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Another Limit Theorem for Local Time

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1. Introduction and Results

Let X be a real valued process with stationary independent increments having right continuous paths with left limits. Let $X=(\Omega, \mathscr{F}, \mathscr{F}_t, X_t, \theta_t, P^x)$ be the usual canonical representation of the process X. Then, as is well known, X is determined by

(1.1) $$E^0 \exp(i\lambda X_t) = E^0 \exp[i\lambda(X_{t+s}-X_s)] = e^{-t\phi(\lambda)},$$

(1.2) $$\phi(\lambda) = ia\lambda + \frac{\sigma^2}{2}\lambda^2 + \int \left[1 - e^{i\lambda y} + \frac{i\lambda y}{1+y^2}\right] \nu(dy)$$

where $a \in \mathbb{R}$, $\sigma^2 \geqq 0$, and ν is a Borel measure on $\mathbb{R}$ not charging $\{0\}$ such that $\int \min(1, y^2)\, \nu(dy) < \infty$. *We assume throughout this paper that* 0 *is regular for* $\{0\}$ *and that either* $\sigma^2 > 0$ *or* $\nu(\mathbb{R}) = \infty$. Under these assumptions each x is regular for $\{x\}$ and there exists a bounded positive continuous function $u(x) \equiv u^1(x)$ on $\mathbb{R}$ such that

(1.3) $$U^1 f(x) \equiv E^x \int_0^\infty e^{-t} f(X_t)\, dt = \int u(y-x) f(y)\, dy$$

for all bounded Borel f. (The symbol "$\equiv$" means "defined to be equal to".) See [2] or [11] for these results. Moreover, there exists a local time $l^x = (l_t^x)$ at each x which we normalize by

(1.4) $$E^x \int_0^\infty e^{-t}\, dl_t^y = u(y-x).$$

Each l^x is a continuous additive functional of X which is jointly measurable in (t, x, ω) and satisfies under the normalization (1.4)

(1.5) $$\int_0^t 1_B(X_s)\, ds = \int_B l_t^x\, dx$$

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almost surely simultaneously for all $t \geqq 0$ and Borel sets $B \subset \mathbb{R}$. See [5]. For each $x \in \mathbb{R}$, let $T_x = \inf\{t>0: X_t = x\}$ be the hitting time of x. Then it is immediate from (1.4) that

(1.6) $\psi(x, y) \equiv E^x\{e^{-T_y}\} = u(y-x)/u(0).$

In particular, $0 \leqq u(x) \leqq u(0)$ for all x. (See [5] for these facts.)

Lévy conjectured that if X is Brownian motion and $D^\varepsilon(t)$ is the number of times that X crosses down from $\varepsilon > 0$ to 0, during $[0, t]$, then $\lim_{\varepsilon \to 0} 2\varepsilon D^\varepsilon(t) = l_t^0$ almost surely. This result is proved in Ito-McKean [7], although I must confess that I do not completely understand their proof. Recently, Chung informed me that their proof contains several serious gaps. However, it turns out that this result is a simple consequence of "general potential theory", and, in fact, there is an analogous result for processes with independent increments satisfying the conditions in the previous paragraph.

Fix $a \leqq 0 \leqq b$ with $a < b$. Define $\tau_0 = 0$, $\tau_1 = T_b$, $\tau_2 = \tau_1 + T_a \circ \theta_{\tau_1}, \ldots,$

(1.7)
$$\begin{aligned} \tau_{2n+1} &= \tau_{2n} + T_b \circ \theta_{\tau_{2n}}, \quad n \geqq 0 \\ \tau_{2n+2} &= \tau_{2n+1} + T_a \circ \theta_{\tau_{2n+1}}, \quad n \geqq 0. \end{aligned}$$

Then τ_{2n}, $n \geqq 1$ are the times of successive downcrossings from (a hitting of) b to (a hitting of) a. Of course, if the paths are not continuous there will, in general, be many downcrossings *over* the interval $[a, b]$ between τ_{2n} and τ_{2n+2}. Let

(1.8) $D(t) = D^{a,b}(t) \equiv \sum_k 1_{(0,t]}(\tau_{2k})$

be the number of downcrossings from b to a during the time interval $(0, t]$. Since each τ_{2k} is a stopping time, $D(t)$ is an adapted, right continuous, increasing process, $D(0) = 0$, and $D(t) < \infty$ for each $t \in \mathbb{R}^+$. Note, however, that D is *not* an additive functional of X. We may now state our main result.

(1.9) **Theorem.** (i) *The function u in* (1.3) *is uniformly continuous on* $\mathbb{R}$ *and satisfies* $\lim_{x \to \pm\infty} u(x) = 0$.

(ii) *The increasing process D is previsible.*

(iii) *Let*

(1.10) $$g(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} (1 - \cos x\lambda) \operatorname{Re}([1 + \phi(\lambda)]^{-1})\, d\lambda.$$

Then for each finite T, $\sup_{t \leqq T} |g(b-a) D^{a,b}(t) - l_t^0|$ approaches zero in $L^2(P^0)$ as $a \uparrow 0$, $b \downarrow 0$ with $b - a > 0$.

(iv) *Let δ be the modulus of continuity of u. Let (a^n) and (b^n) be sequences with $a^n \uparrow 0$, $b^n \downarrow 0$, $b^n - a^n > 0$ and satisfying $\sum \delta(b^n - a^n) < \infty$. Then for each finite T,*

$$\sup_{t \leqq T} |g(b^n - a^n) D^{a^n, b^n}(t) - l_t^0| \to 0$$

almost surely P^0 as $n \to \infty$.

Remark. It is well known (see [2] or [8]) that under the present assumptions $\mathrm{Re}([1+\phi(\lambda)]^{-1})$ is integrable over $\mathbb{R}$. Also it is immediate from (1.2) that $\mathrm{Re}([1+\phi(\lambda)]^{-1})>0$ and that this is an even function of λ. Consequently $g(x)$ defined in (1.10) is an even, continuous, nonnegative function of x satisfying $g(x)\to 0$ as $x\to 0$, and one has

$$(1.11)\quad g(x)=\frac{2}{\pi}\int_0^\infty (1-\cos x\lambda)\,\mathrm{Re}([1+\phi(\lambda)]^{-1})\,d\lambda.$$

Of course, in Theorem (1.9) one may replace the normalizing function $g(x)$ by any function $h(x)$ defined for $x>0$ which satisfies $g(\varepsilon)\sim h(\varepsilon)$ as $\varepsilon\downarrow 0$. With this in mind we list the following examples, all of which satisfy our hypotheses.

(1.12) Let X be Brownian motion; that is, $\phi(\lambda)=\lambda^2/2$. Then $g(\varepsilon)\sim 2\varepsilon$ as $\varepsilon\downarrow 0$.

(1.13) Let X be a stable process of index α, $1<\alpha<2$; that is

$$\phi(\lambda)=c|\lambda|^\alpha\left[1+i\beta\,\mathrm{sgn}(\lambda)\tan\frac{\pi\alpha}{2}\right]$$

where $|\beta|\leqq 1$ and $c>0$. Then $g(\varepsilon)\sim\varepsilon^{\alpha-1}\left[c(1+h^2)\,\Gamma(\alpha)\sin\frac{(\alpha-1)\pi}{2}\right]^{-1}$ where $h=\beta\tan\frac{\pi\alpha}{2}$.

(1.14) Let X be an asymmetric Cauchy process; that is,

$$\phi(\lambda)=c|\lambda|[1+i\beta\,\mathrm{sgn}(\lambda)\log|\lambda|]$$

where $c>0$, $0<|\beta|\leqq\frac{2}{\pi}$. Then $g(\varepsilon)\sim 2[c\pi\beta^2|\log\varepsilon|]^{-1}$.

In the case of Brownian motion there is a simple martingale argument (see p. 48 of [7]) that enables one to conclude that almost surely for all t, $2\varepsilon D^{0,\varepsilon}(t)\to l_t^0$ as $\varepsilon\downarrow 0$. Notice that in [7], the local time is normalized to satisfy (1.5) with dx replace by $2dx$. Consequently in [7] the normalizing factor is "ε" rather than "2ε".

We have confined our attention entirely to real valued processes with stationary independent increments since these are by far the most interesting examples of our results. However, our methods obviously extend to a wider class of processes. We leave such extensions to the interested reader.

Other limit theorems for local time are contained in [6]. For example, let X be a stable process with index α, $1<\alpha<2$, and $\beta\neq -1$. Let $N^\varepsilon(t)$ be the number of jumps from the interval $J_\varepsilon=(\varepsilon,\lambda\varepsilon)$, $\lambda>1$ to $(-\lambda\varepsilon,-\varepsilon)$ in $[0,t]$, that is, $N^\varepsilon(t)=\sum_{s\leqq t}1_{J_\varepsilon}(X_{s-})\,1_{J_\varepsilon}(-X_s)$. Then it follows from (2.1) of [6], that $\sup_{t\leqq T}|k\varepsilon^{\alpha-1}N^\varepsilon(t)-l_t^0|\to 0$ in $L^2(P^0)$ as $\varepsilon\downarrow 0$ for each $T<\infty$, where

$$k=\pi(\alpha-1)\left\{2c(1+\beta)\,\Gamma(\alpha)\sin\frac{\pi\alpha}{2}\,[2^{-\alpha}(1+\lambda^{1-\alpha})-(1+\lambda)^{1-\alpha}]\right\}^{-1}.$$

This should be compared with (1.13). Note also the limiting value of k as $\lambda\to\infty$.

2. Proofs

We begin with (1.9i). Recall from (4.9) of [5] that under our assumptions on X

$$(2.1)\quad u(x)+u(-x)=\frac{1}{\pi}\int_{-\infty}^{\infty}\cos x\lambda\,\mathrm{Re}([1+\phi(\lambda)]^{-1})\,d\lambda$$

where ϕ is the exponent of X defined in (1.2) and $\mathrm{Re}([1+\phi]^{-1})$ is integrable. Thus by the Riemann Lebesgue lemma $u(x)+u(-x)\to 0$ as $x\to\pm\infty$ and since $u\geqq 0$ this implies that $u(x)\to 0$ as $x\to\pm\infty$. Consequently u is uniformly continuous on $\mathbb{R}$ establishing (1.9i).

Next we shall prove (1.9ii). To this end we fix a and b with $a\leqq 0\leqq b$ and $a<b$. Using the notation of (1.6) and (1.7) we have for $k\geqq 2$

$$\begin{aligned}E^x\{e^{-\tau_{2k}}\}&=E^x\{e^{-\tau_{2k-1}}E^{X(\tau_{2k-1})}(e^{-T_a})\}\\&=\psi(b,a)\,E^x\{e^{-\tau_{2k-2}}E^{X(\tau_{2k-2})}(e^{-T_b})\}\\&=\psi(b,a)\,\psi(a,b)\,E^x\{e^{-\tau_{2k-2}}\},\end{aligned}$$

and similarly $E^x\{e^{-\tau_2}\}=\psi(b,a)\,\psi(x,b)$. As a result for all $k\geqq 1$

$$(2.2)\quad E^x\{e^{-\tau_{2k}}\}=[\psi(b,a)\,\psi(a,b)]^{k-1}\,\psi(b,a)\,\psi(x,b).$$

Now define $A=A^{a,b}$ by

$$(2.3)\quad A_t=\int_{(0,t]}e^{-s}\,dD(s).$$

Then $A=(A_t)$ is a right continuous, increasing, adapted process which is integrable since by (2.2)

$$E^0(A_\infty)=\sum_{k\geqq 1}E^0\{e^{-\tau_{2k}}\}=\frac{\psi(b,a)\,\psi(0,b)}{1-\psi(b,a)\,\psi(a,b)}<\infty.$$

Note that $\psi(x,y)<1$ if $x\neq y$.

Statement (ii) of Theorem (1.9) is contained in the following lemma.

(2.4) **Lemma.** *D and A are previsible.*

Proof. Clearly it suffices to show that D is previsible. Of course, this means that D is $(\Omega,\mathscr{F}^\mu,\mathscr{F}_t^\mu,P^\mu)$ previsible for each μ. (Our terminology for the general theory of processes follows [3], and for Markov processes [1] and [4].) From (1.8)

$$D=\sum_{k\geqq 1}1_{[[\tau_{2k},\infty))},$$

and so it suffices to show that each τ_{2k} is a previsible stopping time. But this is immediate once we show that T_a is a previsible stopping time. The fact that T_a is previsible under our present assumptions is certainly known, but we shall give a proof for the convenience of the reader.

It is well known that T_a is previsible if and only if $X(T_a)=X(T_a-)$ almost surely on $\{0<T_a<\infty\}$. See, for example, (7.6) of [4]. We shall prove the continuity $t\to X_t$ at T_a by an appeal to the theory of Lévy systems which is especially simple for processes with independent increments. For each $x \in \mathbb{R}$ let $v^x(C) = v(C-x)$ be the translate by x of the Lévy measure v defined in (1.2). Let B be the set of atoms of v, that is,

$$B = \{x \in \mathbb{R} \colon v(\{x\}) > 0\}.$$

Then B is countable. Fix $\eta > 0$ and let $G = \{x \colon |x-a| > \eta\}$. Let 1_a and 1_G be the indicators of $\{a\}$ and G respectively. Then the result on Lévy systems (see [10] or [12]) tells us that for each $x \in \mathbb{R}$

$$E^x \sum_t e^{-t} 1_a(X_t) 1_G(X_{t-}) = E^x \int_0^\infty e^{-t} \int 1_a(y) 1_G(X_t) v^{X(t)}(dy)\, dt$$
$$= E^x \int_0^\infty e^{-t} 1_G(X_t) v^{X(t)}(\{a\})\, dt.$$

But $v^{X(t)}(\{a\}) = v(\{a - X_t\}) > 0$ if and only if $X_t \in (-B) + a \equiv B_a$. Of course, B_a is countable. Also if $X_t \in G$, then $|a - X_t| > \eta$ and so

$$1_G(X_t) v^{X(t)}(\{a\}) \leqq v(\{y \colon |y| > \eta\}) \equiv M < \infty.$$

Combining these observations with $X(T_a) = a$ almost surely on $\{T_a < \infty\}$ we find

$$E^x\{e^{-T_a} 1_G(X_{T_a -})\} \leqq E^x \sum_t e^{-t} 1_a(X_t) 1_G(X_{t-})$$
$$\leqq M E^x \int_0^\infty e^{-t} 1_{G \cap B_a}(X_t)\, dt$$
$$= M \int_{G \cap B_a} u(y-x)\, dy = 0,$$

because $G \cap B_a$ is countable. Letting $\eta \downarrow 0$ we see that $P^x[X(T_a -) \neq a, T_a < \infty] = 0$, completing the proof of Lemma (2.4).

We are now going to compute the potential, in the sense of martingale theory, of the process A defined in (2.3) relative to the measure P^0, that is

$$(2.5) \quad Y_t \equiv E^0\{A_\infty - A_t | \mathscr{F}_t\} = E^0\{\int_{(t,\infty)} e^{-s}\, dD(s) | \mathscr{F}_t\}.$$

See [3] or [9]. Actually we need a *right continuous* version of the martingale defined in (2.5) and that is what we shall obtain. For the computation one first observes that if $n \geqq k \geqq 1$ and $\tau_{2k-2} \leqq t \leqq \tau_{2k-1}$, then

$$(2.6) \quad \tau_{2n} = t + \tau_{2(n+1-k)} \circ \theta_t,$$

while if $n \geqq k \geqq 1$ and $\tau_{2k-1} < t < \tau_{2k}$, then

$$(2.7) \quad \tau_{2n} = t + T_a \circ \theta_t + \tau_{2(n-k)} \circ \theta_{T_a} \circ \theta_t.$$

The reader will find a picture helpful in verifying (2.6) and (2.7). Also recall that b is regular for $\{b\}$. Now

$$(2.8) \quad Y_t = E^0\{\int_{(t,\infty)} e^{-s}\, dD_s | \mathscr{F}_t\} = \sum_{n \geqq 1} E^0\{e^{-\tau_{2n}}; \tau_{2n} > t | \mathscr{F}_t\}$$
$$= \sum_{k \geqq 1} \sum_{n \geqq k} E^0\{e^{-\tau_{2n}}; \tau_{2k-2} \leqq t < \tau_{2k} | \mathscr{F}_t\}.$$

Let $I(k,t)$ be the indicator of $\{\tau_{2k-2} \leqq t \leqq \tau_{2k-1}\}$ and $J(k,t)$ be the indicator of $\{\tau_{2k-1} < t < \tau_{2k}\}$. Then $I(k,t)$ and $J(k,t)$ are $\mathscr{F}_t$ measurable since each τ_j is a stopping time. Using (2.6) and (2.2) one finds

$$E^0\{e^{-\tau_{2n}}; \tau_{2k-2} \leqq t \leqq \tau_{2k-1} | \mathscr{F}_t\} = e^{-t} I(k,t) E^{X(t)}\{e^{-\tau_{2(n+1-k)}}\}$$
$$= e^{-t} I(k,t) [\psi(b,a)\psi(a,b)]^{n-k} \psi(b,a) \psi(X_t, b).$$

Similarly using (2.7)

$$E^0\{e^{-\tau_{2n}}; \tau_{2k-1} < t < \tau_{2k-2} | \mathscr{F}_t\} = e^{-t} J(k,t) [\psi(b,a)\psi(a,b)]^{n-k} \psi(X_t, a).$$

Finally let $I(t) = \sum_{k \geqq 1} I(k,t)$ and $J(t) = \sum_{k \geqq 1} J(k,t)$ so that $I(t) + J(t) = 1$. Combining these last two computations with (2.8) gives

(2.9) $$Y_t = e^{-t} K(a,b) [I(t)\psi(b,a)\psi(X_t,b) + J(t)\psi(X_t,a)]$$

where using (1.6)

(2.10) $$K(a,b) \equiv \frac{1}{1-\psi(a,b)\psi(b,a)} = \frac{u(0)^2}{u(0)^2 - u(b-a)u(a-b)}.$$

Since $x \to \psi(x,y)$ is continuous, both $t \to \psi(X_t, a)$ and $t \to \psi(X_t, b)$ are right continuous. Moreover we claim that the term in square brackets in (2.9) is right continuous. It is clearly right continuous at any $t \neq \tau_{2k-1}$ for some $k \geqq 1$, but using $\psi(b,b) = 1$ one easily checks right continuity at τ_{2k-1}. Consequently the right side of (2.9) is a right continuous version of the potential Y defined in (2.5). From here on Y will denote the process on the right side of (2.9). Clearly Y is bounded.

Let

(2.11) $$k(b-a) \equiv [K(a,b)]^{-1} = \frac{u(0)^2 - u(b-a)u(a-b)}{u(0)^2}.$$

The $k(b-a) \to 0$ as $b-a \to 0$. From (2.9), $\psi(0,0) = 1$, and $I(t) + J(t) = 1$ we see that

(2.12) $$k(b-a) Y_t \to W_t \equiv e^{-t} \psi(X_t, 0)$$

as $b - a \to 0$ with $a \leqq 0 \leqq b$. Also writing $Y^{a,b} \equiv Y$ to denote its dependence on a and b

(2.13) $$k(b-a) Y_t^{a,b} - W_t = e^{-t} I(t) [\psi(b,a)\psi(X_t,b) - \psi(X_t,0)]$$
$$+ e^{-t} J(t) [\psi(X_t,a) - \psi(X_t,0)],$$

and so $|k(b-a) Y_t^{a,b} - W_t| \leqq 4$.

Next let $L_t = \int_0^t e^{-s} dl_s^0$ where $l^0 = (l_t^0)$ is the local time at 0. Then using (1.4) and (1.6)

(2.14) $$E^0\{L_\infty - L_t | \mathscr{F}_t\} = E^0\left\{\int_t^\infty e^{-s} dl_s^0 | \mathscr{F}_t\right\}$$
$$= e^{-t} u(-X_t) = u(0) W_t.$$

Finally let

(2.15) $$B_t^{a,b} \equiv u(0)\,k(b-a)\,A_t = u(0)\,k(b-a) \int_{(0,t]} e^{-s}\,dD(s).$$

Then $Z_t^{a,b} \equiv u(0)\,k(b-a)\,Y_t^{a,b}$ is a right continuous version of the potential generated by $B^{a,b}$, and by (2.12) and (2.14), $Z_t^{a,b} \to Z_t \equiv u(0)\,W_t$ as $b-a \to 0$ with $a \leqq 0 \leqq b$ and $Z=(Z_t)$ is the potential generated by $L=(L_t)$. Of course, Z is continuous.

Let $\delta(h)$ be the modulus of continuity of u. It now follows readily from (2.13) and the definitions of $Z^{a,b}$ and Z that

(2.16) $$\begin{aligned} &|Z_t^{a,b} - Z_t| \leqq K\,\delta(b-a) \\ &|Z_t^{a,b}| \leqq K \quad \text{and} \quad |Z_t| \leqq K, \end{aligned}$$

where K is a constant independent of a, b, and t. In the sequel K, K_1, K_2, etc. will always denote such a constant.

Now $B^{a,b}$ and L are integrable previsible increasing processes with potentials $Z^{a,b}$ and Z respectively. It follows from the energy inequality (T24, p. 116 of [9]) and (2.16) that $E^0([B_\infty^{a,b}]^2) \leqq 2K^2$ and $E(L_\infty^2) \leqq 2K^2$. It is now a standard argument (see the top of p. 126 of [9]) that

(2.17) $$\begin{aligned} E^0[(B_\infty^{a,b} - L_\infty)^2] &= E^0 \int_0^\infty (Z_t^{a,b} - Z_t + Z_{t-}^{a,b} - Z_{t-})\,d(B_t^{a,b} - L_t) \\ &\leqq 2K\,\delta(b-a)\,E^0(B_\infty^{a,b} + L_\infty) \leqq 4K^2\,\delta(b-a), \end{aligned}$$

where we have used (2.16) and the fact that $E^0(L_\infty) = E^0(Z_0)$ and $E^0(B_\infty^{a,b}) = E^0(Z_0^{a,b})$. Introducing the martingales

$$\begin{aligned} M_t^{a,b} &= E^0(B_\infty^{a,b} \mid \mathscr{F}_t) = Z_t^{a,b} + B_t^{a,b}, \\ M_t &= E^0(L_\infty \mid \mathscr{F}_t) = Z_t + L_t, \end{aligned}$$

it follows from Doob's maximal inequality and (2.17) that

(2.18) $$E^0(\sup_t |M_t^{a,b} - M_t|^2) \leqq 16\,K^2\,\delta(b-a),$$

and combining this with (2.16) it is easy to see that

(2.19) $$E^0[\sup_t |B_t^{a,b} - L_t|^2] \leqq K_1\,\delta(b-a).$$

Let (a^n) and (b^n) be sequences such that

(2.20) $$a^n \uparrow 0, \quad b^n \downarrow 0, \quad \varepsilon_n \equiv b^n - a^n > 0, \quad \text{and} \quad \sum \delta(b^n - a^n) = \sum \delta(\varepsilon_n) < \infty.$$

Now it is an immediate consequence of (2.19), the Čebyšev inequality, and the Borel-Cantelli lemma that

(2.21) $$\sup_t |B_t^{a^n,b^n} - L_t| \to 0$$

almost surely P^0 as $n \to \infty$ whenever (a^n) and (b^n) are sequences satisfying (2.20).

It remains to transform (2.19) and (2.21) into statements about $D^{a,b}$ and l^0. Using the definition of $B^{a,b}$ and L and integrating by parts one finds

$$l_t^0=\int_0^t e^s\,dL_s=e^t L_t-\int_0^t e^s L_s\,ds,$$

$$u(0)\,k(b-a)\,D_t^{a,b}=\int_{(0,t]} e^s\,dB_s^{a,b}=e^t B_t^{a,b}-\int_0^t e^s B_s^{a,b}\,ds.$$

Statements (iii) and (iv) of Theorem (1.9) are an immediate consequence of these relationships, (2.19), (2.21), and the following lemma.

(2.22) **Lemma.** *As* $x\to 0$,

$$u(0)\,k(x)\sim 2u(0)-(u(x)+u(-x))=\frac{1}{\pi}\int_{-\infty}^{\infty}(1-\cos x\lambda)\,\mathrm{Re}([1+\phi(\lambda)]^{-1})\,d\lambda.$$

Proof. Recall from (2.11) that $u(0)^2\,k(x)=u(0)^2-u(x)\,u(-x)$. Making use of the identity

$$\alpha^2-\beta\gamma=\left(\alpha-\frac{\beta+\gamma}{2}\right)\left(\alpha+\frac{\beta+\gamma}{2}\right)+\tfrac{1}{4}(\beta-\gamma)^2$$

valid for all real numbers α, β, and γ, we see that in light of (2.1) the desired conclusion will follow once we show that

$$\frac{[u(x)-u(-x)]^2}{2u(0)-u(x)-u(-x)}=\frac{[u(x)-u(-x)]\,[u(0)-u(-x)-(u(0)-u(x))]}{u(0)-u(-x)+u(0)-u(x)}\to 0$$

as $x\to 0$. But $u(0)>u(y)$ for all $y\neq 0$ and so the quantity in the previous display is dominated by $|u(x)-u(-x)|$ which approaches zero as $x\to 0$. Thus Lemma (2.22) and Theorem (1.9) are established.

Next we turn our attention to the examples. For (1.13) with $h=\beta\tan\dfrac{\pi\alpha}{2}$, $1<\alpha\leqq 2$

$$\begin{aligned}g(x)&=\frac{2}{\pi}\int_0^\infty\frac{(1-\cos x\lambda)(1+c\lambda^\alpha)}{(1+c\lambda^\alpha)^2+c^2h^2\lambda^{2\alpha}}\,d\lambda\\&=\frac{2x^{\alpha-1}}{\pi}\int_0^\infty\frac{(1-\cos\lambda)(x^\alpha+c\lambda^\alpha)\,d\lambda}{(x^\alpha+c\lambda^\alpha)^2+c^2h^2\lambda^{2\alpha}}\\&\sim\frac{2x^{\alpha-1}}{\pi}\int_0^\infty\frac{(1-\cos\lambda)\,d\lambda}{c(1+h^2)\,\lambda^\alpha}\\&=x^{\alpha-1}\left[c(1+h^2)\,\Gamma(\alpha)\sin\frac{(\alpha-1)\pi}{2}\right]^{-1}.\end{aligned}$$

Note that (1.12) is a special case of this: take $\alpha=2$ and $c=1/2$.

Lastly we consider (1.14). The computation here seems to be a bit more involved and since I do not know an explicit reference I shall give the argument. In this case

$$g(x)=\frac{2}{\pi}\int_0^\infty\frac{(1-\cos x\lambda)(1+c\lambda)\,d\lambda}{(1+c\lambda)^2+(c\beta\lambda\log\lambda)^2} \tag{2.23}$$

where $c>0$ and $0<|\beta|\leqq 2/\pi$. Let $L(x)=(-\log x)^{-1}=|\log x|^{-1}$ for $0<x<1$ and let $M(x)=(L(x))^{\alpha}=|\log x|^{-\alpha}$ where $1/2<\alpha<1$. Split the integral defining g in (2.23) into an integral over $(0, M(x)/x)$ denoted by $g_1(x)$ and an integral over $(M(x)/x, \infty)$ denoted by $g_2(x)$ so that $g(x)=g_1(x)+g_2(x)$. Using the inequality $1-\cos u\leqq Ku^2$ it is easy to see that

$$g_1(x)\leqq Kx^2 \int_0^{M(x)/x} \lambda\, d\lambda \leqq KM(x)^2=K(L(x))^{2\alpha}$$

and consequently $g_1(x)=o(L(x))$ as $x\downarrow 0$ since $\alpha>1/2$. Next write $g_2=g_3+g_4$ where

$$g_3(x)=\frac{2}{\pi}\int_{M(x)/x}^{\infty} \frac{(1-\cos\lambda x)\, d\lambda}{(1+c\lambda)^2+(c\beta\lambda\log\lambda)^2}\leqq K\int_{M(x)/x}^{\infty}\frac{d\lambda}{\lambda^2}=K\frac{x}{M(x)},$$

and so $g_3(x)=o(L(x))$. Thus it remains to study

$$\text{(2.24)}\quad g_4(x)=\frac{2c}{\pi}\int_{M(x)/x}^{\infty}\frac{\lambda(1-\cos x\lambda)\, d\lambda}{(1+c\lambda)^2+(c\beta\lambda\log\lambda)^2}.$$

The following lemma is the heart of the matter.

(2.25) **Lemma.** *Let*

$$h(x)=\int_{M(x)/x}^{\infty}\frac{(1-\cos\lambda x)\, d\lambda}{\lambda[1+(\gamma\log\lambda)^2]}\quad \textit{where } \gamma\neq 0.$$

Then $h(x)\sim\gamma^{-2}L(x)$ *as* $x\downarrow 0$.

Proof. As $x\to 0$, $M(x)/x\to\infty$ and so we shall suppose throughout this proof that x is small enough so that $\log\lambda\geqq 1$ for $\lambda\geqq M(x)/x$. Write $h=h_1-h_2$ where

$$h_1(x)=\int_{M(x)/x}^{\infty}\frac{d\lambda}{\lambda[1+(\gamma\log\lambda)^2]};\qquad h_2(x)=\int_{M(x)/x}^{\infty}\frac{\cos x\lambda\, dx}{\lambda[1+(\gamma\log\lambda)^2]}.$$

In h_1 make the change of variable $u=\log\lambda$ to obtain

$$h_1(x)=\int_{\log(M(x)/x)}^{\infty}\frac{du}{1+\gamma^2u^2}=\gamma^{-1}[\pi/2-\arctan(\gamma\log M(x)/x)].$$

But $\pi/2-\arctan t\sim t^{-1}$ as $t\to\infty$ and $(\log M(x)/x)^{-1}\sim(-\log x)^{-1}=L(x)$ as $x\to 0$, and so $h_1(x)\sim\gamma^{-2}L(x)$ as $x\to 0$. In h_2 we integrate by parts, integrating $\cos x\lambda$ and differentiating $(\lambda[1+(\gamma\log\lambda)^2])^{-1}$. We obtain a "boundary term" $B(x)$ and a new integral $I(x)$.

$$\begin{aligned}-B(x)&=\frac{\sin M(x)}{x}\left\{\frac{M(x)}{x}\left[1+\left(\gamma\log\frac{M(x)}{x}\right)^2\right]\right\}^{-1}\\&=\frac{\sin M(x)}{M(x)}\left[1+\left(\gamma\log\frac{M(x)}{x}\right)^2\right]^{-1}\\&\sim\left(\gamma\log\frac{M(x)}{x}\right)^{-2}\sim\gamma^{-2}(L(x))^2\qquad\text{as } x\to 0.\end{aligned}$$

Consequently $B(x)=o(L(x))$ as $x\to 0$. Finally

$$I(x)=\int_{M(x)/x}^{\infty}\frac{\sin x\lambda}{x}\,\frac{1+(\gamma\log\lambda)^2+2\gamma^2\log\lambda}{\lambda^2[1+(\gamma\log\lambda)^2]^2}\,d\lambda.$$

But $\log\lambda\leqq(\log\lambda)^2$ since $\log\lambda\geqq 1$, and so

$$|I(x)|\leqq\frac{K}{x}\int_{M(x)/x}^{\infty}\frac{(\gamma\log\lambda)^2\,d\lambda}{\lambda^2[1+(\gamma\log\lambda)^2]^2}\leqq\frac{K}{x}\int_{M(x)/x}^{\infty}\frac{d\lambda}{\lambda^2[1+(\gamma\log\lambda)^2]}$$

$$\leqq\frac{K}{x}(\gamma\log M(x)/x)^{-2}\,\frac{x}{M(x)}\sim K\gamma^{-2}\,\frac{L(x)^2}{M(x)}=K\gamma^{-2}\,L(x)^{2-\alpha}.$$

Consequently $|I(x)|/L(x)\to 0$ as $x\to 0$ since $\alpha<1$. This completes the proof of Lemma 2.25.

We leave it to the reader to verify that (2.25) implies that g_4 defined in (2.24) satisfies $g_4(x)\sim 2(\pi c\beta^2)^{-1}L(x)$. This completes the verification in example (1.14).

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