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Paths of Random Evolutions

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We study the paths of general random evolution processes obtained by piecing together deterministic evolution functions according to the dictates of a regular step process. If the state space is metrizable we show that such processes are strong Markov; and they are even standard under a certain continuity condition on paths. We apply this result to solutions of stochastic delayed differential equations, and we make a connection between our processes and random evolutions associated with classes of semigroups.

1. Introduction

An often used model for the content of a dam is that which assumes content is piecewise linear with slope given by an underlying Markov chain. Writing Y for content, this is phrased as $dY_t/dt = Z_t$, Z a Markov chain. (See Pinsky (1968) and Brockwell (1972) for treatments of this model.)

More generally, we have considered the case $dY_t/dt = a(Z_t)\, Y_t + b(Z_t)$, where Z is a finite state Markov chain, and have shown that $X = (Y, Z)$ is a Hunt process if Y is pieced together using solutions of appropriate initial value problems (Erickson (1972)).

This idea leads to very general results. Below we replace Z by a regular step process, and we show how to piece together rather general deterministic evolution functions to yield a process Y such that the process $X = (Y, Z)$ is a standard process.

Our results will imply that a "solution Y of an autonomous delayed differential equation driven by a regular step process Z" is such that $X = (Y, Z)$ is a standard process. We also show that paths of random evolution processes associated with families of semigroups (Griego and Hersh (1971), p. 407) and multiplicative operator functions (Pinsky (1973)) yield standard processes.

These facts are given in Section 7 which deals with applications. In that section, in addition, we raise a question concerning processes with deterministic germ fields (Knight (1972)).

In sections (2) through (6) we consider evolution data, the evolution process, the Markov property, strong Markov property, and finally quasi-left-continuity.

The pattern of the construction and proof is much like that used in Blumenthal and Getoor (1968) to construct regular step processes. We follow this reference for notation and several results. To facilitate citation we give chapter and paragraph number of this book in the form BGI 7.12, for example.

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2. Evolution Data

We think of random evolution as follows. Let Z be a piecewise constant process whose states are thought of as giving rules for evolution. For each rule z and initial position y there is a deterministic evolution function $k_1(y, z, t)$. The evolution process Y is obtained by piecing together the functions k_1 appropriately: if $Y_0 = y_0$ and Z passes through states z_0 to $z_1 \ldots$ to z_n at times $0 < \tau_1 < \cdots < \tau_n \leqq t < \tau_{n+1}$ then Y evolves from y_0 to $k_1(y_0, z_0, \tau_0), \ldots$, jumps to y_n and evolves to $k_1(y_n, z_n, t - \tau_n)$ at time t.

From this it is clear that to specify Y as a stochastic process it is necessary to specify k_1, the jump probabilities for Y and Z and the holding probabilities for Z. This we do now, and in section (3) we construct the process $X = (Y, Z)$.

Let E_1 and E_2 denote metrizable spaces with Borels $\mathscr{E}_1$ and $\mathscr{E}_2$. Evolution occurs in E_1 and the rules governing evolution are denoted by elements of E_2.

Set $R_+ = [0, \infty)$ with Borels $\mathscr{R}_+$, $T = [0, \infty]$ with Borels $\mathscr{T}$. Let $\Delta = (\Delta_1, \Delta_2)$ be a point isolated from $E = E_1 \times E_2$ and define $E_\Delta = E \cup \{\Delta\}$. E and E_Δ are metrizable with Borels $\mathscr{E} \supset \mathscr{E}_1 \times \mathscr{E}_2$ and $\mathscr{E}_\Delta$. Finally define $F = E \times R_+ \cup \{\delta\}$ with Borels $\mathscr{F}$, where $\delta = (\Delta, \infty)$, and "$\infty$" is the one point compactification of R_+ with $\infty > r$ for all r in R_+.

A function $k\colon F \to E_\Delta$ is called a (deterministic) *evolution function* if it has the general properties

(2.1) $k(x, 0) = x$, $k(x, t)$ is in E for all x in E, t in R_+, and $k(\Delta, \infty) = \Delta$,

(2.2) $k(x, t+s) = k(k(x, s), t)$ for all x in E, s, t in R_+,

(2.3) k is $\mathscr{F}/\mathscr{E}_\Delta$ measurable and

(2.4) $t \to k(x, t)$ is right continuous for all x in E, t in R_+, and $x \to k(x, t)$ is $\mathscr{E}/\mathscr{E}$ measurable for each t in R_+.

So that the evolution state can be distinguished from the evolution rule we require, in addition, the special property

(2.5) $k(x, t) = (k_1(y, z, t), z)$ for all $x = (y, z)$ in E and all t in R_+.

In applications the function k_1 is typically the solution of an initial value problem in E_1, indexed by a symbol from E_2. In such a case assumptions (2.1) to (2.4) can be deduced from (2.5) and usual theorems concerning such solutions, if enough conditions are placed on the equations defining the initial value problems. Actually, the function k_1 is basic in our considerations and the function k defined in (2.5) merely simplifies notation.

We may sometimes demand that

(2.6) $t \to k(x, t)$ has left limits for all x in E, t in R_+, or

(2.7) $t \to k(x, t)$ is continuous for all x in E, t in R_+,

and we then say simply that k has *left limits* (is *continuous*).

Concerning the independence of the hypotheses on k we note

(2.8) **Proposition.** *Assumption* (2.4) *implies* (2.3) *and even that* $k|E \times R_+$ *is* $\mathscr{E} \times R_+/\mathscr{E}$ *measurable. The converse fails.*

The implication is proved exactly as is the well known fact that "right continuous processes (in a Hausdorff space) are progressively measurable", [Meyer (1966) IV. 47]. One might conjecture the converse, and even (2.7), if one defines the contraction semigroup $\{T_t, t \geqq 0\}$ by $T_t f(x) = f(k(x,t))$, where x is in E, t in E_+ and f is in the Banach space L of bounded measurable functions from E to R, with supremum norm. Apparently neither (2.3) nor (2.4) nor even (2.7) is enough to guarantee strong measurability of $t \to T_t$ as the following example shows: Set $E = [-1, 1]$ and define k using the following table, where $0 \leqq a \leqq b \leqq 1$.

		$x<0$			$x \geqq 0$	
(2.9) (x,t)	$x+t<0$	$x+t=0$	$x+t>0$	$x \neq a$	$x=a, t=0$	$x=a, t>0$.
$k(x,t)$	$x+t$	a	b	x	a	b

Notice that k satisfies (2.1), (2.2) and (2.3) for any a, b. Now (2.7) holds if $a=0=b$; (2.4) holds and (2.7) fails if $a=1=b$; and even (2.4) fails if $a=\frac{1}{2}$, $b=1$.

In this setup L is a separable Banach space, and if $\{T_t\}$ were strongly measurable it would also be strongly continuous, $t>0$. (See, for example, Dynkin (1965), p. 35.) But then the action of $\{T_t\}$ on the identity function would imply that $k(x,t)$ is continuous for $t>0$, uniformly in x. Even when $a=0=b$, so that k is continuous (2.7), we see that $\{T_t\}$ fails to be strongly measurable: just consider its action on the *signum* function.

Let us return to giving the evolution data.

Having specified k we still must specify holding and jump distributions for the rules governing evolution. To do this we specify two functions, λ and Q.

The function λ: $E \to (0, \infty)$ will be used as a parameter for the exponential density and will be assumed to satisfy the conditions

(2.10) λ is $\mathscr{E}/\mathscr{R}_+$ measurable, $\lambda(x) = \lambda_2(z)$ for all $x=(y,z)$ in E.

Extend λ to F by setting $\lambda(\Delta)=0$. Again λ_2 is basic and λ saves much writing.

To specify jumps we assume given a Markov transition function $Q: E \times \mathscr{E} \to [0,1]$ such that

(2.11) $Q(x, \cdot)$ is a probability measure on $\mathscr{E}$ for each x in E,

(2.12) $Q(\cdot, A)$ is $\mathscr{E}/\mathscr{R}_+$ measurable for each A in $\mathscr{E}$, and

(2.13) $Q((y,z), E_1 \times \{z\}) = 0$ for all (y,z) in E.

Extend Q to $F \times \mathscr{F}$ by setting $Q(\Delta, \{\Delta\}) = 1$. Condition (2.13) guarantees that "evolution rules change when jumps occur".

When $\mathscr{E} = \mathscr{E}_1 \times \mathscr{E}_2$ (for example, when E_1 and E_2 are separable), one way to obtain such a Q is to be given Markov transition functions Q_i on E_i, $i=1,2$, with the property that $Q_2(z, \{z\}) = 0$ for all z in E_2. Use these to define $Q((y,z), A \times B) = Q_1(y, A)\, Q_2(z, B)$, and extend this to Q on $E \times \mathscr{E}$. In applications Q_2 is often thought of as basic: for example it might give the jumps of an underlying Markov chain or regular step process. In this case, when only Q_2 is specified, we assume that $Q_1(y, A) = \varepsilon_y(A)$ is unit mass at y.

Obviously our Q is somewhat more general than $Q_1 \times Q_2$ and will have the effect of allowing the evolution state to "undergo a sudden shift when the evolution rule changes". Thus in our setup k_1, λ_2 and Q are basic.

In case Q_2 is basic we could say that the random evolution is driven by the regular step process specified by λ_2 and Q_2.

From now on the data E_1, E_2, k, λ, and Q will be assumed given and assumed to satisfy all the conditions listed in this section, except (2.6), (2.7) which will be invoked explicitly when used. We will refer to this data, or simply to k, λ and Q, as *basic data*.

3. The Process

In this section we use the basic data k, λ and Q to construct the process of interest.

For $N=\{0,1,\dots\}$ set $\Omega=F^N$ and let $\mathscr{G}=\mathscr{F}^N$. For $\omega=\{(y_n, z_n, t_n)\}_0^\infty$ in Ω define the projections

$$\pi_n(\omega)=(y_n, z_n, t_n)=w_n, \quad a_n(\omega)=(y_n, z_n)=x_n, \quad b_n(\omega)=y_n, \quad c_n(\omega)=z_n, \quad \tau_n(\omega)=t_n. \tag{3.1}$$

Define a Markov kernel K: $F\times\mathscr{F}\to[0,1]$ using the basic data: for (x,t) in $E\times R_+$

$$K(x,t; du, ds)=Q\big(k(x,s-t), du\big)\,\lambda(x)\,e(x; s-t)\,ds, \tag{3.2}$$

where $\lambda(x)\,e(x;s)=\lambda(x)\exp[-\lambda(x)s]$ for $s\geqq 0$, zero otherwise, is the exponential density with mean $1/\lambda(x)$. Also define

$$K(\delta,\{\delta\})=1. \tag{3.3}$$

This is a Markov kernel (has properties (2.11) and (2.12) on $F\times\mathscr{F}$) because of Fubini's theorem and the assumptions on k, λ and Q (see Neveu (1965), p. 74).

By the theorem of Ionescu-Tulcea (Neveu (1965), p. 162), for every measure γ on $(F,\mathscr{F})$ there exists a measure P^γ on $(\Omega,\mathscr{G})$ such that the sequence $\{\pi_n\}_0^\infty$ is a Markov chain with initial distribution γ.

The joint distributions are computed using the operators γK^n defined on bounded $\mathscr{F}^{n+1}/\mathscr{R}$ measurable functions f by

$$\int f\,dP^\gamma=\gamma K^n f=\int\gamma(w_0)\int K(w_0, dw_1)\dots\int K(w_{n-1}, dw_n)\,f(w_0,\dots,w_n). \tag{3.4}$$

When we calculate using (3.4) we will write $w_j=(x_j, t_j)=(y_j, z_j, t_j)$ for generic variables.

Write $P^x=P^{yz0}$ if $\gamma=\varepsilon_{x,0}$ is unit mass at the point $(x,0)=(y,z,0)$, and write E^γ for expectation relative to P^γ.

Define $\nu(\omega)=\inf\{n|\pi_n(\omega)=\delta\}$ ($=\infty$ if no such n) as the first time π hits δ. By simple calculations we see that

$$P^x(\tau_0=0)=1=P^x(\nu=\infty), \qquad P^\gamma(\tau_{n+1}\leqq\tau_n)=0=P^\gamma(c_n=c_{n+1}) \tag{3.5}$$

for all $x\in E$ and all γ such that $\gamma(\{\delta\})=0$. For example, if $\alpha>0$ then

$$E^\gamma\big(\exp[-\alpha(\tau_{n+1}-\tau_n)]\big)=E^\gamma\big(\lambda(a_n)/[\alpha+\lambda(a_n)]\big),$$

and the third equality follows by letting $\alpha\to\infty$.

Because of (3.5) we see that nothing is lost if we restrict Ω to sequences $\{w_n\}_0^\infty$, $w_n=(y_n, z_n, t_n)$, where

$$(3.6)\qquad \begin{aligned} &0=t_0<t_1<\cdots<t_{\nu-1}<t_\nu=\infty=t_{\nu+1}\ \ldots \quad \text{and} \\ &z_n \neq z_{n+1} \quad \text{for } 0\leqq n<\nu, \qquad w_{\nu+j}=\delta \quad \text{for } j=0,1,\ldots. \end{aligned}$$

We interpret the conditions after index ν as vacuous if $\nu=\infty$. Actually we could omit all ω's for which $\nu\neq\infty$, but our proof of the strong Markov property depends on having sequences of the type given in (3.6). (See (5.6) and (5.7).) We designate the particular sequence with $\nu=0$ by $\omega_\Delta=\{\delta_n\}_0^\infty$, $\delta_n\equiv\delta$.

Denote the restricted set of sequences satisfying (3.6) by Ω' and let $\mathscr{G}'$ be the trace of $\mathscr{G}$ on Ω'. We still have $\mathscr{G}'=\sigma\{\pi_0, \pi_1, \ldots\}$, and $\{\pi_n\}_0^\infty$ remains a Markov process on $(\Omega', \mathscr{G}', P^\gamma)$. Now drop the primes.

Define $\zeta(\omega)=\lim_n \tau_n(\omega)$ and piece together the evolution process for t in $T=[0,\infty]$: $X_t(\omega_\Delta)=\Delta$ for all t, and for $\nu\geqq 1$,

$$(3.7)\qquad X_t=(Y_t, Z_t)=\begin{cases} k(a_n, t-\tau_n) & \text{if } \tau_n\leqq t<\tau_{n+1}, \\ \Delta & \text{if } \zeta\leqq t\leqq\infty. \end{cases}$$

Notice that $t\to X_t(\omega)$ is right continuous for every ω, t in R_+; and it has left limits for every ω if k has left limits (2.6).

Finally, define translation operators θ_t: $\Omega\to\Omega$ by setting $\theta_t(\omega_\Delta)=\omega_\Delta$ for all t, $\theta_t(\omega)=\omega_\Delta$ if $t\geqq\zeta(\omega)$, and for $\tau_n(\omega)\leqq t<\tau_{n+1}(\omega)$ set $\theta_t(\omega)=\{\omega_j'\}_0^\infty$, where $\omega_0'=(X_t(\omega), 0)$ and $\omega_j'=(a_{n+j}(\omega), \tau_{n+j}(\omega)-t)$, $j=1,2,\ldots$. We have $X_s\circ\theta_t=X_{s+t}$ for all s, t in T.

4. The Markov Property

In this section we show that X is a normal process (in the sense of BGI 3.1 and 5.16).

To be more precise, define $\mathscr{F}_t^0=\sigma\{X_s, s\leqq t\}$, $\mathscr{F}_\infty^0=\sigma\{X_s, s\in T\}$ and $\mathscr{F}_{t+}^0=\bigcap_{s>t}\mathscr{F}_s^0$.

(4.1) **Theorem.** *The collection $X=(\Omega, \mathscr{F}_\infty^0, \mathscr{F}_{t+}^0, X_t, \theta_t, P^x)$ is a normal Markov process with state space $(E, \mathscr{E})$ (augmented by Δ). Every path $t\to X_t(\omega)$ is right continuous; and every path has left limits if k does* (2.6).

The collection X is called the *random evolution determined by the data k, λ, Q.*

To prove the Markov property we must show

$$(4.2)\qquad P^x(X_{t+s}\in B; \Lambda)=E^x\big(P^{X(t)}(X_s\in B); \Lambda\big)$$

for all x in E, s, t in R_+, B in $\mathscr{E}$ and Λ in $\mathscr{F}_{t+}^0$. Since E is a metric space, the σ-algebra of Baire sets coincides with the Borels $\mathscr{E}$, and because of the Lebesgue monotone convergence theorem it suffices to prove

$$(4.3)\qquad E^x\big(f(X_{t+s}); \Lambda\big)=E^x\big(E^{X(t)}f(X_s); \Lambda\big)$$

for all bounded continuous functions f: $E\to R$, (extended by $f(\Delta)=0$) (see, for example, Neveu (1965), II.7.1 and 2). But for such functions f, $u\to f\big(X_u(\omega)\big)$ is bounded and right continuous for every ω; thus both sides of (4.3) are right continuous functions of s. Hence, it suffices to verify the equality of the corresponding Laplace transforms, which, by Fubini's theorem, demands

$$(4.4)\qquad E^x\left(\int_0^\infty e^{-\alpha s} f(X_{t+s})\,ds;\,\Lambda\right)=E^x\left(E^{X(t)}\int_0^\infty e^{-\alpha s} f(X_s)\,ds;\,\Lambda\right).$$

In terms of the *resolvent operators* U^α, $\alpha\geqq 0$, defined by $U^\alpha f(x)=E^x\int_0^\infty e^{-\alpha t} f(X_t)\,dt$, we see that we must prove

$$(4.5)\qquad E^x\left(\int_t^\infty e^{-\alpha u} f(X_u)\,du;\,\Lambda\right)=E^x\left(e^{-\alpha t}\,U^\alpha f(X_t);\,\Lambda\right)$$

for all x, s, t, all Λ in $\mathscr{F}_{t+}^0$ and all bounded continuous $f\,(f(\Delta)=0)$.

In their construction of regular step processes Blumenthal and Getoor (BGI 12) make no metrizability assumption on E nor continuity assumption on f. They are able to do this because $t\to X_t(\omega)$ is piecewise constant and right continuous, thus implying right continuity of $t\to f(X_t(\omega))$. To retain this last property we chose to restrict attention to continuous f and thus were forced to argue as above.

The first step in the proof of (4.1) is to prove

(4.6) **Lemma.** *In the above notation we have* $\mathscr{F}_{t+}^0=\mathscr{F}_t^0$.

Proof. First introduce the right continuous jump process $J_t(\omega_\Delta)\equiv\Delta$, and for $v\geqq 1$, $J_t=a_n$ if $\tau_n\leqq t<\tau_{n+1}=\Delta$ if $\zeta\leqq t\leqq\infty$. We claim that $\mathscr{H}_t=\sigma\{J_s,\,s\leqq t\}$ is such that for all t (a) $\mathscr{H}_t=\mathscr{H}_{t+}$ and (b) $\mathscr{H}_t=\mathscr{F}_t^0$.

To prove (a) note first that for each given t and ω there exists an ω' so that $J_s(\omega')=J_{s\wedge t}(\omega)$ for all $s\leqq t+1$. Now use BGI 6.17 to characterize sets Λ in $\mathscr{H}_{t+\varepsilon}$, $0\leqq\varepsilon<1$: $\Lambda\in\mathscr{H}_{t+\varepsilon}$ iff (i) $\Lambda\in\mathscr{H}_{t+1}$ and (ii) if $\omega_0\in\Lambda$ and $J_s(\omega_0)=J_s(\omega)$ for all $s\leqq t+\varepsilon$, then $\omega\in\Lambda$. Given $\Lambda\in\mathscr{H}_{t+}\subset\mathscr{H}_{t+\varepsilon}$, $0<\varepsilon<1$, let $\omega_0\in\Lambda$ and suppose ω is such that $J_s(\omega_0)=J_s(\omega)$ for $s\leqq t$; then by right continuity and piecewise constancy of J there exists an $\varepsilon=\varepsilon(\omega_0,\omega,t)$, $0<\varepsilon<1$ such that $J_s(\omega_0)=J_s(\omega)$ for $s\leqq t+\varepsilon$. For this ε, $\Lambda\in\mathscr{H}_{t+\varepsilon}$; thus $\omega\in\Lambda$ and $\Lambda\in\mathscr{H}_t$, by two applications of the above characterization. This proves (a). For A in $\mathscr{E}$ the event

$$\Lambda=\{X_s\in A\}=\bigcup_0^\infty\{X_s\in A\}\cap\{\tau_n\leqq s<\tau_{n+1}\}=\bigcup_0^\infty\Lambda_n,$$

where

$$\Lambda_n=\{k(J(\tau_n),s-\tau_n)\in A\}\cap\{\tau_n\leqq s<\tau_{n+1}\}.$$

Similarly

$$\{J_s\in A\}\cap\{\tau_n\leqq s<\tau_{n+1}\}=\{X(\tau_n)\in A\}\cap\{\tau_n\leqq s<\tau_{n+1}\}.$$

Since (2.4) implies k is $\mathscr{E}\times\mathscr{R}_+/\mathscr{E}$ measurable we have (b) if we prove that each τ_n is a stopping time relative to both $\{\mathscr{F}_t^0\}$ and $\{\mathscr{H}_t\}$. This is done in (4.7) below.

(4.7) **Lemma.** *For each n the projection* τ_n *is a stopping time relative to the* σ*-fields* $\mathscr{A}_t=\sigma\{Z_s,\,s\leqq t\}\subset\mathscr{F}_t^0\cap\mathscr{H}_t$.

Proof. First notice that our arguments for (4.6.a) show that $\mathscr{A}_t=\mathscr{A}_{t+}$. The inclusion is clear: for B in $\mathscr{E}_2$, $\{Z_s\in B\}=\{X_s\in E_1\times B\}=\{J_s\in E_1\times B\}$, s in R_+. Next $\tau_1=\inf\{t>0\,|\,Z_t\neq Z_0\}$, and right continuity gives $\{\tau_1>t\}=\bigcap_{r\in Q(t)}\{Z_r=Z_0\}$, where

$Q(t)=\{r=qt|0\leqq q\leqq 1,\ q \text{ rational}\}$. Thus τ_1 is an $\mathscr{A}_t$ stopping time. Since $\tau_{n+1}=\tau_n+\tau_1\circ\theta_{\tau_n}$, we have $\{\tau_{n+1}<t\}=\bigcup_{r\in Q}\{\tau_n<t-r,\ \tau_1\circ\theta_{\tau_n}<r\}$, Q=rationals. If we show that $\{\tau_1\circ\theta_{\tau_n}<r\}=\theta_{\tau_n}^{-1}\{\tau_1<r\}\in\mathscr{A}_{\tau_n+r}$ we are done by induction, since $\mathscr{A}_{t+}=\mathscr{A}_t$. In general $\theta_{\tau_n}^{-1}\{Z_s\in B\}=\{Z_{s+\tau_n}\in B\}$, and the result follows. (These arguments are those of BGI 7.5 and 8.7, but the hypotheses in those theorems are made for ease of formulation and do not cover our case.)

We turn now to the proof of the Markov property and (4.5). Since $f(\Delta)=0$, it suffices to show, for Λ in $\mathscr{F}_t^0$,

$$E^x\left(\int_t^\infty e^{-\alpha u}f(X_u)\,du;\ \Lambda_n\right)=E^x\{e^{-\alpha t}\,U^\alpha f(X_t);\ \Lambda_n\} \tag{4.8}$$

where $\Lambda_n=\Lambda\cap\{\tau_n\leqq t<\tau_{n+1}\}=A_n\cap B_n$, $B_n=\{t<\tau_{n+1}\}$, $A_n\in\sigma\{\pi_0,\dots,\pi_n\}$ and $A_n\subset\{\tau_n\leqq t\}$. The representation of Λ_n is proved by observing that the generators $\{X_s\in A\}$ of $\mathscr{F}_t^0$ are such that

$$\{X_s\in A\}\cap\{\tau_j\leqq s<\tau_{j+1}\}=\{k(a_j,\,s-\tau_j)\in A\}\cap\{\tau_j\leqq s<\tau_{j+1}\}=A_{sj}\cap\{s<\tau_{j+1}\},$$

with $A_{sj}\in\sigma\{\pi_0,\dots,\pi_n\}$ for $j\leqq n$, $s\leqq t$.

Fix x, t, α and $\Lambda_n=A_n\cap B_n$, $A_n\in\sigma\{\pi_0,\dots,\pi_n\}$, $A_n\subset\{\tau_n\leqq t\}$, $B_n=\{t<\tau_{n+1}\}$. Let $A_n^*\in\mathscr{F}^{n+1}$, $B_n^*\in\mathscr{F}^{n+2}$ be appropriate sets corresponding to A_n, B_n in Ω. In the notation of (3.4) we have

$$\begin{aligned}E^x\{e^{-\alpha t}\,U^\alpha f(X_t);\ \Lambda_n\}&=\varepsilon_{x0}\,K^{n+1}\big[e^{-\alpha t}\,U^\alpha f\big(k(x_n,\,t-t_n)\big)\,I(A_n^*)\,I(B_n^*)\big]\\&=\varepsilon_{x0}\,K^n\big[I(A_n^*)\,e(x_n,\,t-t_n)\,e^{-\alpha t}\,U^\alpha f\big(k(x_n,\,t-t_n)\big)\big],\end{aligned} \tag{4.9}$$

since the only dependence on $(x_{n+1},\,t_{n+1})$ occurs in the indicator function $I(B_n^*)=I(t<t_{n+1})$. Next split the left side of (4.8) into the sum $\sum_0^\infty L_j$, where

$$L_j=\varepsilon_{x0}\,K^{n+j+1}\,I(A_n^*)\,I(B_n^*)\left[\int_{t_{n+j}}^{t_{n+j+1}}e^{-\alpha u}f\big(k(x_{n+j},\,u-t_{n+j})\big)\,du\right],$$

for $j\geqq 1$, and L_0 is defined similarly but with $\int_t^{t_{n+1}}$ replacing $\int_{t_n}^{t_{n+1}}$. Introduce the new time variables $v=u-t$, $s_i=t_{n+i}-t$, $i\geqq 1$. Because of $I(B_n^*)$ and the exponential density in K, integration in the new variables is over $0<v<s_1$ in L_0 and $0=s_0<s_1<\dots<s_j<v<s_{j+1}$ in L_j. Notice that in these variables B_n^* is transformed into $\{0<s_1\}$ and is superfluous next to an exponential density. Finally, introduce the new state variables $x_i^*=x_{n+i}$, $i\geqq 1$, $x'=x_0^*=k(x_n,\,t-t_n)$, $k(x_{n+i},\,u-t_{n+i})=k(x_i^*,\,v-s_i)$. Since $\lambda(x_i^*)=\lambda(x_{n+i})$ by (2.10), and $k(x_{n+i},\,t_{n+i+1}-t_{n+i})=k(x_i^*,\,s_{i+1}-s_i)$, it is easy to check that

$$L_j=\varepsilon_{x0}\,K^n\left[I(A_n^*)\,e(x_n,\,t-t_n)\,e^{-\alpha t}\,\varepsilon_{x'0}\,K^{j+1}\int_{s_j}^{s_{j+1}}e^{-\alpha v}f\big(k(x_j^*,\,v)\big)\,dv\right]. \tag{4.10}$$

Then $\sum_0^j L_j$ equals the right side of (4.9) and (4.8) holds. This proves the Markov property.

To complete the proof of Theorem (4.1) we observe that for all x $P^x(X_0=x)=P^x(k(x,\tau_0)=x)\geqq P^x(k(x,0)=x,\tau_0=0)=1$, by (2.1) and (3.4). Thus X is a normal Markov process.

5. The Strong Markov Property

In this section we prove

(5.1) **Theorem.** *The random evolution determined by* k, λ *and* Q, *that is*

$$X=\{\Omega,\mathscr{F}_\infty^0,\mathscr{F}_{t+}^0,X_t,\theta_t,P^x\},$$

is a strong Markov process.

This means that for every bounded $\mathscr{E}_\Delta/\mathscr{R}$ measurable function f, every $\{\mathscr{F}_{t+}^0\}$ stopping time T, $t\geqq 0$ and x in E

$$E^x\{f\circ X_{t+T}|\mathscr{F}_T^0\}=E^{X(T)}f\circ X_t. \tag{5.2}$$

(See BGI, p. 12, for the interpretation of equality of such statements as (5.2).) As in the proof of Theorem 4.1, it suffices to show that

$$E^x\left\{\int_T^\infty e^{\alpha u}f(X_u)\,du;\Lambda\right\}=E^x\{e^{-\alpha T}\,U^\alpha f(X_T);\Lambda\} \tag{5.3}$$

for all $\alpha\geqq 0$, all bounded continuous f with $f(\Delta)=0$, and all Λ in $\mathscr{F}_T^0$ (i.e. $\Lambda\in\mathscr{F}_\infty^0$ such that $\Lambda\cap\{T\leqq s\}\in\mathscr{F}_{s+}^0=\mathscr{F}_s^0$, all s).

By observing that (5.3) is Eq. XIV.11.3 of Meyer (1967), we see that (5.3) also implies XIV.7.1 of Meyer (1967):

$$E^x\{f\circ X_S|\mathscr{F}_T^0\}=E^{X(T)}f\circ X(S-T) \tag{5.4}$$

for all random variables $S\geqq T$ which are $\mathscr{F}_T^0/\mathscr{R}_+$ measurable. (Notice that (5.2) follows from (5.4) with $S=t+T$.)

From this observation we have

(5.5) **Corollary.** *For all* $\alpha>0$ *and all continuous mappings* $f\colon E\to R$, *with compact support, the map* $t\to U^\alpha f(X_t)$ *is right continuous on* $[0,\zeta)$ *almost surely, if* E *is homeomorphic to a universally measurable subset of a compact metrizable space.*

If we could deduce (5.5) directly we could then invoke BGI 8.11 to infer the strong Markov property of X. This is the technique employed by Blumenthal and Getoor in their construction of regular step processes (BGI 12). But we are unable to do this. Instead, we prove (5.3) directly and then (5.5) is a corollary of Meyer (1967) XIV, T 11. (See (6.3) for stronger assumptions on E that are usually fulfilled in applications.)

To show that (5.3) is true it evidently suffices to prove its validity with $\Lambda_n=\Lambda\cap\{\tau_n\leqq T<\tau_{n+1}\}$ in place of Λ, for we may assume $f(\Delta)=0$. Write (5.3.n) for this new equation. We would like to say that (5.3.n) follows by exactly the calculations (4.9) and (4.10) used to prove (4.8). Indeed it does once we prove the next two lemmas. The effect of these lemmas is to show that on the event $\{\tau_n\leqq T<\tau_{n+1}\}$, the value T can be treated as a constant when performing the inner (or first) few iterated integrations of γK^{n+j}. (Recall that $\mathscr{F}_t^0=\mathscr{F}_{t+}^0$ (4.6).)

(5.6) **Lemma.** *Let T be an $\{\mathscr{F}^0_{t+}\}$ stopping time, $\Lambda \in \mathscr{F}^0_T$, $\Lambda_n = \Lambda \cap \{\tau_n \leqq T < \tau_{n+1}\}$.*

(a) *$\Lambda_n \in \sigma\{\pi_0, \ldots, \pi_n, \tau_{n+1}\}$ for $n=0, 1, \ldots$.*

(b) *Let $\omega \in \Lambda_n$ and suppose ω' is such that $\pi_j(\omega') = \pi_j(\omega)$, $j=0, \ldots, n$. Then $\omega' \in \Lambda_n$ iff $\tau_{n+1}(\omega') > T(\omega)$, and in this case $T(\omega') = T(\omega)$.*

Proof. For (a) notice that $\Lambda_n = \bigcup_r (\Lambda \cap \{\tau_n \leqq T \leqq r < \tau_{n+1}\})$, where the union is over nonnegative, rational r. But we have seen (just following (4.8)) that every set $C \in \mathscr{F}_r^0$ has the property that $C \cap \{\tau_n \leqq r < \tau_{n+1}\} = C_n \in \sigma\{\pi_0, \ldots, \pi_n, \tau_{n+1}\}$. Thus (a) follows if we choose $C = \Lambda \cap \{\tau_n \leqq T\} \cap \{T \leqq r\} \in \mathscr{F}_r^0$. Now let ω, ω' be as in (b). If $v(\omega) \leqq n$ then $\omega = \omega'$. Assume $v(\omega) \wedge v(\omega') \geqq n+1$ so that $T(\omega) < \infty$. If $\omega' \in \Lambda_n$ then $\tau_{n+1}(\omega') > T(\omega') \geqq \tau_n(\omega') = \tau_n(\omega)$, which implies that $X_s(\omega') = X_s(\omega)$ for all $s \leqq T(\omega') \wedge T(\omega) < \infty$. Set $t' = T(\omega')$, $t = T(\omega)$. If $t' \leqq t$, then $\omega' \in \{T \leqq t'\} \in \mathscr{F}^0_{t'}$ and $X_s(\omega') = X_s(\omega)$ for $s \leqq t'$, giving $\omega \in \{T \leqq t'\}$ and $t \leqq t' \leqq t$. (See BGI 6.17.a "only if", which follows without assuming the existence of an ω' such that $X_s(\omega') = X_{s \wedge t}(\omega) \ldots$.) If $t \leqq t'$ then we show that $t' = t$ similarly. This gives

$$T(\omega) = T(\omega') \quad \text{and} \quad \tau_{n+1}(\omega') > T(\omega') = T(\omega).$$

Conversely, if $\tau_{n+1}(\omega') > T(\omega)$, then $X_s(\omega') = X_s(\omega)$ for $s \leqq T(\omega) < \tau_{n+1}(\omega) \wedge \tau_{n+1}(\omega')$. Arguing as above we show $T(\omega) = T(\omega')$ and $\omega' \in \Lambda_n$. This completes the proof.

Now define the map $d_n: \Omega \to \Omega$ so that $\pi_j \circ d_n = \pi_j$ for $0 \leqq j \leqq n$ and $\pi_j \circ d_n(\omega) = \delta = (\Delta, \infty)$ for $j \geqq n+1$ and all ω. Thus d_n leaves the first $n+1$ terms of the sequence ω unchanged and replaces the remaining terms with δ's. Clearly d_n is $\mathscr{G}_n/\mathscr{G}_\infty$ measurable, $\mathscr{G}_n = \sigma\{\pi_0, \ldots, \pi_n\}$ (Meyer (1966) article I.8). Lemma (5.7) now gives us a crucial factorization.

(5.7) **Lemma.** *Let T, Λ, Λ_n be as in (5.6). Then $\Lambda_n = A_n \cap B_n$, where $A_n = d_n^{-1}\Lambda_n \in \mathscr{G}_n$, $A_n \subset \{\tau_n \leqq T \circ d_n\}$ and $B_n = \{T \circ d_n < \tau_{n+1}\}$. Also, $T \circ d_n$ is $\mathscr{G}_n/\mathscr{R}_+$ measurable.*

Proof. If $\omega \in \Lambda_n$ then so is $d_n(\omega)$, for $\tau_{n+1} \circ d_n(\omega) = \infty > T(\omega)$, and $\tau_{n+1}(\omega) > T(\omega) = T(d_n(\omega))$. This shows $\Lambda_n \subset A_n \cap B_n$. Conversely, if $\omega \in A_n \cap B_n$ then $d_n(\omega) \in \Lambda_n$, $T \circ d_n(\omega) < \tau_{n+1}(\omega)$ and $\omega \in \Lambda_n$ by (5.6.b). Next, $d_n(\omega) \in \Lambda_n$ implies $\tau_n(\omega) = \tau_n(d_n(\omega)) \leqq T(d_n(\omega))$; thus $d_n^{-1}\Lambda_n \subset \{\tau_n \leqq T \circ d_n\}$. Finally, T is $\mathscr{G}_\infty/\mathscr{R}_+$ measurable, since $\mathscr{G}_\infty = \mathscr{F}^0_\infty$, making the $T \circ d_n$ measurability clear. This completes the proof.

Now fix $x, \alpha \geqq 0$, n and $T, \Lambda, \Lambda_n, A_n$ and B_n as in (5.6) and (5.7). Using the results of (5.6) and (5.7), the proof of (5.3.n) (i.e. (5.3) with Λ_n replacing Λ) is essentially identical with the proof of (4.8) given in (4.9) and (4.10): In (4.9) replace t by T in the first term and by T_n in the next two terms, where $T_n(x_0, t_0, \ldots, x_n, t_n) = T \circ d_n(\omega)$ if $\pi_j(\omega) = (x_j, t_j)$, $j=0, \ldots, n$. The set B_n^* equals $\{T_n < t_{n+1}\}$. Concerning the L_j in (4.10), replace t by T_n and x_0^* by $k(x_n, T_n - t_n)$. Since T_n does not depend on w_{n+j}, $j=1, 2, \ldots$, argue exactly as in (4.8) and (4.9). This completes the proof of (5.1).

To prove Corollary (5.5) we note first that

$$x \to P_t f(x) = E^x f(X_t) = \sum_0^\infty \varepsilon_{x0} K^{n+1}(f(k(a_n, t-t_n)); \{t_n \leqq t < t_{n+1}\})$$

is $\mathscr{E}_\Delta/\mathscr{R}$ measurable for all bounded $\mathscr{E}_\Delta/\mathscr{R}$ measurable f (Neveu (1965), p. 74). Since $U^\alpha f$ is α excessive for nonnegative $\mathscr{E}/\mathscr{R}$ measurable f (BGII 2.2e), that $t \to U^\alpha f(X_t)$ is right continuous almost surely follows from (5.1) and Meyer's (1967) Theorem XIV, T 11. (Certain completions may be necessary for these theorems to apply. We could introduce these now, but we prefer to wait until the paragraphs before (6.2) below.) This completes the proof of (5.5).

6. Quasi-Left-Continuity and Standard Processes

Let $\{T_n\}$ be an increasing sequence of $\{\mathscr{F}_{t+}^0\}$ stopping times with limit T. We wish to show that $X(T_n)\to X(T)$ on $\{T<\zeta\}$ almost surely. This property is called *quasi-left-continuity on* $[0,\zeta)$, and we abbreviate this by $q-l-c-\zeta$.

(6.1) **Theorem.** *Let X be the random evolution determined by k, Q and λ. Then X is $q-l-c-\zeta$ iff k is continuous* (2.7).

Proof. Suppose X is $q-l-c-\zeta$. Let x be in E, $0<t<\infty$ and set $x'=k(x,t)$. Let $t_n\uparrow t$ be any sequence along which $x_n=k(x,t_n)\to x_\infty$, some point x_∞ in E. For the sequence of stopping times $T_n=t_n\wedge\tau_1\uparrow T=t\wedge\tau_1$ we have $X(T_n)\to X(T)$ a.s. P^x on $\{T<\zeta\}$. Define $\Lambda=\{\tau_1>t, a_0=x\}$ so that $P^x\Lambda=\exp[-\lambda(x)t]>0$. Then for almost all $\omega\in\Lambda$ the above convergence gives $x_n=k(x,t_n)=X(T_n,\omega)\to X(T,\omega)=k(x,t)=x'$, and continuity of k follows. Conversely, suppose k is continuous and let $T_n\uparrow T$. Then $q-l-c-\zeta$ can fail for at most those ω such that for some j and all large n, $T=\tau_{j+1}$ and $\tau_j\leqq T_n<\tau_{j+1}$. On the set $\Lambda_{nj}=\{\tau_j\leqq T_n<\tau_{j+1}\}=\Lambda_j\cap\{T_n<\tau_{j+1}\}$ we calculate as in the proof of the strong Markov property following (5.7) (T_{nj} is defined using $T_n\circ d_j$):

$$\begin{aligned}&E^x(e^{-\alpha(\tau_{j+1}-T_n)};\Lambda_{nj})\\&\quad=\varepsilon_x K^{j+1}\exp[-\alpha(t_{j+1}-T_{nj})]\,I(A_j^*)\,I(T_{nj}<t_{j+1})\\&\quad=\varepsilon_x K^j I(A_j^*)\,\lambda_2(z_j)/[\alpha+\lambda_2(z_j)]\leqq E^x(\lambda(X(\tau_j))/[\alpha+\lambda(X(\tau_j))]),\end{aligned}$$

since $\lambda(x)=\lambda_2(z)$ for all $x=(y,z)$ in E and $A_j^*\subset\{t_j\leqq T_{nj}\}$. (See (3.4) and (5.7) for notation.) Now argue exactly as in the penultimate paragraph of BGI 12: for $\Lambda=\{\tau_j\leqq T_n<\tau_{j+1}$ for all large $n\}$ we have (for $\alpha\to\infty$)

$$\begin{aligned}P^\mu(T=\tau_{j+1};\Lambda)&=\lim_\alpha E^\mu(e^{-\alpha(\tau_{j+1}-T)};\Lambda)\\&\leqq\lim_\alpha\liminf_n E^\mu(e^{-\alpha(\tau_{j+1}-T_n)};\Lambda_{nj})\\&\leqq\lim_\alpha E^\mu(\lambda(X(\tau_j))/[\alpha+\lambda(X(\tau_j))])=0.\end{aligned}$$

From this follows $q-l-c-\zeta$, and the proof is complete.

We recall that k can be right continuous and fail to be continuous (2.7), so that for our processes the strong Markov property may hold and $q-l-c-\zeta$ fail.

From (5.1) and (6.1) we see that X is very nearly a standard process. The only requirements lacking are (a) completeness of the σ-algebras $\{\mathscr{F}_{t+}^0\}$ and (b) suitable topological properties on the state space E.

To take care of (a) define $\mathscr{F}$ to be the completion of $\mathscr{F}_\infty^0$ with respect to the family $\{P^\mu;\mu$ a finite measure on $\mathscr{E}_\Delta\}$ and define $\mathscr{F}_t$ to be the completion of $\mathscr{F}_{t+}^0=\mathscr{F}_t^0$ in $\mathscr{F}$ (see BGI 5.3 and paragraphs before BGI 5.7). Then $\mathscr{F}_t=\mathscr{F}_{t+}$ (BGI 8.12) and $X^\sim=(\Omega,\mathscr{F},\mathscr{F}_{t+},X_t,\theta_t,P^x)$ is strong Markov (BGI 8.3). Using BGI 7.3 our proof of $q-l-c-\zeta$ holds as well for $X^\sim$.

The following is now immediate.

(6.2) **Theorem.** *Let k, Q and λ be given as in Section 2. The process*

$$X^{\sim}=(\Omega, \mathscr{F}, \mathscr{F}_{t+}, X_t, \theta_t, P^x)$$

is a standard process if k is continuous and if E_Δ is homeomorphic to a Borel subset of a compact metric space.

Here we have used Meyer's (1967, XIV.20) definition of a standard process. Our process is standard relative to the definition of BGI 9.2 if we assume E is locally compact with countable base (L.C.C.B.).

We remark that the topological condition on E_Δ can be phrased by saying simply that E_Δ is a Lusin space. To see the relationship between this assumption and L.C.C.B., we state the following well known facts. Not all of these are explicitly given in Bourbaki (1966), but they are all simple corollaries of Sections 2.8, 2.9, 6.2, 6.4 and 6.7 of Chapter IX of this reference.

(6.3) **Proposition.** *Let X be a topological space and let $Y=[0,1]^N$ be the countable product of the closed unit interval. Then X has property* (i) *iff X is homeomorphic to a subset of Y of the type specified in* (i′), $i=1,2,\dots,5$:

(1) *compact metric*	(1′) *closed,*
(2) *L.C.C.B.*	(2′) $F\setminus\{y\}$ *(F closed in Y, $y\in Y$),*
(3) *Polish*	(3′) G_δ *(countable intersection of opens),*
(4) *Lusin*	(4′) *Borel,*
(5) *metrizable of countable type*	(5′) *arbitrary.*

7. Applications

In this section we apply Theorem (6.2) to processes often met in practice.

(7.1) **Functional Differential Equations.** Let $0\leqq r<\infty$, $C=\{x\colon [-r,0]\to R^n$, x continuous$\}$, and for any function $y\colon [-r,\infty)\to R^n$ and any $t\geqq 0$, let y_t denote the shifted restriction defined by $y_t(\theta)=y(t+\theta)$, $-r\leqq\theta\leqq 0$. We wish to consider initial value problems for autonomous functional differential equations

$$\dot{y}(t)=f(y_t), \qquad y_0=\varphi \tag{7.2}$$

where $\varphi\in C$ and $f\colon C\to R^n$ is continuous. It is shown in Hale (1971), Section 9, that if f maps closed bounded sets of C into bounded sets in R^n and if each solution $y(\varphi)$ to (6.2) is unique and defined on $[-r,\infty)$ then $y(\varphi)(t)$ is continous in (t,φ) and satisfies $y_0(\varphi)=\varphi$ and $y_{t+s}(\varphi)=y_t(y_s(\varphi))$, $s,t\geqq 0$.

Now let $f_z\colon C\to R^n$ be a collection of functions so that all of the above results hold for each fixed z in some set E_2. For each $\varphi\in C$ and $z\in E_2$ define $k_1(\varphi,z,t)=y(\varphi,z)_t\in C$, where $y(\varphi,z)$ is a solution of $\dot{y}(t)=f_z(y_t)$, $y_0=\varphi$. Finally, define $E=C\times E_2$ and $k(x,t)=(k_1(\varphi,z,t),z)$ for $x=(\varphi,z)\in E$. Then k satisfies (2.1) to (2.7) if we extend to F by the definition $k(\Delta,\infty)=\Delta$ and give E_2 the discrete topology (or any topology with fewer open sets).

Now suppose that the index z changes according to a regular step process Z with data $\lambda_2\colon E_2\to(0,\infty)$ and $Q_2\colon E_2\times\varepsilon_2\to[0,1]$, and define

$$\lambda(\varphi,z)=\lambda_2(z), \qquad Q(\varphi,z;A\times B)=\varepsilon_\varphi(A)\,Q_2(z;B),$$

where ε_φ is unit mass at φ.

(7.3) **Theorem.** *If E_2 is a Lusin space, then the data (f_z, λ_2, Q_2) leads to a standard process X with data (k, λ, Q). Roughly, the process $X=(Y, Z)$ satisfies the functional differential equation*

$$\dot{Y}(t)=f(Y_t, Z(t)),$$

where $f_z(y_t)=f(y_t, z)$.

Remark on notation: The process $X=(Y, Z)$ takes values in $C\times E_2$. We write Y_t for the function in $C=C[-r, 0]$ and $Y(t)=Y_t(0)$ for the value of Y_t at 0 (suppressing all ω's). No such distinction is necessary for the Z component but we write $Z(t)$ to emphasize the meaning.

Proof. Since k is continuous, the only part which needs to be proved is that $E=C\times E_2$ is a Lusin space. This is obvious from (6.3) and the fact that C is Polish. This completes the proof.

Notice that C is not locally compact with countable base so that even when E_2 is finite the process in (7.2) is not standard according to the definition of BGI 9.2.

Certainly initial value problems for ordinary as well as Banach valued differential equations can lead to standard processes.

(7.4) **Multiplicative Operator Functionals.** In this section we show the connection between our random evolutions and those associated with families of semigroups of operators.

First, suppose we are given data λ_2 and Q_2 which generate a regular step process Z on a Lusin space E_2. Suppose also given a separable Banach space L and strongly continuous semigroups of bounded linear operators on L, say $\{T_z(t), t\geqq 0\}$, z in E_2. Define $E_1=L$ and $k_1(y, z, t)=T_z(t)\,y$. Then k_1, λ_2 and Q_2 give fundamental data for our process. Notice that $k(y, z, t)=(k_1(y, z, t), z)$ satisfies a stronger continuity condition then imposed in (2.7).

Conversely, suppose given fundamental data consisting of Lusin spaces E_1 and E_2 and functions k_1, λ_2 and Q_2 which generate k, λ, Q satisfying (2.1), (2.2), (2.7) and (2.10) to (2.13). (In particular $Q(y, z; A\times B)=\varepsilon_y(A)\,Q_2(z, B)$.) Define the Banach space $L(L^\sim)$ of bounded measurable functions from E_1 to R ($E_1\times E_2$ to R); and define the semigroups $T_z(t)$ on L by $[T_z(t)f](y)=f(k_1(y, z, t))$. Use the notation of (3.1) and define the "multiplicative operator functional" or "random evolution" (see Pinsky (1973) or Griego and Hersh (1971), Eq. (2.1)),

$$M(t, \omega)=T_{z_0}(t_1)\,T_{z_1}(t_2-t_1)\dots T_{z_n}(t-t_n) \tag{7.5}$$

if $\tau_n(\omega)=t_n\leqq t<\tau_{n+1}(\omega)$. Then $M(t, \omega)\,f^\sim(y, Z_t(\omega))=f^\sim(Y_t(\omega), Z_t(\omega))$ for all ω in $\Lambda_y=\{\omega|b_0(\omega)=y,\ Y(\tau_j(\omega)-)=Y(\tau_j(\omega)), j=1, 2, \dots\}$. But when k_1 is continuous (2.7) and Q is constructed from Q_2 as above, we have $P^{yz}\Lambda_y=1$ for all y, z.

To see a further connection between our processes (taking paths as basic) and the semigroup approach to random evolution, write $f_z(\cdot)=f^\sim(\cdot, z)$ in L for $f^\sim$ in $L^\sim$. Then the "expectation semigroup" $\{T^\sim(t), t\geqq 0\}$ is defined by

$$[T^\sim(t)f^\sim](y, z)=E^{yz}M(t)\,f_{Z(t)},$$

and the above arguments show it has the further representation $E^{yz}f^\sim(Y(t), Z(t))$.

One final remark: Pinsky (1973) begins with axioms defining a multiplicative operator functional M. He shows that such objects can be factored as in (7.5) under certain continuity assumptions, but that in general this factorization will involve, in addition, contraction operators between the T terms. Random evolutions of this generality also generate processes of our type with k_1 as above but with Q having a certain form which allows jumps. Conversely, our general processes also yield such operators M. We leave the details of this for the interested reader.

(7.6) **Right Deterministic Germ Fields.** Recently Knight (1972) considered Hunt processes with right deterministic germ fields, i.e., processes X for which $\bigcap_{\varepsilon>0} \sigma\{X(T+s), 0 \leqq s < \varepsilon\}$ is "nearly" equal to $\sigma\{X(T)\}$ for all $\{\mathscr{F}_{t+}^0\}$ stopping times T. He shows this property to be equivalent to the property that there exists a function k which satisfies our (2.1) to (2.3) and is such that

$$P^x\{X(s)=k(x,s), 0 \leqq s < \varepsilon \text{ for some } \varepsilon > 0\} = 1 \tag{7.7}$$

for all x in E. Thus, modulo the Hunt assumption (which entails that E be locally compact with countable base), our processes have right deterministic germ fields. Knight's proof of the equivalence rests on quasi-left-continuity of X which implies continuity of k (2.7). Since we have constructed a large class of processes which satisfy (7.7) but for which quasi-left-continuity fails, it is natural to ask if general strong random evolution processes have right deterministic germ fields. We hope to consider this elsewhere.

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