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Convergence Properties of Random T-Fractions

Sheila Foster and Tom Pitcher*

I. Introduction

T-fractions are analytic continued fractions which were first introducted by Thron in 1948 [3]. They have the general form

$$1+d_0 z+\frac{z}{1+d_1 z}\,\frac{z}{+\,1+d_2 z}\,+\cdots.$$

One reason for interest in the convergence behavior for T-fractions is that there is a one-to-one correspondence between the set of T-fractions and the set of all formal power series. However, the convergence behavior of T-fractions can be very strange. For example, if $d_n=-1$ for all n, the T-fraction

$$1-z+\frac{z}{1-z}\,\frac{z}{+\,1-z}\,+\cdots$$

converges to 1 for $|z|<1$. On the unit circle, the T-fraction diverges except at the point $z=-1$ where it converges to 1. On the region $|z|>1$, this same T-fraction converges to the value $-z$.

In this paper, we will consider $\{d_n\}$, $n=0,1,2,\ldots$, to be an independent identically distributed sequence of random variables. Through use of the ergodic theorem, we shall show that a T-fraction has an exponential rate of point-wise convergence with probability one provided that the support of the measure on the d_n, $n=0,1,2,\ldots$, is sufficiently large. On the other hand we will show in Theorem 4.3 that they do not converge uniformly in general.

T-fractions have approximants for any positive integer n, given by

$$\frac{A_n(z)}{B_n(z)}=1+d_0 z+\frac{z}{1+d_1 z}\,+\cdots\,\frac{z}{+\,1+d_n z}$$

where the numerators, $A_n(z)$, and the denominators, $B_n(z)$, are given by the following recursion formulas:

$$\begin{aligned}
A_{-1}(z)&=1\\
A_0(z)&=1+d_0 z\\
&\ \vdots\\
A_n(z)&=(1+d_n z)\,A_{n-1}(z)+z\,A_{n-2}(z),
\end{aligned}$$

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and

$$\begin{aligned} B_{-1}(z) &= 0 \\ B_0(z) &= 1 \\ &\vdots \\ B_n(z) &= (1+d_n z)\, B_{n-1}(z) + z\, B_{n-2}(z). \end{aligned}$$

Applying these recursion relationships, we see that the difference between the n-th and $(n-1)$-th approximants is given by

$$\begin{aligned} \left|\frac{A_n(z)}{B_n(z)} - \frac{A_{n-1}(z)}{B_{n-1}(z)}\right| &= \left|\frac{A_n(z)\, B_{n-1}(z) - A_{n-1}(z)\, B_n(z)}{B_n(z)\, B_{n-1}(z)}\right| \\ &= \frac{|z|^n}{|B_n(z)\, B_{n-1}(z)|}. \end{aligned}$$

Looking more closely at the recursion relationships for the denominators, we have

$$\begin{pmatrix} B_n(z) \\ B_{n-1}(z) \end{pmatrix} = \begin{pmatrix} 1+d_n z & z \\ 1 & 0 \end{pmatrix} \begin{pmatrix} B_{n-1}(z) \\ B_{n-2}(z) \end{pmatrix}$$

or

$$\begin{aligned} \begin{pmatrix} B_n(z) \\ B_{n-1}(z) \end{pmatrix} &= \prod_{k=1}^{n} \begin{pmatrix} 1+d_k z & z \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= z^{n/2} \prod_{k=1}^{n} M(d_k, z) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{aligned}$$

where the matrices

$$M(d, z) = \begin{pmatrix} (1+dz)\, z^{-\frac{1}{2}} & z^{\frac{1}{2}} \\ z^{-\frac{1}{2}} & 0 \end{pmatrix}$$

are unimodular. If

$$\begin{pmatrix} g_{11}^{(n)} & g_{12}^{(n)} \\ g_{21}^{(n)} & g_{22}^{(n)} \end{pmatrix} = \prod_{k=1}^{n} M(d_k, z),$$

then

$$\left|\frac{A_n(z)}{B_n(z)} - \frac{A_{n-1}(z)}{B_{n-1}(z)}\right| = \frac{|z|^n}{|B_n(z)\, B_{n-1}(z)|} = \frac{1}{|g_{11}^{(n)}\, g_{21}^{(n)}|}.$$

We will prove convergence of A_n/B_n by proving that the term $|g_{11}^{(n)} g_{21}^{(n)}|$ grows exponentially.

II. Growth of the Column Vectors of a Product of Random Matrices

Let μ be a probability measure on $SL(2, C)$, the group of all unimodular 2×2 matrices over the complex field, and let G be the smallest closed subgroup containing the support of μ. G operates as a matrix group on the space C^2, the 2-dimensional vector space over the complex field. If, in the space $C^2-\{0\}$ we identify any two vectors that are positive multiples of each other, we obtain the space V. Using

$$\left\{ v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \,\middle|\, v_1, v_2 \in C, |v_1|^2 + |v_2|^2 = 1 \right\}$$

as the representation of V, we see that V can be imbedded in C^2 in an obvious way. We will denote the operation of G on V as $\hat{g}v = \frac{gv}{\|gv\|}$.

We will need the following theorem which is an easy consequence of some results of Furstenberg (Theorem 8.5, p. 424 of [1] is the real version of it). We will write $\mu * \xi$ for the convolution of a measure μ on $SL(2, C)$ and a measure ξ on V.

Theorem 2.1. *There exists a stationary ergodic probability measure ξ for μ on V, that is a solution of $\mu * \xi = \xi$. If G does not leave a subspace of C^2 fixed and if*

$$0 < \alpha_\mu = \int_V \int_G \log \|gv\| \, d\mu(g) \, d\xi(v) < \infty,$$

then for the sequence $g_1, g_2, \ldots$ of independent, μ-distributed matrices

$$\lim \frac{1}{n} \log \|g_n \ldots g_1 u\| = \alpha_\mu$$

with probability one for every $u \in C^2$.

We intend to use this result to show that the product $|g_{11}^{(n)} g_{21}^{(n)}|$ grows exponentially. We shall consider $d_n, n = 0, 1, 2, \ldots$ to be a sequence of independent identically distributed random variables so that the matrices $M(d_k, z)$ are independent with some distribution μ induced by the distribution of the d's. Note that the resultant distribution μ gives rise to a G which leaves no subspace fixed except in the degenerate case of constant d. In fact if G had an eigenvalue v and eigenvector λ, this would lead to the equation

$$(1 + dz)\lambda + z^{\frac{1}{2}} = \lambda^2 z^{\frac{1}{2}}$$

which determines d.

According to Theorem 2.1 $|g_{11}^{(n)}|^2 + |g_{21}^{(n)}|^2$ grows like $e^{2n\alpha}(\mu)$. We assume throughout this section that $0 < \alpha_\mu < \infty$. We will say more about this condition later.

Lemma 2.1. *If the distribution of the d_n's has a bounded density and K and L are positive constants, then for $z \neq 0$*

$$|(1 + d_n z) z^{-\frac{1}{2}}| < L e^{-Kn}$$

only finitely often with probability one.

Proof. A simple application of Borel-Cantelli.

The rather awkward looking conditions of the next lemma are satisfied whenever d has a distribution which is bounded and compactly supported.

Lemma 2.2. *Suppose $z \neq 0$, $\alpha > 0$, $\varepsilon > \varepsilon' > 0$. If the densities of d_n and of* $\mathrm{Arg}((1 + d_n z) z^{-\frac{1}{2}})$ *are bounded, then for n sufficiently large, with probability one,*

$$|g_{11}^{(n+1)}| \geqq e^{(n+1)(\alpha - \varepsilon)}$$

if either $|g_{11}^{(n)}| \geqq e^{n(\alpha - \varepsilon')}$ or $|g_{21}^{(n)}| \geqq e^{n(\alpha - \varepsilon')}$.

Proof. We see that

$$g_{11}^{(n+1)} = (1 + d_{n+1} z) z^{-\frac{1}{2}} g_{11}^{(n)} + z^{\frac{1}{2}} g_{21}^{(n)} = A + B.$$

If δ is the angle between A and B, then for $-90^\circ \leqq \delta \leqq 90^\circ$

$$|g_{11}^{(n+1)}| \geqq \max(|A|, |B|)\, |2 \sin(\delta/2)|$$

so that, for some $k>0$,

$$|g_{11}^{(n+1)}| \geqq k \max(|A|, |B|)\, |\delta|.$$

We will prove the lemma for the case $|g_{11}^{(n)}| \geqq e^{n(\alpha-\varepsilon')}$, the other case is proved similarly. In this case

$$|g_{11}^{(n+1)}| \geqq k\, |(1+d_{n+1} z)\, z^{-\frac{1}{2}}|\, e^{n(\alpha-\varepsilon')}\, |\delta| \geqq k\, e^{n(\alpha-\varepsilon'-\varphi)}\, |\delta|$$

for large n, applying Lemma 2.1 with $L=1$ and $K=\varphi>0$. If M is the bound for the distribution of the argument and if $\psi>0$, then

$$P(|\delta|<e^{-n\psi}) = P(|\operatorname{Arg}(1+d_{n+1} z)\, z^{-\frac{1}{2}} + \operatorname{Arg} g_{11}^{(n)} - \operatorname{Arg} z^{\frac{1}{2}} g_{21}^{(n)}| < e^{-n\psi}) \leqq 2 M e^{-n\psi}.$$

Hence, by Borel-Cantelli, for large enough n

$$|g_{11}^{(n+1)}| \geqq k\, e^{n(\alpha-\varepsilon'-\varphi-\psi)} \geqq e^{n(\alpha-\varepsilon)}$$

if φ and ψ are taken small enough.

Theorem 2.2. *Let $z \neq 0$ be fixed, and suppose that the distribution of the d's and of $\operatorname{Arg}((1+dz)\, z^{-\frac{1}{2}})$ are bounded. If $0<\alpha_\mu<\infty$ and if $\varepsilon>0$, then*

$$|g_{11}^{(n)} g_{21}^{(n)}| \geqq e^{2n(\alpha-\varepsilon)}$$

for large n with probability one. Consequently the T-fraction

$$1+d_0 z + \frac{z}{1+d_1 z} + \frac{z}{1+d_2 z} + \cdots$$

converges with probability one.

Proof. By Theorem 2.1, for $\varepsilon>0$

$$|g_{11}^{(n)}|^2 + |g_{21}^{(n)}|^2 \geqq e^{2n(\alpha-\varepsilon)}$$

so that either

$$|g_{11}^{(n)}| \geqq e^{n(\alpha-2\varepsilon)} \quad \text{or} \quad |g_{21}^{(n)}| \geqq e^{n(\alpha-2\varepsilon)}$$

for large n. Therefore, by Lemma 2.2

$$|g_{11}^{(n)}| \geqq e^{n(\alpha-3\varepsilon)}$$

and

$$|g_{21}^{(n)}| = |g_{11}^{(n-1)}|\, |z|^{-\frac{1}{2}} \geqq e^{(n-1)(\alpha-3\varepsilon-\log|z|/2n)}$$

from which the theorem follows.

III. The Sign of α_μ

In light of Theorem 2.2 it is natural to ask when α_μ is positive. We define a measure λ on V by setting $d\lambda = (2\pi)^{-2} \rho\, d\rho\, d\theta_1\, d\theta_2$ where

$$v = \begin{pmatrix} \rho & e^{i\theta_1} \\ \sqrt{1-\rho^2} & e^{i\theta_2} \end{pmatrix}$$

is an element of V. λ is a probability on V and the following result implies that it is invariant under unitary transformation.

Lemma 3.1. $\dfrac{d(\hat{g}\lambda)}{d\lambda}(v)=\|g^{-1}v\|^{-4}$.

Proof. For any continuous f on V

$$\int_V f(v)\,dg\lambda(v)=\int_V f(\hat{g}v)\,d\lambda(v)$$

$$=\frac{1}{2\pi^2}\int_0^{2\pi}\int_0^{2\pi}\int_0^1 f\left(\begin{pmatrix}\sigma & e^{iz_1}\\ \sqrt{1-\sigma^2} & e^{iz_2}\end{pmatrix}\right)\rho\,d\rho\,d\theta_1\,d\theta_2$$

$$=\frac{1}{2\pi^2}\int_0^{2\pi}\int_0^{2\pi}\int_0^1 f\left(\begin{pmatrix}\sigma & e^{iz_1}\\ \sqrt{1-\sigma^2} & e^{iz_2}\end{pmatrix}\right)\rho\frac{\partial(\rho,\theta_1,\theta_2)}{\partial(\sigma,z_1,z_2)}\,d\sigma\,dz_1\,dz_2.$$

By a straightforward calculation

$$\frac{\partial(\rho,\theta_1,\theta_2)}{\partial(\sigma,z_1,z_2)}=\frac{\sigma}{\rho}\left\|g^{-1}\begin{pmatrix}\sigma & e^{iz_1}\\ \sqrt{1-\sigma^2} & e^{iz_2}\end{pmatrix}\right\|^{-4}$$

which gives the result.

Using this we can prove, just as in [1] (Lemma 8.9, p. 425) the following result.

Lemma 3.2. *If the stationary measure ξ for μ on V is equivalent to λ, then*

$$\alpha_\mu=-\tfrac{1}{4}\int_G\int_V \log\frac{d(\hat{g}^{-1}\xi)}{d\xi}(v)\,d\xi(v)\,d\mu(g).$$

Theorem 3.1. *If the stationary measure ω for μ is equivalent to λ, $\alpha_\mu>0$ unless $\hat{g}^{-1}\xi=\xi$ for $\mu-$almost every $g\in G$.*

Proof. By Jensen's inequality

$$\int_V \log\frac{d(\hat{g}^{-1}\xi)}{d\xi}(v)\,d\xi(v)\leq\log\int\frac{d(\hat{g}^{-1}\xi)}{d\xi}(v)\,d\xi(v)=0.$$

Equality will hold only if $\dfrac{d(\hat{g}^{-1}\xi)}{d\xi}=1$, i.e. only for those g for which $\hat{g}^{-1}\xi=\xi$.

The question of the existence of a stationary measure of the form $d\xi=p\,d\lambda$ is easily seen, with the help of Lemma 3.1, to involve the solution of

$$Tp(v)=\int_G\frac{p(\hat{g}^{-1}v)}{\|g^{-1}v\|^4}\,d\mu(g)=p(v).$$

The transformation T preserves norm in $L_1(d\lambda)$ and also preserves positivity so we have a familiar, though intractable, problem from ergodic theory to deal with.

IV. Uniform Convergence of A_n/B_n

If we suppose that the conditions of Theorem 2.2 are satisfied throughout a region D, then A_n/B_n converges with probability one at each point. By a straightforward application of Fubini's theorem we can assert that A_n/B_n converges almost everywhere in D with probability one. It would seem natural to expect that in most cases this convergence would be uniform on compact subsets and hence that the limit function would be holomorphic. Surprisingly this is not the case in general as is shown by Theorem 4.3.

First we note two results which are direct translations of known theorems about T-fractions.

Theorem 4.1. *If $|d_n| \leqq M$ with probability one, then A_n/B_n is uniformly bounded with probability one in some neighborhood of the origin depending only on M and hence converges uniformly on compact subsets to a holomorphic function.*

Proof. This follows from Theorem 3.1 of [3].

Theorem 4.2. *If $d_1 \geqq -1$ with probability one and the distribution of d_1 is unbounded, then, with probability one, A_n/B_n does not converge uniformly in any neighborhood of the origin.*

Proof. This follows from Theorem 3.1 of [2].

Theorem 4.3. *If the distribution of d_1 has a density ϕ with $\phi(\omega)>0$, a.e. then, with probability one, A_n/B_n does not converge uniformly in any open set.*

Proof. Suppose $f_n = A_n/B_n$ converges uniformly in a disk U_0 with positive probability and hence, by the $0-1$ law, with probability one. We can find a smaller disk U with $0 \notin U \subset \bar{U} \subset U_0$. Then

$$P\big(c \in f_n(U)\big) = P\left(d_n = -\frac{1}{z} - \frac{A_{n-2} - cB_{n-2}}{A_{n-1} - cB_{n-1}} \text{ for some } z \in U\right)$$

$$= P\left(d_n = -\frac{1}{z} - \frac{1}{1+d_{n-1}z} + \frac{z}{1+d_{n-2}z} + \cdots + \frac{z}{1+d_0 z - c} \text{ for some } z \in U\right).$$

Since the d_n are independent and identically distributed

$$P\big(c \in f_n(U)\big) = P\big(d_n \in g_n(U)\big)$$

where

$$g_n(z) = -\frac{1}{z} - \frac{1}{1+d_0 z} + \frac{z}{1+d_1 z} + \cdots + \frac{z}{1+d_{n-1} z - c}$$

$$= -\frac{1}{z} - \frac{1}{\tilde{f}_n(z)}.$$

Lemma 4.1. *$\tilde{f}_n$ has a subsequence $\tilde{f}_{n_k}$ converging uniformly on U to $f = \lim f_n$ with probability one.*

Proof. Let Q be the measure on $U \times \Omega$ (Ω being the underlying probability space) which is the product of Lebesgue measure $d|z|$ and the probability measure P and let Q_n be the measure which results from the transformation $d_n \to d_n - (c/z)$. Then the Q_n are absolutely continuous with respect to Q,

$$\frac{dQ_n}{dQ}(z, \omega) = \frac{\varphi(d_n + c/z)}{\varphi(d_n)},$$

and moreover for any positive δ one can find an ε such that $Q(A)<\varepsilon$ implies $Q_n(A)<\delta$ for all n. If the last statement were not true, there would be sets A_k and numbers n_k with $Q(A_k)<2^{-k}$ and $Q_{n_k}(A_k)\geqq\delta$. If A'_k is the set that results from interchanging d_1 and d_{n_k}, then $Q(A'_k)<2^{-k}$ and $Q_1(A'_k)\geqq\delta$ which violates the absolute continuity of Q_1 with respect to Q.

Now $\|f_n-f\|$ (uniform norm) is measurable since it only depends on the rational points and for any positive ε, $Q(\|f_n-f\|>\varepsilon)$ goes to zero so $Q_n(\|f_n-f\|>\varepsilon)$ $=Q(\|\tilde{f}_n-f\|>\varepsilon)$ also goes to zero. The lemma now follows from a standard measure theoretic argument.

Lemma 4.2. *For almost every c,*

$$P(c\in f(U))\geqq\overline{\lim}\, P(c\in f_n(U)).$$

Proof. We will write A_ε for the ε neighborhood of the set A and ∂A for its boundary. For any $\varepsilon>0$

$$\overline{\lim}\, P(c\in f_n(U))\leqq P(c\in f(U)_\varepsilon)$$

so

$$\begin{aligned}\overline{\lim}\, P(c\in f_n(U))&\leqq P(c\in\overline{f(U)})\\ &\leqq P(c\in(f(U)\cup f(\partial U)))\\ &\leqq P(c\in f(U))+P(c\in f(\partial U)).\end{aligned}$$

But $f(\partial U)$ has measure zero for almost all f so by a Fubini type argument $P(c\in f(\partial U))=0$ for almost all c.

Lemma 4.3. *Let d have the same distribution as d_1 but be independent of all the d_n's. Let $g(z)=-\frac{1}{z}-1/f(z)$. Then*

$$\underline{\lim}\, P(d_{n_k}\in g_{n_k}(U))\geqq P(d\in g(U)).$$

Proof. By Rouche's theorem and Lemma 4.1 $\underline{\lim}\, g_{n_k}(U)\supset g(U)-(\infty)$ so that

$$\begin{aligned}\underline{\lim}\, P(d_{n_k}\in g_{n_k}(U))&\geqq P(d\in\underline{\lim}\, g_{n_k}(U))\\ &\geqq P(d\in g(U)).\end{aligned}$$

We can now complete the proof of Theorem 4.3. For almost every c

$$\begin{aligned}P(c\in f(U))&\geqq\overline{\lim}\, P(c\in f_n(U))\\ &\geqq\underline{\lim}\, P(d_{n_k}\in g_{n_k}(U))\\ &\geqq P(d\in g(U))\\ &=\int\Big(\int_{g(U)}\phi(w)\,d|w|\Big)\,dP\\ &=\gamma>0.\end{aligned}$$

But if N is large enough, $P(\|f\|\geqq N)<\gamma$ so $P(c\in f(U))<\gamma$ whenever $|c|\geqq N$.

Thus with probability one f_n does not converge uniformly in any disk with rational center and radius and hence does not converge uniformly in any open set.

References

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Tom Pitcher
Department of Mathematics
University of Hawaii
Honolulu, Hawaii 96822, USA

Sheila Foster
California State College
at Long Beach
Long Beach, California, USA

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