda(t_n))$ is

$$\int f^2(t_1-s)\dots f^2(t_n-s)\,ds$$

and using again the result in [1], we conclude that f^2 is determined up to a translation.

Then we could consider the problem of determining a real valued rapidly decreasing function f for which we know both its absolute value and the modulus of its Fourier transform.

It is not hard to see that these two conditions alone do not give enough information about the function f.

If g_1 and g_2 are even and odd functions respectively, and their supports are disjoint, the functions

$$f_1=g_1+g_2,\qquad f_2=g_1-g_2$$

satisfy

$$|f_1|=|f_2|\ \text{a.e.}\quad\text{and}\quad|\hat f_1|=|\hat f_2|\ \text{a.e.}$$

but do not stand in the desired relation $f_1(t)=\pm f_2(t+a)$. But one should notice that we've not used all of the information contained in $E(X^2(t_1)\dots X_\lambda^2(t_n))$. Only the coefficients of λ^n and λ have been looked at and they turn out to be useful by referring to the results contained in [1] and [5] respectively. This indicates that many useful theorems ought to lie between [1] and [5], and on this hopeful note we close this paper.

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