Z. Wahrscheinlichkeitstheorie verw. Gebiete 29, 273–293 (1974)

Local Times and Supermartingales

Donald Geman and Joseph Horowitz

Introduction

Let $X=(X_t)$ be a random process with state space $(E, \mathscr{E})$ on which a σ-finite measure π is given. This paper deals with the problem of finding conditions under which, with probability 1, the process spends (Lebesgue measure) zero time in any π-negligible set.

The density $\alpha_t^x(\omega)=\mu_t(dx,\omega)/\pi(dx)$, $\mu_t(\Gamma,\omega)=\int_0^t I_\Gamma(X_s(\omega))\,ds$, if it exists, is called the local time at x, and has been studied mostly for Markov processes [3, 11, 12], but also for certain Gaussian processes [1, 2, 9, 21].

In the Markov case, the local time is characterized as the unique additive functional whose fine support [3] is $\{x\}$; its connection with occupation times has been rather secondary. Here we apply the so-called "general theory of processes" (developed largely by Meyer and the "Strasbourgeoisie" along lines suggested by Markov theory) to non-Markovian processes. This gives a systematic method for dealing with occupation times.

In § 1 we give a necessary and sufficient condition for a local time which consists of three parts: the first two are absolute continuity requirements on various measures, while the last is that certain potentials Z^x, $x\in E$ (in the sense of Meyer [17]) be of class (D). We also describe some general circumstances in which the absolute continuity conditions are verified, and investigate the continuity in t of the local time. The situations for nondeterministic Gaussian processes and Markov processes are explained briefly – the potentials Z^x are new in the Gaussian case, but in the Markov case reduce to those well-known in the Blumenthal and Getoor theory. Since the Z^x's cannot all be of class (D) when no local time exists, we have an easy method of constructing potentials not of class (D). In particular, for Brownian motion in $\mathbb{R}^3$, we show that the example of such due to Johnson and Helms [14] arises in essentially this manner.

The results of § 1 are applied in § 2 to stationary, especially Gaussian, processes. If the mean-square prediction error for a stationary, continuous, Gaussian process satisfies a certain integrability condition, there exists a continuous (in t) local time. Also, a condition used by Berman and Orey is shown to be necessary and sufficient for the energy of α_t^x (in the sense of [18]) to be finite for a.e. x; otherwise it is infinite a.e. (in particular, if the second spectral moment is finite, the energy is infinite). The section concludes with several examples, including two complex Gaussian processes, one with and one without a local time, and a short computation of the first passage time distribution for the stationary Ornstein-Uhlenbeck process. At present we are investigating the possibility of extending this method to non-Markov situations.

We have placed several measurability arguments in an Appendix, along with some results in the "general theory of processes" which are necessary for § 2 and may be of some independent interest.

Our notation is a conglomeration of that of [3] and [17]. We write $\mathbb{R}$ ($\mathbb{R}_+$) for the real line (positive half-line $[0, \infty)$), with Borel sets $\mathscr{B}$ ($\mathscr{B}_+$). If $(M, \mathscr{M})$ is a measurable space, we write (ambiguously) $f \in (\mathscr{M})$ to mean that f is an $\mathscr{M}$-measurable function on M (whose range will always be clear); in particular $f \in (\mathscr{M})_+$ means the range is $\mathbb{R}_+$. For a family $\{\mathscr{M}_t\}_{t \in I}$ of σ-fields, $\bigvee_{t \in I} \mathscr{M}_t$ denotes $\sigma(\bigcup_{t \in I} \mathscr{M}_t)$, i.e. the σ-field generated by $\bigcup_{t \in I} \mathscr{M}_t$. An *increasing process* $\alpha = (\alpha_t)_{t \in \mathbb{R}_+}$ is a real-valued random process with $\alpha_0 = 0$ and almost every trajectory right continuous and nondecreasing; we often consider the trajectory of such a process as a measure on $\mathbb{R}_+$. Let $(E, \mathscr{E})$ and $(F, \mathscr{F})$ be measurable spaces. A (*Markov*) *kernel* on $E \times \mathscr{F}$ is a mapping $(x, A) \to p(x, A)$ such that $p(\cdot, A) \in (\mathscr{E})_+$ for each $A \in \mathscr{F}$ and $p(x, \cdot)$ is a (probability) measure for each $x \in E$.

Finally, we make the convention that $\int_a^b$ means the integral over $(a, b]$.

1. Existence of Local Times

a) Let $X = (X_t)_{t \in \mathbb{R}_+}$ be a measurable random process on a probability space $(\Omega, \mathscr{F}, P)$ with state space $(E, \mathscr{E})$. Fix a σ-finite measure π on $\mathscr{E}$. A *local time* of X (relative to π) is a family of increasing processes $\alpha^x = (\alpha_t^x)$, $x \in E$, such that $(t, x, \omega) \to \alpha_t^x(\omega)$ is in $(\mathscr{B}_+ \times \mathscr{E} \times \mathscr{F})$ and

$$(1) \qquad \int_\Gamma \alpha_t^x(\omega)\, \pi(dx) = \int_0^t I_\Gamma(X_s(\omega))\, ds, \qquad t \in \mathbb{R}_+,\ \Gamma \in \mathscr{E} \text{ a.s.}$$

(I_Γ is the indicator of the set Γ.) The process α^x is called the local time *at* x, and it can be shown that, if E is a locally compact Hausdorff space with countable base, and $M_x(\omega) = \{t\colon X_t(\omega) = x\}$ is closed a.s., then, for π-a.e. x, the support of $\alpha_\cdot^x(\omega)$ (construed as a measure on $\mathbb{R}_+$) is a.s. contained in $M_x(\omega)$. The existence of a local time means that a.s. a particle travelling along the trajectory $X_\cdot(\omega)$ spends (Lebesgue measure) zero time in any π-negligible set in E.

Suppose we start with a family of processes (α_t^x) satisfying (1). It is not difficult to "regularize" the family to obtain right continuous, nondecreasing paths, hence it is no real restriction to require that α^x be an increasing process. As we shall see in Section d), we may also require that $\alpha_\cdot^x$ be predictable relative to a certain "natural" increasing family of σ-fields $\{\mathscr{F}_t\}$ defined below.

b) We will always assume (and this is crucial) that the σ-fields $\mathscr{E}, \mathscr{F}_t^0 = \sigma(X_s\colon s \leq t)$, and $\mathscr{F}^0 = \bigvee_{t \in \mathbb{R}_+} \mathscr{F}_t^0$ are all *separable*, i.e. generated by a countable subfamily. Further, we take $\mathscr{F}$ to be the P-completion of $\mathscr{F}^0$. These restrictions are fairly harmless: for example, if X is continuous in probability, then a $(\mathscr{B}_+ \times \mathscr{F})$ measurable version of X may be chosen with each $\mathscr{F}_t^0$ separable ([20, 91]). Define

$$(2) \qquad K_t(\Gamma, A) = \int_0^t P(A, X_s \in \Gamma)\, ds, \qquad \Gamma \in \mathscr{E},\ A \in \mathscr{F}^0,\ t \in \mathbb{R}_+.$$

(3) **Lemma.** *Suppose $K_t(\cdot, A) \ll \pi$ for each $t \in \mathbb{R}_+$, $A \in \mathscr{F}^0$, and let $k(t, x, A)\,\pi(dx) = K_t(dx, A)$. Assume the density $k(t, x, A)$ may be chosen so that*

(i) *it is a measure on $\mathscr{F}^0$ for each fixed (t, x).*

Then $k(t, x, A)$ may be chosen so that it is also

(ii) *(t, x)-measurable for each A*

(iii) *a right-continuous non-decreasing function of t for every x, A.*

Proof. Appendix.

The choice of $k(t, x, A)$ which satisfies (i)–(iii) will be termed the "perfect version". When a local time $\{\alpha^x\}$ exists, we have $K_t(dx, A) = E(\alpha_t^x; A)\,\pi(dx)$ from (1), so $k(t, x, A) = E(\alpha_t^x; A)$ is the perfect version. Finally, notice that $K_t(\cdot, A) \ll \pi$ for each A iff $K_t(\cdot, \Omega) \ll \pi$.

(4) **Lemma.** *Suppose $K_t(\cdot, \Omega) \ll \pi$ for all t, and let $k(t, x, A)$ satisfy* (i)–(iii). *Then a local time exists iff, for* a.e. *x, $k(t, x, \cdot) \ll P$ (on $\mathscr{F}^0$) for all $t \in \mathbb{R}_+$.*

In fact, by (iii), we could say instead: for each t, $k(t, x, \cdot) \ll P$ on $\mathscr{F}^0$ for a.e. x.

The proof is easy, but depends on the following lemma, which results immediately from [17, 154].

(5) **Lemma.** *Let $(M_i, \mathscr{M}_i)$, $i = 1, 2$ be measurable spaces, $\mathscr{M}_2$ separable, let $p(m_1, A)$ be a kernel on $M_1 \times \mathscr{M}_2$ with $p(m_1, \cdot)$ a finite measure on $\mathscr{M}_2$ for each $m_1 \in M_1$, and let λ be a σ-*finite *measure on $\mathscr{M}_2$. If, for each m_1, $p(m_1, \cdot) \ll \lambda$, the density may be chosen jointly measurable.*

Proof of (4). The "only if" part having already been shown, let $k(t, x, A)$ be as stated, but redefine $k(t, x, A) \equiv 0$ on the exceptional x-set. This still gives a perfect version. Now applying (5) to the kernel $(x, A) \to k(t, x, A)$ with t fixed, we get an (x, ω)-measurable density $\tilde{\alpha}_t(x, \omega) = k(t, x, d\omega)/P(d\omega)$. As in the proof of (3), we can choose a version $\alpha_t^x(\omega)$ of $\tilde{\alpha}_t(x, \omega)$ which is (x, ω)-measurable, nondecreasing, right-continuous in t for each (x, ω), hence (t, x, ω)-measurable. Thus,

$$\int_A \int_\Gamma \alpha_t^x\, \pi(dx)\, dP = K_t(\Gamma, A) = \int_A \int_0^t I_\Gamma(X_s)\, ds\, dP,$$

so

$$\int_\Gamma \alpha_t^x\, d\pi = \int_0^t I_\Gamma(X_s)\, ds \quad \text{a.s.} \tag{6}$$

for each $t \in \mathbb{R}_+$, $\Gamma \in \mathscr{E}$. Now let Γ run through a countable generating set closed under finite intersections and let t run through the rationals to get a single set of probability zero off which (1) holds.

c) We sketch here two very general situations in which $k(t, x, A)$ exists and may be chosen perfect. Recall that a class $\mathscr{C}$ of sets in $\mathscr{F}^0$ is called *compact* if, for any sequence $C_n \in \mathscr{C}$ with $\bigcap_1^\infty C_n = \varnothing$ there is an integer $N \geqq 1$ for which $\bigcap_1^N C_n = \varnothing$. Assume that $\mathscr{F}^0$ is separable, and contains a compact class $\mathscr{C}$ with the property

$$P(A) = \sup\{P(B) \colon B \subset A, B \in \mathscr{C}\}, \qquad A \in \mathscr{F}^0.$$

(If Ω is a Polish space and $\mathscr{F}^0$ is the Baire σ-field, this assumption holds.)

Now a "natural" choice of π for local time problems is the "1-potential" measure $\pi(\Gamma)=\int_0^\infty e^{-s} P(X_s\in\Gamma)\,ds$. Then, obviously, $K_t(\cdot,\Omega)\ll\pi$. Define a linear mapping T_t: $L^1(E,\mathscr{E},\pi)\to L^1(\Omega,\mathscr{F}^0,P)$ as follows: for $f\in L^1(E,\mathscr{E},\pi)$,

$$T_t f(\omega)=t^{-1}\int_0^t f(X_s(\omega))\,ds.$$

This is a Markov operator, and [20, 192–193] there is a (Markov) kernel $m_t(x,A)$ such that

$$\int_A T_t f\,dP=\int_E m_t(x,A)\,f(x)\,\pi(dx).$$

Taking $k(t,x,A)=t\,m_t(x,A)$ we obtain a density for $K_t(\cdot,A)$ satisfying (i).

Next, with the same assumption on $\mathscr{F}^0$, assume that π is any σ-finite measure on $\mathscr{E}$. Consider the conditional probabilities $P(A|X_s)$ for each $A\in\mathscr{F}^0$, $s\in\mathbb{R}_+$, and say they admit a *perfect version* if a Markov kernel $p(s,x,A)$ on $(\mathbb{R}_+\times E)\times\mathscr{F}^0$ may be chosen so that, for a.e. s (relative to Lebesgue measure)

$$\int_\Gamma p(s,x,A)\,P(X_s\in dx)=P(A,X_s\in\Gamma),\qquad \Gamma\in\mathscr{E},\ A\in\mathscr{F}^0. \tag{7}$$

We then use the more suggestive notation $P(A|X_s=x)$ instead of $p(s,x,A)$.

(8) **Theorem.** *There exists a perfect version* $P(A|X_s=x)$.

Let $\pi_s(\Gamma)=P(X_s\in\Gamma)$, $s\in\mathbb{R}_+$, $\Gamma\in\mathscr{E}$. This is a kernel on $\mathbb{R}_+\times\mathscr{E}$, so we may define a unique probability measure μ on $\mathscr{B}_+\times\mathscr{E}$ by

$$\int f(s,x)\,\mu(ds,dx)=\int_0^\infty e^{-s}\int_E f(s,x)\,\pi_s(dx)\,ds.$$

Now define a Markov operator

$$T\colon L^1(\mathbb{R}_+\times E,\mathscr{B}_+\times\mathscr{E},\mu)\to L^1(\Omega,\mathscr{F}^0,P)\quad\text{by}\quad Tf(\omega)=\int_0^\infty e^{-s} f(s,X_s(\omega))\,ds.$$

There is, again, a Markov kernel $p(s,x,A)$ on $(\mathbb{R}_+\times E)\times\mathscr{F}^0$ such that

$$\int_A Tf\,dP=\int_{\mathbb{R}_+\times E} p(s,x,A)\,f(s,x)\,\mu(ds,dx).$$

Putting $f(s,x)=I_B(s)\,I_\Gamma(x)$, where $B\in\mathscr{B}_+$, $\Gamma\in\mathscr{E}$, we find

$$\int_B e^{-s} P(A,X_s\in\Gamma)\,ds=\int_B e^{-s}\int_\Gamma p(s,x,A)\,\pi_s(dx)\,ds,$$

so

$$P(A,X_s\in\Gamma)=\int_\Gamma p(s,x,A)\,\pi_s(dx)\quad \text{a.e. } s.$$

Now, using the separability of $\mathscr{F}^0$ and $\mathscr{E}$, one easily finds a single exceptional s-set off of which (7) holds, thus establishing (8).

Finally, suppose each π_s has a density relative to π: $\pi_s(dx)=\phi_s(x)\,\pi(dx)$. By (5), $\phi_s(x)$ may be chosen (s,x)-measurable, and, if $P(A|X_s=x)$ has been chosen perfect, it is immediate that $K_t(\cdot,A)\ll\pi$ for all $A\in\mathscr{F}^0$, and $k(t,x,A)=\int_0^t P(A|X_s=x)\,\phi_s(x)\,ds$ is a perfect version of the density.

Since we will return to the situation described in this section several times, we summarize the assumptions and results:

(i) the separable σ-field $\mathscr{F}^0$ contains a compact subfamily

(ii) for each $s \in \mathbb{R}_+$, the one-dimensional distribution $\pi_s(dx)$ has a density $\phi_s(x)$ relative to π which is jointly (s, x)-measurable.

Under condition (i), the conditional probabilities $P(A|X_s = x)$ always admit a perfect version, and $k(t, x, A)$ may be chosen perfect if either π is the 1-potential measure or (ii) holds.

d) In this section we state and prove the main theorem. Before doing so, we need the following concept. Let $\{Q_t: t \in \mathbb{R}_+\}$ be a family of finite measures on $\mathscr{F}^0$ such that

(a) $Q_t(A)$ is a decreasing, right-continuous function of t for each A

(b) $Q_t(\Omega) \to 0$ $(t \to \infty)$.

We call the family *progressively absolutely continuous along* $\{\mathscr{F}_t^0\}$ (written $PAC\{\mathscr{F}_t^0\}$ and understood relative to P) if $Q_t \ll P$ on $\mathscr{F}_t^0$ for each t. Let $Z_t = dQ_t/dP$ (relative to $\mathscr{F}_t^0$), i.e. $Q_t(A) = \int_A Z_t \, dP$, $Z_t \in (\mathscr{F}_t^0)$. It is immediate that $Z = (Z_t)$ is a non-negative supermartingale relative to $\{\mathscr{F}_t^0\}$, and $E(Z_t) \to 0$ $(t \to \infty)$ by (b). We now enlarge the σ-fields to get a right-continuous version of Z. Recall that $\mathscr{F}$ is the P-completion of $\mathscr{F}^0$. Let $\mathscr{N} \subset \mathscr{F}$ be the family of P-negligible sets, and define $\mathscr{F}_t = \mathscr{F}_{t+}^0 \vee \mathscr{N}$. It may be shown that $\{\mathscr{F}_t\}$ is right-continuous, hence satisfies the "conditions habituelles" of [5]; and later, when we use concepts such as predictability, we mean in terms of this family. Then Z, being a supermartingale relative to $\{\mathscr{F}_t\}$, admits a right-continuous version (see [17, 95]), again denoted Z_t, and adapted to $\{\mathscr{F}_t\}$, though not to $\{\mathscr{F}_t^0\}$ in general.

The new Z_t is still a density for Q_t on $\mathscr{F}_t^0$, but now we get

$$Q_t(A) = \int_A Z_t \, dP \qquad \text{for } A \in \mathscr{F}_{t+}^0,$$

and we may extend Q_t to $\mathscr{F}_t$ in an obvious way; then $\{Q_t\}$ is $PAC\{\mathscr{F}_t\}$. Using Meyer's [17] terminology, $Z = (Z_t)$ is a *potential*.

Returning to the question of local times, assume that $K_t(\cdot, \Omega) \ll \pi$ and the perfect version of $k(t, x, A)$ has been chosen. Define the measure

$$Q_t^x(A) = \int_t^\infty e^{-s} k(ds, x, A), \qquad A \in \mathscr{F}^0, \ t \in \mathbb{R}_+, \ x \in E.$$

From the definition of k we see Q_t^x is a finite measure for a.e. x – indeed, we may and do assume for *every* x – and (a), (b) above are satisfied. When $\{Q_t^x\}$ is PAC we write $Z^x = (Z_t^x)$ for the corresponding potential and note the relation

$$E(Z_t^x; A) = Q_t^x(A) = \int_t^\infty e^{-s} k(ds, x, A) \qquad \text{for } A \in \mathscr{F}_{t+}^0. \tag{9}$$

Assuming $\{Q_t^x\}$ is PAC for a.e. x, let $Z^x \equiv 0$ for the exceptional x's. From (9) follows

(10) the map $x \to E(Z_t^x; A)$ is measurable for $A \in \mathscr{F}_{t+}^0$, hence for $A \in \mathscr{F}_t$.

For later use we now have:

(11) **Lemma.** *Suppose* $\{Z^x\colon x\in E\}$ *is a family of class* (D) *potentials for which* (10) *holds. Then there exist predictable* (=*natural*), *integrable, increasing processes* $A^x=(A_t^x)$ *such that* A^x *generates* Z^x *and* $(t,x,\omega)\to A_t^x(\omega)$ *is measurable.*

Both the proof and an explanation of the terminology are in the Appendix. Consider now the following three conditions:

(I) $K_t(dx,A)=k(t,x,A)\,\pi(dx)$ with $k(t,x,A)$ perfect,

(II) $\{Q_t^x\}$ is PAC for a.e. x,

(III) Z^x is of class (D) for a.e. x.

(12) **Theorem.** *A local time exists iff* (I)–(III) *hold.*

Proof. Given a local time $\{\alpha^x\}$, we already have (I) with $k(t,x,A)=E(\alpha_t^x;A)$ (see Section b)). Then for a.e. x,

$$Q_t^x(A)=E\left(\int_t^\infty e^{-s}\,d\alpha_s^x;\,A\right)$$

is clearly PAC, and $Z_t^x=E\left(\int_t^\infty e^{-s}\,d\alpha_s^x|\mathscr{F}_t\right)$ is a class (D) potential [17, 106].

For the converse we may assume Z^x is class (D) for all x; if (A_t^x) is the predictable, integrable, increasing process which generates Z^x, we may assume (by (11)) that $A_t^x(\omega)$ is (t,x,ω)-measurable. Define $\alpha_t^x(\omega)=\int_0^t e^s\,dA_s^x(\omega)$, giving again a predictable, increasing process. By (4) it suffices to show that $k(t,x,M)=E(\alpha_t^x;M)$ for every $t\in\mathbb{R}_+$, $M\in\mathscr{F}^0$, π-a.e. Equivalently, we may show $E(A_t^x;M)=\int_0^t e^{-s}\,k(ds,x,M)$, $t\in\mathbb{R}_+$, $M\in\mathscr{F}^0$, π-a.e. Finally, this is equivalent – using the separability of $\mathscr{F}^0$ – to

$$E(A_t^\Gamma;M)=\int_0^t e^{-s}P(M,X_s\in\Gamma)\,ds,\quad t\in\mathbb{R}_+,\ M\in\mathscr{F}^0,\ \Gamma\in\mathscr{E},\tag{13}$$

where $A_t^\Gamma=\int_\Gamma A_t^x\,\pi(dx)$.

From the decomposition $Z_t^x=E(A_\infty^x-A_t^x|\mathscr{F}_t)$ we have

$$E(A_\infty^\Gamma-A_t^\Gamma;M)=\int_t^\infty e^{-s}P(M,X_s\in\Gamma)\,ds,\qquad \Gamma\in\mathscr{E},\ M\in\mathscr{F}_t.\tag{14}$$

Put $B_t^\Gamma=\int_0^t e^{-s}I_\Gamma(X_s)\,ds$. Then (14) reads $E(A_\infty^\Gamma-A_t^\Gamma;M)=E(B_\infty^\Gamma-B_t^\Gamma;M)$, $M\in\mathscr{F}_t$. Thus $A^\Gamma-B^\Gamma$ is a martingale, and

$$E\int_0^t Y_{s-}\,dA_s^\Gamma=E\int_0^t Y_{s-}\,dB_s^\Gamma\tag{15}$$

for any bounded, positive, right-continuous martingale Y [17, VII. T 17]. Since A^Γ, B^Γ are both *natural* increasing processes, we may replace "$s-$" by s in (15). Now, for any $M\in\mathscr{F}$, take Y_t to be the right continuous modification of the martingale $P(M|\mathscr{F}_t)$ (see [17, VI. T 4]), and (13) follows.

Remarks. 1. Suppose we have a family of increasing processes α_t^x which satisfy (1). Then conditions (I)–(III) are satisfied, so there exist predictable increasing processes satisfying (1). Indeed, $(\alpha_\cdot^x)$ must have already been predictable for π-a.e. x.

2. We shall show later by example that condition (III) may fail even with (I) and (II) in force.

e) Let us return to the situation described at the end of Section c) so that (I) holds. Suppose that regular versions have been chosen for the conditional probabilities $P(X_s \in \Gamma | \mathscr{F}_t)$, $\Gamma \in \mathscr{E}$, $t < s$.

(16) **Theorem.** *If* $P(X_s \in dx | \mathscr{F}_t) \ll \pi(dx)$ a.s., *for* $t < s$, *then* (II) *holds.*

Proof. Let $\psi_t(s, x, \omega)$ be the density: almost surely

(17) $$P(X_s \in \Gamma | \mathscr{F}_t) = \int_\Gamma \psi_t(s, x, \omega)\, \pi(dx), \qquad \Gamma \in \mathscr{E}.$$

As shown in the Appendix, ψ_t can be chosen (s, x, ω) measurable $(s > t)$. Thus, for $A \in \mathscr{F}_t$, $\Gamma \in \mathscr{E}$,

$$\int_\Gamma Q_t^x(A)\, \pi(dx) = \int_t^\infty e^{-s} P(A, X_s \in \Gamma)\, ds$$

$$= \int_\Gamma E\left(\int_t^\infty e^{-s} \psi_t(s, x, \omega)\, ds;\, A\right) \pi(dx),$$

and, consequently,

(18) $$Q_t^x(A) = E\left(\int_t^\infty e^{-s} \psi_t(s, x, \omega)\, ds;\, A\right) \quad \pi\text{-a.e.}$$

Letting A run through a countable generating subfamily of $\mathscr{F}_t^0$ yields a single π-negligible set off of which (18) holds for all A (t fixed). In view of the right continuity of $Q_\cdot^x(A)$, (II) is established.

From (18) it is evident that the potential Z^x is given by

(19) $$Z_t^x(\omega) = \int_t^\infty e^{-s} \psi_t(s, x, \omega)\, ds \qquad \text{for a.e. } t, \text{ a.s.}$$

(We say that $\int_t^\infty e^{-s} \psi_t(s, x, \omega)\, ds$ equals Z_t^x *up to modification.*) The two processes in (19) may not be indistinguishable since the right hand side may not be right-continuous. In many cases, however, it *is* possible to select ψ_t so as to render the right side of (19) right-continuous, and then "a.e. t" can be replaced by "for all t".

f) In this section we investigate briefly the behavior of Z^x at stopping times, and the implications concerning smothness of α^x as a function of t. We retain the assumptions at the end of Section c), noting that

(20) $$Q_t^x(A) = \int_t^\infty e^{-s} P(A | X_s = x)\, \phi_s(x)\, ds.$$

(21) **Theorem.** *Suppose* (I) *and* (II) *hold with* Q_t^x *given by* (20). *For every stopping time* T *of the family* $\{\mathscr{F}_t\}$, *we have*

(22) $$E(Z_T^x) = \int_0^\infty e^{-s} \phi_s(x)\, P(T < s | X_s = x)\, ds.$$

Note. Let $A \in \mathscr{F}_T$. Applying (22) to the stopping time $T_A = T I_A + \infty I_{A^c}$ we obtain

$$E(Z_T^x; A) = \int_0^\infty e^{-s} \phi_s(x) P(T < s, A | X_s = x) \, ds, \qquad A \in \mathscr{F}_T. \tag{23}$$

Proof. We prove (22) first for stopping times whose range is contained in a countable set D in $\mathbb{R}_+$. Since $\{T = t\} \in \mathscr{F}_t$ for $t \in D$, we have

$$\begin{aligned} E(Z_T^x) &= \sum_{t \in D} E(Z_t^x; T = t) \\ &= \sum_{t \in D} \int_0^\infty e^{-s} \phi_s(x) I_{(t,\infty)}(s) P(T = t | X_s = x) \, ds \\ &= \int_0^\infty e^{-s} \phi_s(x) \sum_{t \in D} P(T < s, T = t | X_s = x) \, ds \end{aligned}$$

which is the same as (22). Now for an arbitrary stopping time T, let T_n be the stopping times with values in the dyadic rationals such that $T_n \downarrow T$ for all ω. The right-continuity of Z^x, Fatou's lemma, and the optional sampling theorem [17, VI. T 13] give $EZ_T^x = \lim_n EZ_{T_n}^x$. Since $I_{[0,s)}(T_n) \to I_{[0,s)}(T)$ for all ω, s, we get also

$$\lim_n \int_0^\infty e^{-s} \phi_s(x) P(T_n < s | X_s = x) \, ds = \int_0^\infty e^{-s} \phi_s(x) P(T < s | X_s = x) \, ds,$$

which concludes the proof.

(24) **Corollary.** *With Q_t^x as in (20), the potential Z^x will be regular (hence of class (D)) iff*

$$\int_0^\infty e^{-s} \phi_s(x) P(T = s | X_s = x) \, ds = 0 \tag{25}$$

for every bounded stopping time.

Recall that a potential Z is *regular* [17] iff $\lim_n EZ_{T_n} = EZ_T$ for any bounded stopping times T, T_n with $T_n \uparrow T$. Föllmer [8] has shown that regular $\Rightarrow$ class (D). Finally, we remark that a class (D) potential Z is regular iff it is generated by a continuous increasing process. Hence, condition (25) is necessary and sufficient for a continuous local time at x.

g) Here are two examples.

Example 1. Let X_t be a real-valued process with absolutely continuous trajectories such that $m(t \geqq 0: \dot{X}_t = 0) = 0$ a.s., where m = Lebesgue measure and $\dot{X}_t$ is the (a.e.) derivative of X_t (for instance, a quadratic mean differentiable Gaussian process with certain conditions on the q.m. derivative).

Let $v_t^x(\omega)$ denote the number of times $X_s(\omega) = x$ for $0 < s \leqq t$. It is shown in [9] that a local time (relative to Lebesgue measure) exists and is given for a.e. x by

$$\alpha_t^x(\omega) = \int_0^t |\dot{X}_s(\omega)|^{-1} \, dv_s^x(\omega).$$

The potential is then $Z_t^x = E\left(\int_t^\infty e^{-s}|\dot{X}_s|^{-1}\, dv_s^x|\mathscr{F}_t\right)$ up to modification; it cannot be regular since α_t^x is a step function.

Example 2. Suppose X_t is real-valued Gaussian, continuous in probability, mean 0, and variance $v_t^2 = EX_t^2 > 0$ for all $t > 0$. Let $\tilde{X}_{st} = E(X_s|\mathscr{F}_t)$ (=the orthogonal projection on the Hilbert space spanned by X_r, $r \leqq t$); here $t < s$. The conditional variance $V_{st}^2 = E((X_s - \tilde{X}_{st})^2|\mathscr{F}_t) = v_s^2 - E\tilde{X}_{st}^2$ since orthogonality implies independence in the present situation. We assume that $V_{st}^2 > 0$ for $s > t$, so the process is "non-deterministic" in the obvious sense. Then we have a density (see Section e))

$$P(X_s \in dx|\mathscr{F}_t) = \psi_t(s, x)\, dx,$$
$$\psi_t(s, x) = (2\pi)^{-1/2}\, V_{st}^{-1} \exp[-(x - \tilde{X}_{st})^2/2V_{st}^2].$$

Then, by (16), $\{Q_t^x\}$ is PAC and the potential Z^x is given by

$$Z_t^x = (2\pi)^{-1/2} \int_t^\infty e^{-s}\, V_{st}^{-1} \exp[-(x - \tilde{X}_{st})^2/2V_{st}^2]\, ds \tag{26}$$

up to modification.

Suppose that $V_{st} \geqq g(s-t) > 0$ $(s > t)$ for some function g such that

$$\int_0^\infty e^{-s}(g(s))^{-1}\, ds < \infty.$$

The potential Z^x is then bounded, hence of class (D), and a local time (relative to Lebesgue measure) exists. For a specific case, consider a standard Brownian motion $\mathscr{b}(t)$. Then $V_{st} = (s-t)^{-1/2}$ $(s > t)$ and

$$\begin{aligned} Z_t^x &= (2\pi)^{-1/2} \int_t^\infty e^{-s}(s-t)^{-1/2}\, e^{-(x-\mathscr{b}(t))^2/2(s-t)}\, ds \\ &= (2\pi)^{-1/2}\, e^{-t} \int_0^\infty e^{-s}\, s^{-1/2}\, e^{-(x-\mathscr{b}(t))^2/2s}\, ds \\ &= e^{-t}\, e^{-\sqrt{2}\,|x-\mathscr{b}(t)|}/\sqrt{2}. \end{aligned}$$

This is obviously regular, so there is a continuous (in t) local time. It is well-known, of course, that the local time may be chosen jointly continuous [13].

h) Next we give a general remark which will be used in § 2. The potential Z^x given above might well be called the "1-potential" because of the factor e^{-s}. Replacing this throughout by $e^{-\lambda s}$ $(\lambda > 0)$ we get the λ-*potential* $Z_t^\lambda(x)$ corresponding to the measure

$$Q_t^\lambda(x, A) = \int_t^\infty e^{-\lambda s}\, k(ds, x, A), \qquad A \in \mathscr{F}^0$$

when $\{Q_t^\lambda(x, \cdot)\}$ is PAC. All previous results remain valid for $Z^\lambda(x)$; in particular the local time α^x is the same for all λ (it must satisfy (1)). Under the conditions of c), to total mass of $Q_t^\lambda(x, \cdot)$ (which will be useful later) is given by

$$Q_t^\lambda(x, \Omega) = \int_t^\infty e^{-\lambda s}\, \phi_s(x)\, ds, \tag{27}$$

where $\phi_s(x)$ is the density of X_s. (As an example, the λ-potential for standard Brownian motion $\mathscr{b}(t)$ is $Z_t^\lambda(x)=e^{-\lambda t}\, e^{-\sqrt{2\lambda}|x-\mathscr{b}(t)|}/\sqrt{2\lambda}$.)

i) We conclude this part with a brief discussion of the Markov case. Our results are closely related to those of Getoor and Kesten [11].

Let X be a standard Markov process with state space $(E,\mathscr{E})$ and let π be a reference measure (following the terminology of [3]). Our intention is to verify conditions (I) and (II) and identify the potential.

Suppose $u^1(x,y)$ is a density of the 1-potential relative to π:

$$U^1(x,\Gamma)=E^x\int_0^\infty e^{-s}\, I_\Gamma(X_s)\, ds=\int_\Gamma u^1(x,y)\,\pi(dy),\qquad \Gamma\in\mathscr{E};$$

such a density exists since the sets of potential zero are exactly the π-negligible ones. We will assume

(28) $$x\to u^1(x,y)\quad \text{is 1-excessive for each } y\in E.$$

This holds under the duality hypotheses of [3, Chapter VI] or Hunt's hypothesis (F). Let μ be any starting distribution, and define, for $y\in E$, $t\in\mathbb{R}_+$,

(29) $$\tilde{Q}_t^\mu(y,A)=e^{-t}\,E^\mu[u^1(X_t,y);A],\qquad A\in\mathscr{F}_t.$$

The strong Markov property gives

(30) $$\int_\Gamma \tilde{Q}_t^\mu(y,A)\,\pi(dy)=\int_t^\infty e^{-s}P^\mu(X_s\in\Gamma,A)\,ds,\qquad A\in\mathscr{F}_t,\ \Gamma\in\mathscr{E},\ t\in\mathbb{R}_+.$$

Introduce $\tilde{K}_t^\mu(\Gamma,A)=\int_0^t P^\mu(X_s\in\Gamma,A)\,ds$. This is dominated (as a measure in Γ) by μU^1. The density $\tilde{k}^\mu(t,x,A)$ may be chosen perfect using an argument analogous to that of (8) $\left(\text{take the operator } f\to\int_0^t f(X_s)\,ds\right)$. The measure μU^1 is in turn dominated by π, with density $v_\mu(y)=\int_E \mu(dx)\,u^1(x,y)$. Put

$$k^\mu(t,y,A)=\tilde{k}^\mu(t,y,A)\,v_\mu(y),\qquad Q_t^\mu(y,A)=\int_t^\infty e^{-s}\,k^\mu(ds,y,A).$$

Then (30) holds with Q_t^μ in place of $\tilde{Q}_t^\mu$, from which follows

(31) $$Q_t^\mu(y,\cdot)=\tilde{Q}_t^\mu(y,\cdot)\quad \text{on } \mathscr{F}_t^0 \text{ for } \pi\text{-a.e. } y$$

for each t since $\mathscr{F}_t^0$ is separable. But $Q_t^\mu(y,A)$ and $\tilde{Q}_t^\mu(y,A)$ are right continuous in t (the former because of (28)), so that (31) holds off a single y-set independent of t. Thus we have conditions (I) and (II) with the potential $Z_t^y=e^{-t}\,u^1(X_t,y)$. Now a local time exists iff Z^y is of class (D) for a.e. y. The dependence on the starting distribution μ must be removed as in [3, 165].

We remark that Griego [12] has proven the existence of a local time under the assumptions of hypothesis (F) and all points regular, while Getoor and Kesten [11] prove the same result assuming the existence of a reference measure and all points regular ((28) is a consequence of these assumptions).

For Brownian motion $\mathbb{R}^3$, with $\pi(dx)=dx$, we have $u^1(x,y)=(2\pi|x-y|)^{-1}\times \exp(-\sqrt{2}|x-y|)$; hence

$$Z_t^y=e^{-t}(2\pi|X_t-y|)^{-1}\exp(-\sqrt{2}|X_t-y|).$$

Since the range of X_t has measure zero [13, Chapter 7], no local time exists; thus Z^y cannot be class (D) for a.e. y, not even for all y in a set of positive measure. The potential $|X_t-y|^{-1}$ dominates Z_t^y, hence also is not class (D) for a.e. y – indeed $|X_t-y|^{-1}$ is the Johnson-Helms [14] example of non-class (D) potential. (They show it is not class (D) for *every* y.)

2. Stationary Processes

a) When X is a stationary process, the results of § 1 assume a particularly simple and appealing form. We will outline the general situation briefly, then apply the results to stationary Gaussian processes.

Suppose, then, that $X=(X_t)_{t\in\mathbb{R}}$ is strictly stationary, let $\mathscr{F}_t^0$, $\mathscr{F}^0$, etc. be defined as in § 1(b) (with obvious accommodations for the time set $\mathbb{R}$) and assume in addition that there is a flow $(\theta_t)_{t\in\mathbb{R}}$ compatible with X. By *flow* we mean a one parameter group $(\theta_t)_{t\in\mathbb{R}}$ of bimeasurable, measure-preserving bijections $\theta_t\colon \Omega\to\Omega$ such that (i) $\theta_0=$identity, (ii) the map $(t,\omega)\to\theta_t(\omega)$ is measurable relative to $\mathscr{B}\times\mathscr{F}^0/\mathscr{F}^0$; *compatibility* means $\theta_t^{-1}(\mathscr{F}_s^0)=\mathscr{F}_{s+t}^0$ and $X_t=X_0\circ\theta_t$. In general one must be cautious about (ii) which is violated in the usual function space representation with θ_t being the shift [7]. If, however, the paths are (for example) continuous and the state space $(E,\mathscr{E})$ is a Hausdorff space with its Borel σ-field, then the representation in the space C_E of continuous functions on $\mathbb{R}$ to E, with shift θ_t, will have the desired properties.

b) With the assumptions of § 1(c) in effect, a perfect version of $P(A|X_s=x)$ may be obtained by taking $P(A|X_s=x)=P^x(\theta_s A)$, where $P^x(A)=P(A|X_0=x)$ is a regular version of the conditional probability $P(A|X_0)$. We then have, for $t\in\mathbb{R}_+$,

$$K_t(\Gamma,A)=\int_0^t P(X_0\in\Gamma,\theta_s A)\,ds,\qquad A\in\mathscr{F}^0,\ \Gamma\in\mathscr{E}$$

so that $k(t,x,A)=\int_0^t P^x(\theta_s A)\,ds$ gives a perfect version of the density – the measure π being taken as the one-dimensional distribution $\pi(\Gamma)=P(X_t\in\Gamma)$. Thus we can write $Q_t^x(A)=\int_t^\infty e^{-s}P^x(\theta_s A)\,ds$, and, if $\{Q_t^x\}$ is PAC, the potential Z^x may be chosen to satisfy $Z_t^x=e^{-t}Z_0^x\circ\theta_t$ for all $t\geqq 0$ a.s., using the results of [16]. The local time α^x (if it exists) is then an additive functional (see paragraph e)) whose so-called Palm measure [10] is P^x for a.e. x. Indeed, we proved in [9] that, for X stationary, a local time exists iff a.e. P^x is a Palm measure. Using the theory of Palm measures one may show (under supplementary hypotheses) that P and P^x have the same invariant null sets for a.e. x, and $\lim_{t\to\infty} t^{-1}\int_0^t I_A\circ\theta_s\,d\alpha_s^x=P^x(A)$, thereby exhibiting $P^x(A)$ as a suitable "limit of relative frequencies" sampled along the support of α^x. We shall not pursue these matters here (see [10]).

c) Consider a stationary, real-valued, Gaussian process $X=(X_t)_{t\in\mathbb{R}}$, and for convenience assume $EX_t\equiv 0$ and let the covariance $r(t)=E(X_t X_0)$ satisfy $r(0)=1$ $r\not\equiv 1$. The one-dimensional distribution is, in this case, $\Phi(\Gamma)=P(X_t\in\Gamma)=$ standard normal. Local times for such processes have been studied by Berman [1, 2], Orey [21], and the authors [9]. The measure π may be taken either as Φ or as Lebesgue measure. If $r''(0)$ exists (finite), the paths X_t are a.s. absolutely continuous and the local time is given as in § 1, Example 1. Otherwise the paths are non-differentiable a.e. with probability 1, and the situation becomes more complicated. Berman [1] and Orey [21] have shown that

$$\int_{0+} (1-r(s))^{-1/2}\, ds<\infty \tag{1}$$

is sufficient for the existence of a local time $\left(\text{i.e. } \int_0^\delta \text{ is finite for some } \delta>0\right)$. (In fact they show that for any Gaussian process with covariance $r(s,t)$, the condition

$$\iint_{0+} (r(s,s)+r(t,t)-2r(s,t))^{-1/2}\, ds\, dt<\infty \tag{2}$$

is sufficient.) Under further restrictions on $r(t)$ Berman [1, 2] has obtained numerous sufficient conditions for joint continuity in (t, x) of the local times – always by means of Kolmogorov-type moment conditions. Condition (1) is incompatible with

$$-r''(0)<\infty \tag{3}$$

but it is not known (to us) whether (1) is necessary for the existence of local time when (3) fails. (The process

$$X_t=\sum_{n=1}^{\infty} n^{-3/2}[A_n \cos n t+B_n \sin n t],$$

with A_n, B_n i.i.d. standard normal, falls in the gap between (1) and (3).) Since (1) does not depend on the nondeterminism of the process, the method of Example 2, § 1, does not apply. Nonetheless, we can use that method on a certain class of processes to get a local time which is continuous in t.

d) Let X be as just described, but assume that it may be realized in the space $C_{\mathbb{R}}$ with shift θ_t (see a)). Reverting to the notation of § 1, Example 2, we have $\tilde{X}_{st}=E(X_s|\mathscr{F}_t)=E(X_{s-t}|\mathscr{F}_0)\circ\theta_t$ $(s>t)$. Putting $\tilde{X}_t=E(X_t|\mathscr{F}_0)$, we get $V_{st}^2=\sigma_{s-t}^2$, where $\sigma_t^2=1-E\tilde{X}_t^2$ is the mean-squared error made in predicting X_t by $\tilde{X}_t$. It is known [6] that σ_t^2 is monotone increasing and that X is either *deterministic* $(\sigma_t^2\equiv 0)$ or *nondeterministic* $(\sigma_t^2>0$ for all $t>0)$. We consider only the latter case.

The conditional density $\psi_t(s,x,\omega)$ of (1.16) may be written (omitting ω)

$$\psi_t(s,x)=(2\pi\sigma_{s-t}^2)^{-1/2}\exp\left[-(x-E(X_s|\mathscr{F}_t))^2/2\sigma_{s-t}^2\right], \qquad s>t.$$

Putting $\tilde{Z}_t^x=\int_t^\infty e^{-s}\psi_t(s,x)\,ds$, we conclude that the potential Z^x equals $\tilde{Z}^x$ *up to modification* (see (1.26)). Rewriting, we have

$$\tilde{Z}_t^x=(2\pi)^{-1/2}e^{-t}\int_0^\infty e^{-s}\sigma_s^{-1}\exp\left[-(x-E(X_{s+t}|\mathscr{F}_t))^2/2\sigma_s^2\right] ds. \tag{4}$$

Since the "exp" term is at most 1, we find that Z^x is bounded (hence class (D)) if

$$(5) \qquad \int_{0+} \sigma_s^{-1}\,ds < \infty \quad \left(\text{equivalently: } \int_0^\infty e^{-s}\sigma_s^{-1}\,ds < \infty\right).$$

Thus (5) is sufficient for a local time, although this follows already from the Berman-Orey result, since (5) implies (1) $(\sigma_s^2 \leqq 1 - r^2(s))$. On the other hand,

(6) **Theorem.** *Under* (5), Z^x *is a regular potential (hence* α_t^x *is continuous in* t*).*

Proof. Let $\mathscr{S}$ denote the family of finite $\{\mathscr{F}_t\}$-stopping times, and let T_n, $T \in \mathscr{S}$ be uniformly bounded. For $s \geqq 0$ fixed, if $T_n \uparrow T$ or $T_n \downarrow T$ a.s., then

$$(7) \qquad E(X_{s+T_n} | \mathscr{F}_{T_n}) \xrightarrow{\text{a.s.}} E(X_{s+T} | \mathscr{F}_T) \quad (n \to \infty).$$

Assuming this for the moment, we prove (6) as follows. First we note that a version $\xi_s(t)$ of $E(X_{s+t} | \mathscr{F}_t)$ may be chosen which is (s, t, ω)-measurable, and which is, for each fixed s, the well-measurable projection of the process X_{s+t}; hence $\xi_s(T) = E(X_{s+T} | \mathscr{F}_T)$ a.s. for any $T \in \mathscr{S}$. (This is proven in the Appendix.) We form $\tilde{Z}^x$ with this version, and then, for $T \in \mathscr{S}$ bounded,

$$(8) \qquad E\tilde{Z}_T^x = \int_0^\infty e^{-s}\sigma_s^{-1} E\left(e^{-T}\exp\left[-(x - E(X_{s+T} | \mathscr{F}_T))^2 / 2\sigma_s^2\right]\right) ds$$

by Fubini's theorem. But then EZ_T^x is also given by the right member of (8): for $T \in \mathscr{S}$ bounded and *discrete* one has $EZ_T^x = E\tilde{Z}_T^x$ trivially, and for arbitrary T we may approximate from above by $T_n \in \mathscr{S}$ bounded and discrete. It follows that $EZ_{T_n}^x \to EZ_T^x$ since Z^x is a right-continuous supermartingale, and $E\tilde{Z}_{T_n}^x \to E\tilde{Z}_T^x$ by (5), (7), (8) and dominated convergence. Now for $T_n \uparrow T$ bounded, we get $\lim_n EZ_{T_n}^x = EZ_T^x$ for exactly the same reasons, and that proves the theorem.

To prove (7), define $X_a^* = \sup_{0 \leqq s \leqq a} |X_s|$, $a > 0$. This is finite since X is continuous; hence, by the Landau-Shepp theorem [15], X_a^* is integrable. Suppose $T_n \in \mathscr{S}$, $T_n \leqq a$ for every n. Then, for fixed s, $|X_{s+T_n}| \leqq X_{s+a}^*$.

Now if $T_n \downarrow T$, $\mathscr{F}_T = \bigcap \mathscr{F}_{T_n}$ [17, IV. T 42], and Meyer's convergence theorem [18] implies (7). (Although Meyer's theorem involves an increasing family of σ-fields, the proof works equally well for a decreasing family.) If $T_n \uparrow T$, Meyer's theorem says $E(X_{s+T_n} | \mathscr{F}_{T_n}) \xrightarrow{\text{a.s.}} E(X_{s+T} | \bigvee \mathscr{F}_{T_n})$, so it remains to prove

$$(9) \qquad \mathscr{F}_T = \bigvee \mathscr{F}_{T_n} \quad (T_n \uparrow T).$$

Having realized X in the space $C = C_{\mathbb{R}}$ we note that $\mathscr{F}^0(\mathscr{F}_0^0)$ is the Borel σ-field in C for the topology of uniform convergence on compact subsets of $\mathbb{R}$ (of $(-\infty, 0]$). Consider the shift θ_t as a random process on $(C, \mathscr{F}^0)$ with state space $(C, \mathscr{F}_0^0)$. Then $\mathscr{F}_t$ is the augmentation (by the family $\mathscr{N}$ of P-null sets – see §1(d)) of $\sigma(\theta_s\colon s \leqq t)$. The process θ_t has continuous paths since a continuous function is uniformly approximated by its (small) translates. For $T \in \mathscr{S}$, let $\mathscr{F}_T' = \sigma\{\theta_{T+t}\colon t \in \mathbb{R}_+\} \vee \mathscr{N}$, regarding θ_t as mapping $(C, \mathscr{F}^0) \to (C, \mathscr{F}_0^0)$. According to Lazaro and Meyer [16], given a bounded random variable ξ on $(C, \mathscr{F})$, there is a bounded $g \in (\mathscr{F}_{0+}^0)$ such that $E(\xi \circ \theta_T | \mathscr{F}_T) = g \circ \theta_T$ for every $T \in \mathscr{S}$ bounded. This easily implies

$$(10) \qquad P(M | \mathscr{F}_T) = P(M | \theta_T), \quad M \in \mathscr{F}_T'.$$

Clearly $\theta_{T_n} \in (\mathscr{F}_{T_n})$, so that $\theta_T \in (\bigvee \mathscr{F}_{T_n})$. Finally, a result of Chung [4, 94] establishes (9).

e) We investigate here the *energy* (in the sense of [19]) of the potential Z^x, and show that it provides some insight into the meaning of condition (1). Recall that if a potential $Z=(Z_t)$ is not of class (D), its energy $e(Z)$ is defined to be infinite, while, in the class (D) case, $e(Z)=\frac{1}{2}EA_\infty^2 \leqq \infty$, where $A=(A_t)$ is the predictable increasing process which generates Z. (See [19] for the connection with the classical concept of energy.)

An increasing process $\alpha=(\alpha_t)_{t\in\mathbb{R}}$ is an *additive functional* relative to the flow (θ_t) if, for each $s, t\in\mathbb{R}$, $\alpha_{t+s}=\alpha_t+\alpha_s\circ\theta_t$ a.s. Consider the increasing process $A_t=\int_0^t e^{-s}\,d\alpha_s$ and assume $E\alpha_1<\infty$. This implies $E\alpha_t=tE\alpha_1$ for all $t\in\mathbb{R}$, and $EA_\infty<\infty$. Suppose α, and thus A, to be adapted, and let Z be the class (D) potential $E(A_\infty|\mathscr{F}_t)-A_t$ (up to modification). Then $e(Z)<\infty$ iff $E\alpha_t^2<\infty$ for all t:

$$e(Z)<\infty \Rightarrow \infty > EA_\infty^2 \geqq E\left(\int_0^t e^{-s}\,d\alpha_s\right)^2 \geqq e^{-2t}E\alpha_t^2.$$

Conversely, if $E\alpha_t^2<\infty$ for all t (which is equivalent to $E\alpha_t^2<\infty$ for *some* $t\neq 0$), then

$$EA_\infty^2=\int_0^\infty e^{-t}\int_0^\infty e^{-s}E(\alpha_t\alpha_s)\,ds\,dt \leqq 2\int_0^\infty e^{-t}\int_t^\infty e^{-s}E(\alpha_s^2)\,ds\,dt \leqq 2\int_0^\infty e^{-s}E(\alpha_s^2)\,ds.$$

By the ergodic theorem, $s^{-1}\alpha_s$ converges in L^2 as $s\to\infty$, thus $E\alpha_s^2=O(s^2)$, and the last integral is finite.

As remarked earlier (in b)), the local time α^x – if it exists – will be an additive functional in the present situation. Writing $e(x)$ for $e(Z^x)$ we have:

(11) **Theorem.** *Let X have a local time relative to Lebesgue measure. Then: $\{x\colon e(x)<\infty\}$ has positive measure iff* (1) *holds, in which case* $e(x)<\infty$ a.e. *Further,*

$$e(x)=\frac{1}{4\pi}\int_0^\infty (1-r^2(s))^{-1/2}\exp-\left[s+\frac{x^2}{1+r(s)}\right]ds \leqq \infty. \tag{12}$$

Since Berman [1] has shown that (1) implies $E(\alpha_1^x)^2<\infty$ a.e., we need only prove the "only if" assertion. We require some preparation.

(13) **Lemma.** (i) *Let $\phi(u,v)$ be the bivariate normal density having zero means, unit variances, and correlation $p\geqq 0$. Then, for any (complex-valued) integrable function g,*

$$\iint g(u)\overline{g(v)}\,\phi(u,v)\,du\,dv \geqq 0.$$

(ii) *For $D\in\mathscr{B}$ bounded with $m(D)>0$ (m=Lebesgue measure), there exist $\delta>0$, $\eta>0$, such that $|t-s|<\delta$ implies*

$$E\left[\left(\frac{X_t-X_s}{u(t-s)}\right)^{-1}\sin\left(\frac{X_t-X_s}{u(t-s)}\right);\ X_t\in D,\ X_s\in D\right]\geqq\eta,$$

where $u^2(t-s)=E(X_t-X_s)^2$.

Proof. (i) Trivial if $p=0$. For $p>0$, write

$$\phi(u,v)=(2\pi(1-p^2))^{-1/2}\phi_\sigma(u-v)\,\psi(u)\,\psi(v),$$

where $\phi_\sigma(u)=e^{-u^2/2\sigma^2}$, $\sigma^2=p^{-1}(1-p^2)$, and $\psi(u)=e^{-u^2/2(1+p)}$. The result is now immediate since ϕ_σ is positive definite (see, e.g. [23]).

(ii) Let $q(D)$ be the indicated expectation. For any $B\supset D$, $B\in\mathscr{B}$, we have $|q(B)-q(D)|\leqq 3\Phi(B-D)$, hence we may assume D is open. Put $Y=X_s$, $Z=(X_t-X_s)/u(t-s)$, so (Z,Y) is bivariate normal, zero means, unit variances, and covariance $-u(t-s)/2$. If $f(z,y)$ is the density of (Z,Y),

$$\begin{aligned} q(D) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\sin z}{z} I_D(y)\, I_D\big(u(t-s)\,z+y\big)\, f(z,y)\,dz\,dy \\ &\to \Phi(D)\int_{-\infty}^{\infty}\frac{\sin z}{z}\Phi(dz)>0 \qquad \text{as } |t-s|\to 0 \end{aligned}$$

by dominated convergence; this yields (ii).

We can now prove (11). Suppose $m\{x\colon e(x)<\infty\}>0$. Then there is a bounded Borel set D, $m(D)>0$, for which $\int_D E(\alpha_t^x)^2\,dx<\infty$, $t\leqq 1$. Choose $\delta,\eta>0$ such that (13)(ii) holds *and* $r(s-t)\geqq 0$ whenever $s,t<\delta$. For almost every $\omega\in\Omega$, $\alpha_\delta^x(\omega)\,I_D(x)\in L^1\cap L^2$, and the Parseval relation gives

$$\int_D (a_\delta^x)^2\,dx=\int_{-\infty}^{\infty}|\xi(y)|^2\,dy \qquad \text{a.s.}$$

where $\xi(y)=\int_{-\infty}^{\infty} e^{iyx}\alpha_\delta^x I_D(x)\,dx=\int_0^\delta e^{iyX_s} I_D(X_s)\,ds$. (The last equality is a consequence of (1.1).) Hence,

$$\begin{aligned} \infty>\int_D E(\alpha_\delta^x)^2\,dx &= \int_{-\infty}^{\infty} E|\xi(y)|^2\,dy \\ &= \int_{-\infty}^{\infty}\int_0^\delta\int_0^\delta E(e^{iy(X_t-X_s)};\ X_t\in D,\ X_s\in D)\,ds\,dt\,dy \\ &= \int_0^\delta\int_0^\delta\int_{-\infty}^{\infty} E(e^{iy(X_t-X_s)};\ X_t\in D,\ X_s\in D)\,dy\,ds\,dt \qquad \text{(Fubini and 13(i))} \\ &\geqq \int_0^\delta\int_0^\delta\int_{-(u(t-s))^{-1}}^{(u(t-s))^{-1}} E(e^{iy(X_t-X_s)};\ X_t\in D,\ X_s\in D)\,dy\,ds\,dt \\ &= 2\int_0^\delta\int_0^\delta \big(u(t-s)\big)^{-1} q(D)\,ds\,dt \\ &\geqq 2\eta\int_0^\delta\int_0^\delta \big(u(t-s)\big)^{-1}\,ds\,dt, \end{aligned}$$

and this implies (1).

Using the definition of additive functional, it is easy to show that

$$e(x)=\tfrac{1}{4}\int_0^\infty e^{-t}E(\alpha_t^x)^2\,dt,$$

and we may assume (1) is proving (12). But then Berman [1, 295] gives the formula

$$E(\alpha_t^x)^2 = \frac{1}{2\pi} \int_0^t \int_0^t (1 - r^2(s-u))^{-1/2} \exp\left[-x^2/(1 + r(s-u))\right] ds\, du$$

(stated in [1] for $t=1$) from which (12) follows.

Remark. (i) When (3) holds, (1) fails so $e(x) = \infty$. We indicate a direct proof using Palm measures. Under (3) the local time is given as in §1, Example 1, and obviously P^x is concentrated on $\{X_0 = x\}$ for a.e. x, so the measure $v_\cdot^x(\omega)$ has unit mass on $t=0$ under P^x. An alternate formula for the energy may be derived, namely

$$e(x) = \tfrac{1}{4} \int_{-\infty}^{\infty} e^{-|t|} E^x |\alpha_t^x|\, dt \qquad \text{a.e.}$$

(E^x is integration with P^x), so

$$e(x) \geqq \tfrac{1}{4} E^x \int_{-1}^{0} |\dot{X}_s|^{-1}\, dv_s^x \geqq \tfrac{1}{4} E^x |\dot{X}_0|^{-1}.$$

But $\dot{X}_0$ is Gaussian, mean 0, variance $-r''(0) < \infty$ under P^x, so $e(x) = \infty$.

(ii) Condition (5) implies Z^x bounded which itself implies finite energy. Also, it can be shown that Z^x bounded implies (1). We do not know whether Z^x bounded (for a.e. x) is sufficient for (5) or necessary for (1).

(iii) Examples will be given momentarily in which (a)(1) holds but (5) fails, (b)(1) fails for a nondeterministic process.

f) *Examples.* (i) Let $g(\lambda)$ vary slowly at ∞, put $\psi^2(t) = \int^{1/t} g(\lambda)/\lambda\, d\lambda$, and suppose $\int^{\infty} g(\lambda)/\lambda\, d\lambda = \infty$, $\int_{0+} (t\,\psi(t))^{-1}\, dt = \infty$ (e.g. $g(\lambda) = \log \lambda$ will do). Let $G(\lambda)$ be a spectral distribution function such that $G(\lambda) = 1 - \lambda^{-2} g(\lambda)$ for large $|\lambda|$. Then $\int_{-\infty}^{\infty} \lambda^2\, dG = \infty$ so (3) fails, but (1) also fails because $r(0) - r(t) \sim t^2 \psi^2(t)$ $(t \to 0)$ (see Pitman [22, 397]). Finally G can easily be chosen so that $G' > 0$ a.e. and $(1+\lambda^2)^{-1} \log G'(\lambda)$ is integrable, which gives a nondeterministic process [6, 584].

(ii) We use the terminology of Doob [6, XII.5] in this example. Let $0 \leqq C^* \in L^2$, $C^*(t) = 0$ for $t \in \mathbb{R}_+$. Define $c(\lambda) = \int_{-\infty}^{\infty} e^{2\pi i \lambda t} C^*(t)\, dt$ and $G(\lambda) = \int_{-\infty}^{\lambda} |c(s)|^2\, ds$. Taking G as spectral distribution, we get a nondeterministic process if $|c(\lambda)|^2 > 0$ a.e. and $(1+\lambda^2)^{-1} \log |c(\lambda)|^2$ is integrable, and, in that case $\sigma_t^2 = \int_{-t}^{0} |C^*(s)|^2\, ds$, $t > 0$. If we take $C^* \in L^1 \cap L^2$, Fourier inversion will give $r(t) = \int_{-\infty}^{\infty} C^*(y-t)\, C^*(y)\, dy$ where $r(t) = \int_{-\infty}^{\infty} e^{2\pi i t \lambda}\, dG(\lambda)$. Taking, in particular, $C^*(t) = -t$ on $(-(2\pi)^{-1}, 0)$ and zero elsewhere we find, after some computations, $r(0) - r(t) \sim b\,t$ $(t \to 0)$ where $b > 0$, and $|c(\lambda)|^2 = (2\pi\lambda)^{-4}[2(1 - \cos\lambda - \lambda \sin\lambda) + \lambda^2]$. One verifies now that the conditions for nondeterminism are fulfilled, that (1) holds, and (5) does not.

(iii) Let X be a complex-valued stationary Gaussian process. We give examples in which X has and does not have a local time relative to plane Lebesgue measure. The one-dimensional distribution now is the standard complex normal, given by the density $\phi(z)=(2\pi)^{-1}e^{-|z|^2/2}$, $z\in\mathbb{C}$ (=complex numbers). Following the approach of §1, Example 2, we have a conditional density

$$P(X_s\in dz\,|\,\mathscr{F}_0)=(\pi\sigma_s^2)^{-1}\exp\left[-|z-E(X_s|\mathscr{F}_0)|^2/2\sigma_s^2\right]dz.$$

(σ_s^2 has the same significance as in the real-valued case, dz refers to Lebesgue measure on $\mathbb{C}$.) The potential Z^z is given up to modification by

$$\tilde{Z}_t^z=\pi^{-1}e^{-t}\int_0^\infty e^{-s}\sigma_s^{-2}\exp\left[-|z-E(X_{s+t}|\mathscr{F}_t)|^2/2\sigma_s^2\right]ds.$$

Obviously Z^z will be bounded, hence class (D), if $\int_{0+}\sigma_s^{-2}\,ds<\infty$, and to produce an example we again follow Doob [6, XII.5]. Take a number $\alpha\in(0,1)$, put $a(t)=|t|^{(\alpha-1)/2}$ on $(-1,0)$, $=0$ elsewhere, and let $b(t)$ be a function which is in $L^1\cap L^2$, vanishes on $\mathbb{R}_+$, and $\int_{-\infty}^{\infty}b(t)\,dt=0$. Put $C^*(t)=a(t)+i\,b(t)$. Then $\sigma_s^2=\int_{-s}^{0}|C^*(u)|^2\,du$ satisfies the above condition, so we get a local time. To obtain an example for which no local time exists, let X be a complex stationary Gaussian process with covariance $r(t-s)=E(X_t\overline{X}_s)$ such that $-r''(0)<\infty$. The trajectories can be taken absolutely continuous (in $\mathbb{C}$) with probability 1 [6, 536]. Hence, with probability 1, the trajectory is a rectifiable curve and so has Lebesgue measure zero. It follows also that the potential Z^z cannot be of class (D) for a.e. z – indeed not even for all z in a set of positive measure.

(iv) We now use the machinery that has been developed to compute the Laplace transform of the first passage time distribution for the stationary Ornstein-Uhlenbeck (OU) process U_t, $t\in\mathbb{R}$. (The process U_t is the unique stationary Gaussian Markov process, or, most briefly, $U_t=e^{-t}\mathscr{B}(e^{2t})$ where $\mathscr{B}$ is standard Brownian motion.) One can, of course, use existing methods; we offer the present approach as an illustration of a method which may be applicable to non-Markov processes.

Since the OU process has covariance $e^{-|t|}$ it is easily checked that the conditional density is given by (for $s>t$)

$$P(U_s\in du\,|\,\mathscr{F}_t)=\left(2\pi(1-e^{-2(s-t)})\right)^{-1/2}\exp\left[-(u-e^{-(s-t)}U_t)^2/2(1-e^{-2(s-t)})\right]du.$$

The λ-potential is

$$Z_t^\lambda(u)=(2\pi)^{-1/2}e^{-\lambda t}\int_0^\infty e^{-\lambda s}(1-e^{-2s})^{-1/2}\exp\left[-(u-e^{-s}U_t)^2/2(1-e^{-2s})\right]ds \tag{14}$$

(without modification), so (5) holds and $Z^\lambda(u)$ is a regular, class (D) potential, hence a continuous (in t) local time α^u exists.

Denote by $L_u(\omega)$ the time set $\{t\geqq 0\colon U_t(\omega)=u\}$ and let Δ_u be the support of the measure $\alpha^u_\cdot$: $\Delta_u(\omega)=\{t\geqq 0\colon \alpha^u_{t+\varepsilon}(\omega)-\alpha^u_{t-\varepsilon}(\omega)>0$ for all $\varepsilon>0\}$. As we have noted earlier, $\Delta_u\subset L_u$ a.s. Put $\tau_u=\inf\{t>0\colon U_t=u\}=\inf L_u$. Clearly $\alpha^u_{\tau_u}(\omega)=0$ a.s.

Since $Z^\lambda(u)$ is of class (D),

$$Z_t^\lambda(u)=M_t^\lambda(u)-A_t^\lambda(u)$$

where $M^\lambda(u)$ is a uniformly integrable martingale, and $A_t^\lambda(u)=\int_0^t e^{-\lambda s}\,d\alpha_s^u$. Thus $Z_{\tau_u}^\lambda(u)=M_{\tau_u}^\lambda(u)$, and from (14),

$$(2\pi)^{-1/2} E(e^{-\lambda\tau_u})\int_0^\infty e^{-\lambda s}(1-e^{-2s})^{-1/2}\exp[-u^2(1-e^{-s})^2/2(1-e^{-2s})]\,ds=EM_{\tau_u}^\lambda(u).$$

By optional sampling, $EM_{\tau_u}^\lambda(u)=EM_0^\lambda(u)=EZ_0^\lambda(u)=(2\pi)^{-1/2}\lambda^{-1}e^{-u^2/2}$. Finally, then,

$$E(e^{-\lambda\tau_u})=e^{-u^2/2}\lambda^{-1}\left[\int_0^\infty e^{-\lambda s}(1-e^{-2s})^{-1/2}\exp[-u^2(1-e^{-s})^2/2(1-e^{-2s})]\,ds\right]^{-1}.$$

The integral in brackets may be evaluated in terms of various special functions; when $u=0$ the expression is especially simple: $E(e^{-\lambda\tau_0})=2(\lambda B(\lambda/2,1/2))^{-1}$, where B denotes the beta function.

Appendix

Proof of (1.3). Let $\mathscr{G}$ be a countable field which generates $\mathscr{F}^0$. Since $K_t(\Gamma, A)$ increases with t, there exists $H\in\mathscr{E}$ such that $\pi(H^c)=0$ and $x\in H$ implies $k(s,x,A)\leqq k(t,x,A)$ for every pair of rationals $s\leqq t$, and every $A\in\mathscr{G}$, hence for every $A\in\mathscr{F}^0$, since $k(s,x,\cdot)$ is a measure (use a monotone class argument). Writing Q for the rationals, set, for $A\in\mathscr{F}^0$,

$$\bar{k}(t,x,A)=\begin{cases}\lim\limits_{\substack{t<r\downarrow t\\ r\in Q}} k(r,x,A) & x\in H\\ 0 & x\notin H.\end{cases}$$

Then (Vitali-Hahn-Saks theorem) $\bar{k}(t,x,\cdot)$ is a measure. Since $t\to\bar{k}(t,x,A)$ is right continuous, we get (t,x)-measurability; finally it is clear that

$$K_t(\Gamma,A)=\int_\Gamma \bar{k}(t,x,A)\,\pi(dx),$$

so $\bar{k}$ is the desired version.

Proof of (1.11). The reader is referred to [5] for terminology. We recall briefly the Doob-Meyer decomposition of class (D) potentials. First, a *potential* $Z=(Z_t)$ is a right-continuous, non-negative supermartingale such that $EZ_t\to 0$ $(t\to\infty)$. It is of class (D) if the family $\{Z_T\colon T\in\mathscr{S}\}$ is uniformly integrable, where $\mathscr{S}$ is the family of stopping times. For Z of class (D) there is a unique increasing process $A=(A_t)$ such that A_∞ is integrable, $Z_t=E(A_\infty|\mathscr{F}_t)-A_t$ (up to modification) and A is *predictable* (or *natural*), meaning that the map $(t,\omega)\to A_t(\omega)$ is measurable relative to the σ-field $\mathscr{P}$ on $\mathbb{R}_+\times\Omega$ generated by sets of the form

$$\text{(A 1)}\qquad H=\{0\}\times F\cup\bigcup_{i=1}^n \rrbracket S_i, T_i\rrbracket,$$

where $F\in\mathscr{F}_0$, $S_1\leqq T_1\leqq S_2\leqq\dots\leqq T_n$ are in $\mathscr{S}$, and

$$\rrbracket S_i, T_i\rrbracket=\{(t,\omega)\colon\ t\in\mathbb{R}_+,\ S_i(\omega)<t\leqq T_i(\omega)\}.$$

Coming to the proof, if $T\in\mathscr{S}$ is discrete, the map $x\to EZ_T^x$ is measurable by (1.10); for arbitrary T, approximate with discrete stopping times from above.

Thus

(A 2) $$x \to EZ_T^x \quad \text{is measurable for } T \in \mathscr{S}.$$

Let μ_x be the measure on $\mathscr{P}$ induced by Z^x. It is determined by its values on sets H as in (A 1), namely

$$\mu_x(H) = \sum_{i=1}^{n} E(Z_{S_i}^x - Z_{T_i}^x)$$

(which is non-negative). This is measurable by (A 2), and a monotone class argument shows that $x \to \mu_x(H)$ is measurable for all $H \in \mathscr{P}$.

For any function $u \in (\mathscr{P})_+$ we have $x \to \int u \, d\mu_x$ measurable; hence, for $u \in (\mathscr{B} \times \mathscr{F})_+$,

$$E \int_0^\infty u(t, \omega) \, dA_t^x(\omega) = \int \tilde{u} \, d\mu_x$$

is x-measurable, $\tilde{u}$ being the predictable projection of u [5, Chapter V]. Taking $u(s, \omega) = I_{[0,t]}(s) I_B(\omega)$ $(B \in \mathscr{F})$ we find that $(x, B) \to E[A_t^x; B]$ is a kernel on $E \times \mathscr{F}$, and we may choose a jointly (x, ω)-measurable density $\bar{A}_t(x, \omega)$. This may be "regularized" as in the proof of (1.1) above to be right continuous in t, (t, x, ω)-measurable, and indistinguishable from $A_t^x(\omega)$. We omit the details.

The density $\psi_t(s, x, \omega)$. We prove that this density (used in (1.16)) may be chosen (s, x, ω)-measurable $(s > t)$. Fix t, and define a map $\gamma_{s,t}: \Omega \to E \times \Omega$ by $\gamma_{s,t}(\omega) = (X_s(\omega), \omega)$. This is measurable since X is; indeed,

$$\gamma_{s,t}^{-1}(\mathscr{E} \times \mathscr{F}_t^0) = \sigma(X_s) \vee \mathscr{F}_t^0.$$

Let ν_{st} be the distribution of $\gamma_{s,t}$ over $\mathscr{E} \times \mathscr{F}_t^0$: $\nu_{st}(D) = P(\gamma_{s,t}^{-1}(D))$, $D \in \mathscr{E} \times \mathscr{F}_t^0$. Notice that $\nu_{st}(\Gamma \times A) = P(X_s \in \Gamma, A)$, $A \in \mathscr{F}_t^0$, so $\nu_{st}(D)$ is s-measurable for each D. Moreover, $\nu_{st} \ll \pi \times P_t$ for $s > t$, where P_t is the restriction of P to $\mathscr{F}_t^0$: first choose $\psi_t(s, x, \omega)$ measurable in (x, ω) by (1.5) and note that $\gamma_{s,t}(D) = \iint_D \psi_t(s, x, \omega) \, d\pi \, dP$ when $D = \Gamma \times A$, $\Gamma \in \mathscr{E}$, $A \in \mathscr{F}_t^0$, and then for all $D \in \mathscr{E} \times \mathscr{F}_t^0$. Now apply (1.5) to the kernel $(s, D) \to \gamma_{s,t}(D)$ to finish the proof.

The good version of $E(X_{s+t} | \mathscr{F}_t)$. In the proof of (2.6) we needed a version $\xi_s(t)$ of $E(X_{s+t} | \mathscr{F}_t)$ which was (s, t, ω)-measurable and, for each s, a version of the well-measurable projection [5] of the process X_{s+t}. The existence of such is a consequence of the following results. We refer the reader to [5] for terminology. Let $\mathscr{W}$ be the σ-field of *well-measurable* sets in $\mathbb{R}_+ \times \Omega$: it is generated by all processes $u(t, \omega)$ which have right continuous trajectories a.s. The results below have analogues for the accessible and predictable σ-fields (for the latter $\mathscr{F}_T$ is replaced by $\mathscr{F}_{T-}$ at certain points in the argument), but we shall not dwell on these. The family of σ-fields $\{\mathscr{F}_t\}$ is assumed (as usual) to be right continuous with all P-negligible sets contained in each $\mathscr{F}_t$.

(A 3) **Theorem.** *Let $x_{st}(\omega)$ be an (s, t, ω)-measurable process on two variables which is either bounded or non-negative. The well-measurable projection $\xi_s(t)$ of x_{st} (s fixed) may be chosen (s, t, ω)-measurable.*

Recall that, for a bounded or non-negative measurable process $u(t, \omega)$, its well-measurable projection is the unique well-measurable process $\tilde{u}(t, \omega)$ such that $E[u(T); T < \infty] = E[\tilde{u}(T); T < \infty]$ for every $T \in \mathscr{S}$.

Just as in [5, 99] it suffices, in proving (A3), to consider x_{st} of the form $x_{st}(\omega)=a(s)\,b(t)\,c(\omega)$ where a, b are bounded Borel functions, and $c\in(\mathscr{F})$ is bounded. Let ζ_t be the right continuous modification of the martingale $E(c\,|\,\mathscr{F}_t)$. Then $\xi_s(t)=a(s)\,b(t)\,\zeta_t$ is the desired version.

If, in addition, for each s, the process x_{st} is right continuous a.s., $\xi_s(t)$ will have the same property [5, 101].

Recall that a set $A\subset\mathbb{R}_+\times\Omega$ is *evanescent* if, for a.e. $\omega\in\Omega$, $I_A(t,\omega)\equiv 0$. A property holding outside an evanescent set is said to hold *quasi-surely* (q.s.).

(A4) **Theorem.** *Let x_t be a non-negative, measurable, and separable process such that $x_a^*=\sup_{0\leq s\leq a} x_s$ is integrable for $a>0$. Then the well-measurable projection $\tilde{x}_t$ may be chosen finite* q.s.

Remarks. (a) The separability is only used to guarantee x_a^* measurable.

(b) A similar proof will show: if $x_t^n(\omega)$ are uniformly bounded measurable processes, and $x^n\to x$ q.s., then $\tilde{x}^n\to\tilde{x}$ q.s.

Proof. Let $x_t^n=x_t\wedge n$, so $x_t^n(\omega)\uparrow x_t(\omega)$ for all (t,ω). Then $\tilde{x}^n\uparrow\tilde{x}$ q.s. (see the proof of T 14, 98 of [5]). In fact, $\tilde{x}$ may be *defined* as the indicated limit. The set $A=\{(t,\omega)\colon\ \tilde{x}_t^n(\omega)\uparrow\infty\}$ is in $\mathscr{W}$, and its projection $\pi(A)$ on Ω is $\{\omega\colon\ \tilde{x}_t(\omega)=\infty$ for some $t\}$. By the section theorem [5, IV. T 10], given $\varepsilon>0$, there is a $T\in\mathscr{S}$ such that $(T(\omega),\omega)\in A$ whenever $T(\omega)<\infty$, and $P(\pi(A))\leq P(T<\infty)+\varepsilon$. Now

$$\tilde{x}_T^n I_{[T<\infty]}=E[x_T^n I_{[T<\infty]}\,|\,\mathscr{F}_T] \qquad [5,\,100]$$

which implies for $a>0$,

$$E(\tilde{x}_T^n;\,T\leq a)=E(x_T^n;\,T\leq a)\leq E(x_a^*)<\infty.$$

Since $(T(\omega),\omega)\in A$ when $T(\omega)\leq a$, we must have $P(T\leq a)=0$, hence $P(T<\infty)=0$, and, finally, $P(\pi(A))=0$, which proves (A4).

(A5) **Corollary.** *Let x_t be a measurable, separable process such that $x_a^*=\sup_{0\leq s\leq a}|x_s|$ is integrable for $a>0$. Then a well-measurable projection $\tilde{x}$ of x exists and is finite* q.s.

This is immediate upon splitting x into its positive and negative parts, and using (A4).

Let X be a continuous, stationary, real-valued Gaussian process. Then X_a^* is integrable for each $a>0$, and we may choose $\xi_s^+(t)$ to be (s,t,ω)-measurable and, for each s, the well-measurable projection of X_{s+t}^+; $\xi_s^-(t)$ is chosen similarly. Then $\xi_s(t)=\xi_s^+(t)-\xi_s^-(t)$ is the desired process. (We may set $\xi_s(t)=0$ whenever $\infty-\infty$ arises from the above recipe.) Finally, we note that the set

$$B=\{(s,t,\omega)\colon\ \xi_s^+(t,\omega)+\xi_s^-(t,\omega)=\infty\}$$

is measurable, hence $B_1=\{(s,\omega)\colon\ \xi_s^++\xi_s^-<\infty$ for all $t\}$ is in $\overline{\mathscr{B}\times\mathscr{F}}$ (=completion of $\mathscr{B}\times\mathscr{F}$ under $ds\times dP$) by [5, I. T 32]. Since, for each s, the section $B_1(s)$ at s has probability 1, we get from Fubini's theorem that, for a.e. $\omega\in\Omega$, $\xi_s^++\xi_s^-<\infty$ for all t and a.e. s.

References

1. Berman, S.: Local times and sample function properties of stationary Gaussian processes. Trans. Amer. Math. Soc. **137**, 277–299 (1969)
2. Berman, S.: Gaussian processes with stationary increments: local times and sample function properties. Ann. Math. Statist. **41**, 1260–1272 (1970)
3. Blumenthal, R. M., Getoor, R. K.: Markov Processes and Potential Theory. New York: Academic Press 1968
4. Chung, K. L.: Some universal field equations. Lecture Notes in Math, no. **258**, Berlin-Heidelberg-New York: Springer 1972
5. Dellacherie, C.: Capacités et Processus Stochastiques. Berlin-Heidelberg-New York: Springer 1972
6. Doob, J.: Stochastic Processes. New York: John Wiley 1953
7. Dudley, R.: A counterexample on measurable processes. Proc. 6th Berkeley Sympos. Math. Statist. Probab. Univ. Calif. Vol. **2**, 57–66 (1972) (to appear)
8. Föllmer, H.: The exit measure of a supermartingale. Z. Wahrscheinlichkeitstheorie verw. Gebiete **21**, 154–166 (1972)
9. Geman, D., Horowitz, J.: Occupation times for smooth stationary processes. Ann. Probab. **1**, 131–137 (1973)
10. Geman, D., Horowitz, J.: Remarks on Palm Measures. Ann. Inst. H. Poincaré, Sect. B, **9**, 215–232 (1973)
11. Getoor, R. K., Kesten, H.: Continuity of local times for Markov processes. Compositio Math. **24**, 277–303 (1972)
12. Griego, R.: Local time as a derivative of occupation times. Illinois J. Math. **11**, 54–63 (1967)
13. Itô, K., McKean Jr., H. P.: Diffusion Processes and their Sample Paths. New York: Academic Press 1965
14. Johnson, G., Helms, L. L.: Class (*D*) supermartingales. Bull. Amer. Math. Soc. **69**, 59–62 (1963)
15. Landau, H., Shepp, L.: On the supremum of a Gaussian process. Sankhyā Ser. A, **32**, 369–378 (1970)
16. Lazaro, J., Meyer, P. A.: Méthodes des martingales et theorie des flots. Z. Wahrscheinlichkeitstheorie verw. Gebiete **18**, 116–140 (1971)
17. Meyer, P. A.: Probability and Potentials. Waltham: Blaisdell 1966
18. Meyer, P. A.: Un lemme de théorie des martingales. Lecture Notes in Math., no. **88**. Berlin-Heidelberg-New York: Springer 1969
19. Meyer, P. A.: Interprétation probabiliste de la notion d'énergie, presented at Séminaire de Théorie du Potential, Inst. H. Poincaré, Paris 1962–63
20. Neveu, J.: Mathematical Foundations of the Calculus of Probability. San Fransisco: Holden-Day 1965
21. Orey, S.: Gaussian sample functions and the Hausdorff dimension of level crossings. Z. Wahrscheinlichkeitstheorie verw. Gebiete **15**, 249–256 (1970)
22. Pitman, E. J. G.: Some theorems of characteristic functions of probability distributions. Proc. 4th Berkeley Sympos. Math. Statist. and Probab. Univ. Calif. Vol. **2**, 393–402 (1961)
23. Rudin, W.: Fourier Analysis on Groups. New York: Interscience 1962

Donald Geman and Joseph Horowitz
Department of Mathematics
University of Massachusetts
Amherst, Mass. 01002, USA

(Received May 31, 1973)