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Limiting Behavior of *U*-Statistics for Stationary, Absolutely Regular Processes

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Let $\{\xi_i\}$ be a strictly stationary, absolutely regular processes, i.e., the process satisfying the condition

$$\beta(n)=E\{\sup_{A\in\mathscr{M}_n^\infty}|P\{A\,|\,\mathscr{M}_{-\infty}^0\}-P\{A\}|\}\downarrow 0 \qquad (n\to\infty)$$

where $\mathscr{M}_a^b (a\leqq b)$ is the σ-algebra of events generated by $\xi_a, \ldots, \xi_b$.

If the suitable processes are constructed from the sequence of *W.* Hoeffding's [Ann. Math. Statistics 19, 293–325 (1947; this Zbl. 32, 41)] *U*-statistics for the absolutely regular processes, then weak convergence to Brownian motion processes and the Strassen's version of the law of the iterated logarithm [Z. Wahrscheinlichkeitstheorie verw. Gebiete 3, 211–226 (1964, this Zbl. 132, 129)] are established. The results are extensions of Sen's ones [ibid. 25, 71–82 (1972; this Zbl. 238, 6097)]. Weak convergence of the processes constructed by generalized *U*-statistics analogous to Sen's [Ann. Probab. 2, 90–102 (1974)] and almost sure invariance principle and integral tests [Sen; Ann. Statistics 2, 387–395 (1974), Jain et al; Ann. Probab. 3, 119–145, (1975)] for *U*-statistics defined by some ϕ-mixing sequences are considered. Analogous problems for R. von Mises' [Ann. Math. Statistics 18, 309–348 (1947: this Zbl. 37, 84)] differentiable statistical functionals are also treated.

1. Introduction

Let $\{\xi_i, -\infty<i<\infty\}$ be a p-dimensional strictly stationary sequence of stochastic vectors defined on a probability space $(\Omega, \mathscr{A}, P)$. For $a\leqq b$, let $\mathscr{M}_a^b$ denote the σ-algebra of events generated by $\xi_a, \ldots, \xi_b$. As in [3], we shall say that the sequence is absolutely regular, if

$$\beta(n)=E\{\sup_{A\in\mathscr{M}_n^\infty}|P\{A\,|\,\mathscr{M}_{-\infty}^0\}-P\{A\}|\}\downarrow 0 \tag{1.1}$$

as $n\to\infty$ (see [4] and [11]). In [11], Rozanov and Volkonskii found that

$$\beta(n)=\tfrac{1}{2}V(P_{0n}, P_{1n})=\tfrac{1}{2}\operatorname{Var}[P_{0n}-P_{1n}] \tag{1.2}$$

where P_{0n} is the measure induced by the process $\{\xi_n\}$ on the σ-algebra $\mathcal{M}_{-\infty}^{0} \cup \mathcal{M}_n^{\infty}$, and P_{1n} the measure defined for $A \in \mathcal{M}_n^{\infty}$, $B \in \mathcal{M}_{-\infty}^{0}$ by the equality

$$P_{1n}(A \cap B) = P_{0n}(A)\, P_{0n}(B). \tag{1.3}$$

Further, we shall say that $\{\xi_i\}$ satisfies the ϕ-mixing condition if

$$\phi(n) = \sup_{B \in \mathcal{M}_{-\infty}^{0},\, A \in \mathcal{M}_n^{\infty}} \frac{1}{P_{0n}(B)} |P_{0n}(A \cap B) - P_{1n}(A \cap B)| \downarrow 0 \tag{1.4}$$

and that $\{\xi_i\}$ satisfies the strong mixing condition, if

$$\alpha(n) = \sup_{B \in \mathcal{M}_{-\infty}^{0},\, A \in \mathcal{M}_n^{\infty}} |P_{0n}(A \cap B) - P_{1n}(A \cap B)| \downarrow 0. \tag{1.5}$$

Since $\alpha(n) \leqq \beta(n) \leqq \phi(n)$, it follows that if $\{\xi_i\}$ is ϕ-mixing, then, it is absolutely regular and if $\{\xi_i\}$ is absolutely regular, then it is strong mixing. By the way, we note that in [4] Ibragimov and Solev obtained a complete description of stationary Gaussian processes satisfying Condition (1.1).

Next, we denote the distribution function (df) of ξ_i by $F(x)$, $x \in R^p$, the p-dimensional Euclidean space. Consider a functional

$$\theta(F) = \int \cdots \int_{R^{mp}} g(x_1, \dots, x_m)\, dF(x_1) \dots dF(x_m) \tag{1.6}$$

defined over $\mathscr{F} = \{F: |\theta(F)| < \infty\}$, where $g(x_1, \dots, x_m)$ is symmetric in its $m(\geqq 1)$ arguments. As an estimator of $\theta(F)$, we define a U-statistic

$$U_n = \binom{n}{m}^{-1} \sum_{(i)}^{(n)} g(\xi_{i_1}, \dots, \xi_{i_m}), \qquad n \geqq m \tag{1.7}$$

where the summation $\sum_{(i)}^{(n)}$ extends over all possible $1 \leqq i_1 < \cdots < i_m \leqq n$. As another estimator of $\theta(F)$, we shall consider a von Mises' differentiable statistical functional $\theta(F_n)$ defined by

$$\begin{aligned} \theta(F_n) &= \int \cdots \int_{R^{mp}} g(x_1, \dots, x_m)\, dF_n(x_1) \dots dF_n(x_m) \\ &= n^{-m} \sum_{i_1=1}^{n} \cdots \sum_{i_m=1}^{n} g(\xi_{i_1}, \dots, \xi_{i_m}). \end{aligned} \tag{1.8}$$

In [6], Miller and Sen proved a Donsker-type invariance principle for one-sample U-statistics. But, the treatment, they used, does not work out when we consider U-statistics for general weakly dependent processes. In [12], Sen proved a weak convergence theorem and the law of iterated logarithm for U-statistics defined by $*$-mixing processes. The proof rested on certain basic lemmas on Bernoullian random variables in a $*$-mixing process and those lemmas did not hold for general ϕ-mixing processes and hence, the same technique of proof was not applicable for the latter processes.

On the other hand, in [13], Sen extended Strassens invariance principle on the a.s. convergence of partial sums of independent random variables to a class of $\{U_n\}$ and $\{\theta(F_n)\}$.

In this paper, proving a fundamental lemma (Lemma 1), we shall extend above results to a broad class of $\{U_n\}$ and $\{\theta(F_n)\}$ for strictly stationary, absolutely regular processes. More specifically, we shall prove the following theorems under suitable conditions on g and $\beta(n)$:

(i) asymptotic normality of $n^{\frac{1}{2}}(U_n-\theta(F))$ and $n^{\frac{1}{2}}(\theta(F_n)-\theta(F))$ (Theorem 1),

(ii) weak convergence of continuous sample versions of the processes $\{n^{-\frac{1}{2}}k(U_k-\theta(F)), k\geqq m\}$ and $\{n^{-\frac{1}{2}}k(\theta(F_k)-\theta(F)), k\geqq 1\}$ to processes of Brownian motion (Theorem 2),

(iii) Strassen's versions of the law of the iterated logarithm for U_k and $\theta(F_n)$ (Theorem 3),

(iv) almost sure invariance principles and integral tests for U_n and $\theta(F_n)$ defined by some strictly stationary ϕ-mixing processes (Theorems 4, 5, 6).

In Section 6, we shall extend Theorem 2 to the case of generalized U-statistics. The results are also generalizations of Sen's theorems in [13] (Theorems 7 and 8).

2. Basic Lemmas

In what follows we suppose that $\{\xi_i\}$ is a p-dimensional strictly stationary, absolutely regular process with $df\, F(x)$.

As in [12], for every $c(0\leqq c\leqq m)$, let

$$g_c(x_1, \ldots, x_c)=\int\underset{R^{(m-c)p}}{\cdots}\int g(x_1, \ldots, x_m)\, dF(x_{c+1}) \ldots dF(x_m) \tag{2.1}$$

so that $g_0=\theta(F)$ and $g_m=g$. Let

$$\sigma^2=\sigma^2(F)=\{Eg_1^2(\xi_1)-\theta^2(F)\}+2\sum_{k=1}^{\infty}\{Eg_1(\xi_1)\, g_1(\xi_{k+1})-\theta^2(F)\}. \tag{2.2}$$

We assume that for some $r>2$ (i)

$$\mu_r=\int\underset{R^{pm}}{\cdots}\int|g_1(x_1, \ldots, x_m)|^r\, dF(x_1) \ldots dF(x_m)\leqq M_0<\infty \tag{2.3}$$

and (ii) for all integers $i_1, i_2, \ldots, i_m (i_1<i_2<\cdots<i_m)$

$$v_r=E|g(\xi_{i_1}, \xi_{i_2}, \ldots, \xi_{i_m})|^r\leqq M_0<\infty. \tag{2.4}$$

Let $i_1<i_2<\cdots<i_k$ be arbitrary integers. For any $j(1\leqq j\leqq k-1)$, put

$$P_j^{(k)}(E^{(j)}\times E^{(k-j)})=P(\xi_{i_1}, \ldots, \xi_{i_j})\in E^{(j)})\, P((\xi_{i_{j+1}}, \ldots, \xi_{i_k})\in E^{(k-j)}) \tag{2.5}$$

and

$$P_0^{(k)}(E^{(k)})=P((\xi_{i_1}, \ldots, \xi_{i_k})\in E^{(k)}) \tag{2.6}$$

where $E^{(i)}$ is a Borel set in R^{ip}.

From now on, we shall agree to denote by the letter M some quantity bounded in absolute value.

Lemma 1. *For any $j(0\leqq j\leqq k-1)$, let $h(x_1, \ldots, x_k)$ be a Borel function such that*

$$\int\underset{R^{kp}}{\cdots}\int|h(x_1, \ldots, x_k)|^{1+\delta}\, dP_j^{(k)}\leqq M$$

for some $\delta>0$. *Then*

$$|\int_{R^{kp}}\cdots\int h(x_1,\dots,x_k)\,dP_0^{(k)}-\int_{R^{kp}}\cdots\int h(x_1,\dots,x_k)\,dP_j^{(k)}|$$
$$\leqq 4M^{1/1+\delta}\beta^{\delta/1+\delta}(i_{j+1}-i_j). \tag{2.7}$$

Proof. Let $j(1\leqq j\leqq k)$ be fixed. Let $\gamma=1/(1+\delta)$ and put

$$B=\{(x_1,\dots,x_k):\ |h(x_1,\dots,x_k)|\leqq M^\gamma\beta^{-\gamma}(d)\}$$

where $d=i_{j+1}-i_j$. Then, it follows from the definition of absolute regularity that

$$|\int_B\cdots\int h(x_1,\dots,x_k)\,dP_0^{(k)}-\int_B\cdots\int h(x_1,\dots,x_k)\,dP_j^{(k)}|$$
$$\leqq M^\gamma\beta^{-\gamma}(d)\,V(P_0^{(k)},P_j^{(k)})\leqq 2M^\gamma\beta^{1-\gamma}(d)=2M^\gamma\beta^{\gamma\delta}(d). \tag{2.8}$$

Next, let B' be the complementary set of B. Then

$$|\int_{B'}\cdots\int h(x_1,\dots,x_k)\,dP_i^{(k)}|\leqq M^{-\gamma\delta}\beta^{\gamma\delta}(d)\int_{B'}\cdots\int|h(x_1,\dots,x_k)|^{1+\delta}\,dP_i^{(k)}$$
$$\leqq M^{1-\gamma\delta}\beta^{\gamma\delta}(d)=M^\gamma\beta^{\gamma\delta}(d)\qquad(i=0,j). \tag{2.9}$$

Combining (2.8) and (2.9), we have the lemma.

Let

$$n^{-[r]}=\{n(n-1)\dots(n-r+1)\}^{-1}.$$

As in [12], we put

$$U_n^{(c)}=n^{-[c]}\sum_{1\leqq i_1<\dots<i_c\leqq n}\int_{R^{cp}}\cdots\int g_c(x_1,\dots,x_c)\prod_{j=1}^{c}d[u(x_j-\xi_{i_j})-F(x_j)] \tag{2.10}$$

where $u(v)$ is equal to one when all the p components of v are non-negative; otherwise, $u(v)=0$. Then

$$U_n=\theta(F)+\sum_{c=1}^{m}\binom{m}{c}U_n^{(c)}. \tag{2.11}$$

Lemma 2. *If there is a positive number* δ *such that for* $r=2+\delta$ (2.3) *and* (2.4) *hold, and for some* $\delta'(0<\delta'<\delta)\beta(n)=O(n^{-(2+\delta')/\delta'})$, *then we have*

$$E(U_n^{(c)})^2=O(n^{-1-\gamma})\qquad(2\leqq c\leqq m) \tag{2.12}$$

where $\gamma=2(\delta-\delta')/\delta'(2+\delta)>0$.

Proof. We shall only consider the case $c=2$. The proofs in the cases $c=3,\dots,m$ are analogous and so are omitted.

We first note that

$$U_n^{(2)}=n^{-[2]}\sum_{1\leqq i_1<i_2\leqq n}\{g_2(\xi_{i_1},\xi_{i_2})-g_1(\xi_{i_1})-g_1(\xi_{i_2})+\theta(F)\}.$$

So, we have

$$E(U_n^{(2)})^2=\sum_{1\leqq i_1<i_2\leqq n}\ \sum_{1\leqq j_1<j_2\leqq n}J((i_1,i_2),(j_1,j_2)) \tag{2.13}$$

where

$$J((i_1,i_2),(j_1,j_2))=E\{g_2(\xi_{i_1},\xi_{i_2})-g_1(\xi_{i_1})-g_1(\xi_{i_2})+\theta(F)\}\cdot\{g_2(\xi_{j_1},\xi_{j_2})-g_1(\xi_{j_1})-g_1(\xi_{j_2})+\theta(F)\}. \tag{2.14}$$

Since

$$\int\underset{R^p}{\cdots}\int\{g_2(x,y)-g_1(x)-g_1(y)+\theta(F)\}\,dF(x)=0,$$

so from Lemma 1 we have the following inequalities:

(i) If $1\leqq i_1<i_2\leqq j_1<j_2\leqq n$ and $j_2-j_1\geqq i_2-i_1$, then

$$J((i_1,i_2),(j_1,j_2))\leqq M\beta^{\delta/2+\delta}(j_2-j_1) \tag{2.15}$$

and similarly, if $1\leqq i_1<i_2\leqq j_1<j_2\leqq n$ and $i_2-i_1\geqq j_2-j_1$, then

$$J((i_1,i_2),(j_1,j_2))\leqq M\beta^{\delta/2+\delta}(i_2-i_1). \tag{2.16}$$

Thus, from (2.15), (2.16) and the assumption on $\beta(n)$

$$\begin{aligned}&\Big|\sum_{1\leqq i_1<i_2\leqq j_1<j_2\leqq n}J((i_1,i_2),(j_1,j_2))\Big|\\&\leqq\Big\{\sum_{\substack{1\leqq i_1<i_2\leqq j_1<j_2\leqq n\\ i_2-i_1\geqq j_2-j_1}}+\sum_{\substack{1\leqq i_1<i_2\leqq j_1<j_2\leqq n\\ i_2-i_1\leqq j_2-j_1}}\Big\}|J((i_1,i_2),(j_1,j_2))|\\&\leqq Mn^2\sum_{k=1}^{n}(k+1)\beta^{\delta/2+\delta}(k)=O(n^{3-\gamma}).\end{aligned} \tag{2.17}$$

(ii) Similarly, we have

$$\begin{aligned}&\Big|\sum_{1\leqq i_1<j_1\leqq i_2<j_2\leqq n}J((i_1,i_2),(j_1,j_2))\Big|\\&\leqq\Big\{\sum_{\substack{1\leqq i_1<j_1\leqq i_2<j_2\leqq n\\ j_1-i_1\geqq j_2-i_2}}+\sum_{\substack{1\leqq i_1<j_1\leqq i_2<j_2\leqq n\\ j_1-i_1\leqq j_2-i_2}}\Big\}|J((i_1,i_2),(j_1,j_2))|=O(n^{3-\gamma}),\end{aligned} \tag{2.18}$$

$$\begin{aligned}&\Big|\sum_{1\leqq i_1<j_1<j_2<i_2\leqq n}J((i_1,i_2),(j_1,j_2))\Big|\\&\leqq\Big\{\sum_{\substack{1\leqq i_1<j_1<j_2<i_2\leqq n\\ j_1-i_1\geqq i_2-j_2}}+\sum_{\substack{1\leqq i_1<j_1<j_2<i_2\leqq n\\ j_1-i_1\leqq i_2-j_2}}\Big\}|J((i_1,i_2),(j_1,j_2))|=O(n^{3-\gamma}),\end{aligned} \tag{2.19}$$

$$\begin{aligned}&\left|\sum_{1\leqq i_1,j_1\leqq n}\sum_{i_2=1}^{n}J((i_1,i_2),(j_1,i_2))\right|\\&\leqq\sum_{i_1=1}^{n}\sum_{i_2=1}^{n}J((i_1,i_2),(i_1,i_2))+2\sum_{1\leqq i_1<j_1\leqq n}\sum_{i_2=1}^{n}|J((i_1,i_2),(j_1,i_2))|\\&\leqq Mn^2\left(1+\sum_{k=1}^{n}\beta^{\delta/2+\delta}(k)\right)=O(n^2),\end{aligned} \tag{2.20}$$

$$\left|\sum_{1\leqq i_2,j_2\leqq n}\sum_{i_1=1}^{n}J((i_1,i_2),(i_1,j_2))\right|\leqq Mn^2\sum_{k=1}^{n}\beta^{\delta/2+\delta}(k)=O(n^2). \tag{2.21}$$

Hence, from (2.17)–(2.21) and (2.13), we have (2.12) with $c=2$.

Lemma 3. *If there is a positive number* δ *such that for* $r=4+\delta$ (2.3) *and* (2.4) *hold and for some* $\delta'(0<\delta'<\delta)\beta(n)=O(n^{-3(4+\delta')/(2+\delta')})$, *then we have*

$$E(U_n^{(2)})^4=O(n^{-3-\gamma'}) \tag{2.22}$$

where $\gamma'=6(\delta-\delta')/(4+\delta)(2+\delta')>0$ *and*

$$E(U_n^{(c)})^2=O(n^{-3}) \quad (3\leqq c\leqq m). \tag{2.23}$$

Proof. Let $i_{rs}(\leqq n)$ $(r=1,\dots,4;\ s=1,2)$ be mutually different positive integers. Reorder $\{i_{rs}\}$ as

$$1\leqq k_1<k_2<\cdots<k_8\leqq n$$

and put

$$E\left[\prod_{j=1}^{4}\{g_2(\xi_{i_{j1}},\xi_{i_{j2}})-g_1(\xi_{i_{j1}})-g_1(\xi_{i_{j2}})+\theta(F)\}\right]$$
$$=E[H(\xi_{k_1},\dots,\xi_{k_8})]=J(k_1,\dots,k_8). \tag{2.24}$$

Let $d^{(c)}$ be the c-th largest difference among $(k_{j+1}-k_j)$ $(j=1,\dots,7)$. Since

$$\int_{R^{8p}}\cdots\int H(x_1,\dots,x_8)\,dP_i^{(8)}(x_1,\dots,x_8)=0 \quad (i=1,7),$$

so from Lemma 1

$$J(k_1,\dots,k_8)\leqq M\beta^{2+\delta/4+\delta}(k_8-k_7) \quad \text{if } k_8-k_7=d^{(1)} \tag{2.25}$$

and

$$J(k_1,\dots,k_8)\leqq M\beta^{2+\delta/4+\delta}(k_2-k_1) \quad \text{if } k_2-k_1=d^{(1)}. \tag{2.26}$$

Hence

$$\sum_{\substack{1\leqq k_1<\dots<k_8\leqq n\\ k_8-k_7=d^{(1)} \text{ or } k_2-k_1=d^{(1)}}} J(k_1,\dots,k_8)\leqq Mn^4\sum_{j=1}^{n}(j+1)^3\beta^{2+\delta/4+\delta}(j). \tag{2.27}$$

If for some $j_\alpha(2\leqq j_\alpha\leqq 6;\ 1\leqq\alpha\leqq 4)$ $k_{j_\alpha+1}-k_{j_\alpha}=d^{(\alpha)}$ $(1\leqq\alpha\leqq 4)$, then from Lemma 1

$$J(k_1,\dots,k_8)\leqq M\sum_{\alpha=1}^{4}\beta^{2+\delta/4+\delta}(k_{j_\alpha+1}-k_{j_\alpha}), \tag{2.28}$$

and hence

$$\sum_{\substack{1\leqq k_1<\dots<k_8\leqq n\\ k_{j_\alpha+1}-k_{j_\alpha}=d^{(\alpha)}(1\leqq\alpha\leqq 4)}} J(k_1,\dots,k_8)\leqq 4Mn^4\sum_{j=1}^{n}(j+1)^3\beta^{2+\delta/4+\delta}(j). \tag{2.29}$$

Consequently

$$\sum_{1\leqq k_1<\dots<k_8\leqq n} J(k_1,\dots,k_8)\leqq Mn^4\sum_{j=1}^{n}(j+1)^3\beta^{2+\delta/4+\delta}(j)\leqq Mn^{5-\gamma'}. \tag{2.30}$$

We can use the similar method to estimate the sums in the other cases, and so we have (2.22).

The proof of (2.23) is analogous and so is omitted.

Finally, we define for every $c(1 \leqq c \leqq m)$

$$V_n^{(c)} = \int \underset{R^{cp}}{\cdots} \int g_c(x_1, \ldots, x_c) \prod_{j=1}^{c} d[F_n(x_j) - F(x_j)]. \tag{2.31}$$

Then, we have

$$\theta(F_n) = \theta(F) + \sum_{c=1}^{m} \binom{m}{c} V_n^{(c)}, \quad n \geqq 1 \tag{2.32}$$

(cf. [12]). Note that

$$V_n^{(1)} = U_n^{(1)} = n^{-1} \sum_{i=1}^{n} \{g_1(\xi_i) - \theta(F)\}. \tag{2.33}$$

The proofs of the following lemmas are the same as those of Lemmas 2 and 3, respectively.

Lemma 4. *If the conditions of Lemma 2 are satisfied, then*

$$E(V_n^{(c)})^2 = O(n^{-1-\gamma}) \quad (1 \leqq c \leqq m) \tag{2.34}$$

where γ is the same number in (2.12).

Lemma 5. *If the conditions of Lemma 3 are satisfied, then*

$$E(V_n^{(2)})^4 = O(n^{-3-\gamma'}) \tag{2.35}$$

where γ' is the same number in (2.22)

$$E(V_n^{(c)})^2 = O(n^{-3}) \quad (3 \leqq c \leqq m), \tag{2.36}$$

3. Weak Convergence of U_n and $\theta(F_n)$

The following theorem is an extension of Theorem 1 in [12].

Theorem 1. *If there is a positive number δ such that for $r = 2+\delta$ (2.3) and (2.4) hold and*

$$\beta(n) = O(n^{-(2+\delta')/\delta'}) \quad \textit{for some } \delta'(0 < \delta' < \delta) \tag{3.1}$$

then the series (2.2) converges absolutely; if $\sigma^2 > 0$ holds,

$$\lim_{n\to\infty} P\{n^{\frac{1}{2}}(U_n - \theta(F)) \leqq z m \sigma\} = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{z} e^{-\frac{t^2}{2}} dt \tag{3.2}$$

for all $z(-\infty < z < \infty)$ and

$$n^{\frac{1}{2}} |\theta(F_n) - U_n| \to 0 \textit{ in probability}. \tag{3.3}$$

Hence, (3.2) also holds for U_n being replaced by $\theta(F_n)$.

Proof. Since from the central limit theorem for strong mixing (and hence, absolutely regular) processes (cf. [8] $m n^{\frac{1}{2}} U^{(1)}$ converges in law to a normal distribution with mean zero and variance $m^2 \sigma^2$, so the proof of Theorem 1 is obtained from Lemmas 2 and 4 using the method of the proof of Theorem 1 in [12].

Secondly, let C be the space of all continuous real-valued functions on $[0, 1]$, where we give C the uniform topology, i.e., for $g, h \in C$

$$\rho(g, h) = \sup_{0 \leqq t \leqq 1} |g(t) - h(t)|. \tag{3.4}$$

Let $\sigma > 0$. For every $n \geqq m$, let $X_n = \{X_n(t), 0 \leqq t \leqq 1\}$ be a random element in C defined by

$$X_n(t) = \begin{cases} 0 \quad \text{for } 0 \leqq t \leqq (m-1)/n, \\ k(U_k - \theta(F))/(m\sigma n^{\frac{1}{2}}) \quad \text{for } t = k/n, \quad m \leqq k \leqq n, \\ \text{linearly interpolated for} \quad t \in [k/n, (k+1)/n], \quad m-1 \leqq k \leqq n-1. \end{cases} \tag{3.5}$$

Similarly, let $X_n^* = \{X_n^*(t), 0 \leqq t \leqq 1\}$ be a random element in C defined by

$$X_n^*(t) = \begin{cases} 0 \quad \text{for } t = 0, \\ k(\theta(F_k) - \theta(F))/(m\sigma n^{\frac{1}{2}}) \quad \text{for } t = k/n, \quad 1 \leqq k \leqq n, \\ \text{linearly interpolated for} \quad t \in [k/n, (k+1)/n], \quad 0 \leqq k \leqq n-1. \end{cases} \tag{3.6}$$

Let $W = \{W(t), 0 \leqq t \leqq 1\}$ be a standard Brownian motion.

The following theorem is an extension of Theorem 3 in [12].

Theorem 2. *If there is a positive number δ such that for $r = 4 + \delta$ (2.3) and (2.4) hold, and for some δ' $(0 < \delta' < \delta)$*

$$\beta(n) = O(n^{-3(4+\delta')/(2+\delta')}) \tag{3.7}$$

then, both X_n and X_n^* converge weakly to W and as $n \to \infty$ $\rho(X_n, X_n^*) \to 0$ in probability.

Proof. From Theorem 1 in [8], it follows that $X_n^0 \xrightarrow{\mathscr{D}} W$ as $n \to \infty$, where $X_n^0 = \{X_n^0(t);\ 0 \leqq t \leqq 1\}$ is a random element in C, defined by

$$X_n^0(t) = \begin{cases} 0 \quad \text{for } t = 0 \\ m k U_k^{(1)}/(\sigma n^{\frac{1}{2}}) \quad \text{for } t = k/n, \quad 1 \leqq k \leqq n \\ \text{linearly interpolated for} \quad t \in [k/n, (k+1)/n], \quad 0 \leqq k \leqq n-1. \end{cases} \tag{3.8}$$

So the proof of Theorem 2 is completed, since

$$\rho(X_n, X_n^0) \xrightarrow{P} 0 \quad \text{and} \quad \rho(X_n^*, X_n^0) \xrightarrow{P} 0 \tag{3.9}$$

are proved from Lemmas 3 and 5, using the method of the proof of Theorem 3 in [12].

4. Strassen's Versions of the Iterated Longarithm for U_n and $\theta(F_n)$

Let $C_0 (\subset C)$ be the space of continuous functions on $[0, 1]$ vanishing at 0, with the uniform topology and for each $\omega \in \Omega$, define the functions $Y_n(t, \omega)$ and $Y_n^*(t, \omega)$ in C_0 as follows:

$$Y_n(t, \omega) = \frac{X_n(t, \omega)}{(2 \log\log n\sigma^2)^{\frac{1}{2}}}, \quad n \geqq \max(m, 3/\sigma^2) \tag{4.1}$$

and

$$Y_n^*(t,\omega)=\frac{X_n^*(t,\omega)}{(2\log\log n\sigma^2)^{\frac{1}{2}}},\qquad n\geqq 3/\sigma^2 \tag{4.2}$$

We denote by K the subset of C_0 consisting of all functions $h(t)$ absolutely continuous with respect to Lebesgue measure such that

$$\int_0^1 \dot{h}^2(t)\,dt\leqq 1, \tag{4.3}$$

where $\dot{h}(t)$ stands for the Radon-Nikodym derivative of h. The following theorem is an extension of Theorem 2 in [12].

Theorem 3. *If the conditions in Theorem 2 are satisfied, then for almost all $\omega\in\Omega$, the sequences of functions $\{Y_n(t,\omega),\ n\geqq\max(m,3/\sigma^2)\}$ and $\{Y_n^*(t,\omega), n\geqq 3/\sigma^2\}$ are precompact in C_0 and their derived sets coincides with the set K. Furthermore, $\rho(Y_n, Y_n^*)\to 0$ with probability one.*

Proof. Since $\{g_1(\xi_i), -\infty<i<\infty\}$ is strong mixing and satisfies Concition (IV) of Theorem 1 in [7], we have that for almost every $\omega\in\Omega$, the sequence of functions $\{Y_n^0(t,\omega),\ n\geqq\max(m,3/\sigma^2)\}$ is precompact in C_0 and its derived set is K, where

$$Y_n^0(t,\omega)=\frac{X_n^0(t,\omega)}{(2\log\log n\sigma^2)^{\frac{1}{2}}}. \tag{4.4}$$

Thus, it suffices to prove

$$P\big(\lim_{n\to\infty}\rho(Y_n, Y_n^0)=0\big)=1 \tag{4.5}$$

and

$$P\big(\lim_{n\to\infty}\rho(Y_n^*, Y_n^0)=0\big)=1. \tag{4.6}$$

We shall only prove (4.5). The proof of (4.6) is analogous. To prove (4.5), it is enough to show that for every $\varepsilon>0$

$$P(|Z_n|>\varepsilon\chi(n)\text{ i.o.})=0 \tag{4.7}$$

where

$$Z_n=n(U_n-\theta(F)-m\,U_n^{(1)})=n\sum_{c=2}^{m}\binom{m}{c}U_n^{(c)} \tag{4.8}$$

and

$$P\left(\max_{0\leqq j\leqq m} m\left|\sum_{i=0}^{j}(g_1(\xi_i)-\theta(F))\right|>\varepsilon\chi(n)\text{ i.o.}\right)=0 \tag{4.9}$$

where

$$\chi(n)=(2\sigma^2\log\log n\sigma^2)^{\frac{1}{2}}$$

As (4.9) is obvious, we shall only prove (4.7). Let

$$n_k=[k^{(2+\delta')(4+\delta)/3(\delta-\delta')}]$$

and $n_{k_0} \geqq m$. Then, from Lemma 3 and the Bonferonni inequality,

$$\sum_{k=k_0}^{\infty} P(\max_{m \leqq n \leqq n_k} |Z_n| > \varepsilon \chi(n_k)) \leqq \sum_{k=k_0}^{\infty} \left(\sum_{n=m}^{n_k} P(|Z_n| > \varepsilon \chi(n_k)) \right) \leqq M \sum_{k=k_0}^{\infty} k^{-2} < \infty$$

and so from the Borel-Cantelli lemma we have

$$P(|Z_n| > \varepsilon \chi(n) \text{ i.o.}) \leqq P(\max_{n_k \leqq n \leqq n_{k+1}} |Z_n| > \varepsilon \chi(n_k) \text{ i.o.})$$

$$\leqq P\left(\max_{m \leqq n \leqq n_{k+1}} |Z_n| > \frac{\varepsilon}{4} \chi(n_{k+1}) \text{ i.o.} \right) = 0,$$

which implies (4.5). So, we have the theorem.

5. Almost Sure Invariance Principles and Integral Tests of U_n and $\theta(F_n)$ for Some ϕ-Mixing Processes

In this section, we assume that $\{\xi_j\}$ is a p-dimensional, strictly stationary, ϕ-mixing sequence of stochastic vectors with $\sum \phi^{\frac{1}{2}}(n) < \infty$. If (2.3) and (2.4) hold for some $r = 4 + \delta (\delta > 0)$, then $\{g_1(\xi_j) - \theta(F)\}$ is a strictly stationary ϕ-mixing sequence of random variables with $\sum \phi^{\frac{1}{2}}(n) < \infty$ for which

$$E\{g_1(\xi_1) - \theta(F)\} = 0, \tag{5.1}$$

and

$$E|g_1(\xi_1) - \theta(F)|^{4+\delta} < \infty. \tag{5.2}$$

So, we can use the martingale approximation method in [2, 5] and [9], from which we have the following:

Let T be an ergodic one to one measure preserving transformation defined on the probability space $(\Omega, \mathscr{A}, P)$. Write $L_2(P)$ for the Hilbert space of random variables with finite second moment and define the unitary operator U on $L_2(P)$ by $UX(\omega) = X(T\omega)$ for $X \in L_2(P)$, $\omega \in \Omega$. We define

$$Y_0 = \sum_{j=0}^{\infty} [E\{g_1(\xi_j) - \theta(F) | \mathscr{M}_{-\infty}^{0}\} - E\{g_1(\xi_j) - \theta(F) | \mathscr{M}_{-\infty}^{-1}\}] \in L_2(P),$$

$$Y_k = U^k Y_0, \quad k \geqq 1 \tag{5.3}$$

and

$$Z_0 = \sum_{j=0}^{\infty} E\{g_1(\xi_j) - \theta(F) | \mathscr{M}_{-\infty}^{-1}\}, \quad Z_k = U^k Z_0, \quad k \geqq 1. \tag{5.4}$$

Then, for every non-negative integer k

$$EY_k = EZ_k = 0, \quad E|Y_k|^{4+\delta} < \infty, \quad E|Z_k|^{4+\delta} < \infty, \tag{5.5}$$

and

$$g_1(\xi_k) - \theta(F) = Y_k - UZ_k + Z_k \tag{5.6}$$

and the sequence $(Y_k, \mathscr{M}_{-\infty}^{k})$ is a stationary ergodic martingale difference sequence. (cf. Theorem 8.1 in [5]).

Now, we put

$$V_n = \sum_{i=1}^{n} E\{Y_i^2 \mid Y_1, \ldots, Y_{i-1}\}. \tag{5.7}$$

Finally, we define random processes $S = \{S(t), 0 \leqq t < \infty\}$ and $S^* = \{S^*(t), 0 \leqq t < \infty\}$, respectively, by

$$S(t) = \begin{cases} 0 \quad \text{for } t = k, \quad 0 \leqq k \leqq m-1, \\ k[U_k - \theta(F)] \quad \text{for } t = k, \ k \geqq m, \\ \text{linearly interpolated for } t \in [k, k+1], \quad k \geqq 0, \end{cases} \tag{5.8}$$

and

$$S^*(t) = \begin{cases} k[\theta(F_k) - \theta(F)] \quad \text{for } t = k, \quad k \geqq 0 \\ \text{linearly interpolated for } t \in [k, k+1], \quad k \geqq 0. \end{cases} \tag{5.9}$$

By the same reason as in [5] and [16], we use a phrase "if necessary, redefining the X_i's on a new probability space" will imply that the joint distributions of the X_i's are kept the same. The following result is sharper than those of Strassen [16] and of Sen [14].

Theorem 4. *Let $\{\xi_n\}$ be a p-dimensional, strictly stationary, ϕ-mixing sequence. Suppose that there is a positive constant δ such that*

$$\phi(n) = 0(n^{-3(4+\delta')/(2+\delta')}) \tag{5.10}$$

for some $\delta'(0 < \delta' < \delta)$ and (2.3) *and* (2.4) *hold with $r = 4 + \delta$. For $\alpha \geqq 0$, let*

$$f_\alpha(t) = t(\log\log t)^{-\alpha}, \qquad t > e^e \tag{5.11}$$

and suppose that as $t \to \infty$

$$|V_n - n\sigma^2| = o(f_\alpha(t)) \qquad a.s. \tag{5.12}$$

Then, upon redefining $\{S(t), 0 \leqq t < \infty\}$ and $\{S^*(t), 0 \leqq t < \infty\}$ respectively on a new probability space, if necessary, there exists a Brownian motion $W = \{W(t), 0 \leqq t < \infty\}$ such that as $n \to \infty$

$$|S(t) - m\sigma W(t)| = o(t^{\frac{1}{2}}(\log\log t)^{(1-\alpha)/2}) \qquad a.s., \tag{5.13}$$

$$|S^*(t) - m\sigma W(t)| = o(t^{\frac{1}{2}}(\log\log t)^{(1-\alpha)/2}) \qquad a.s., \tag{5.14}$$

and

$$|S(t) - S^*(t)| = o(t^{\frac{1}{2}}(\log\log t)^{(1-\alpha)/2} \qquad a.s. \tag{5.15}$$

The following is a theorem concerning integral tests for U-statistics and differentiable statistical functions.

Theorem 5. *Under the conditions in Theorem* 4, *we have the followings:*

(a) *For every real function φ, $0 < \varphi \nearrow$,*

$$P(S(n) > V_n^{\frac{1}{2}}\varphi(V_n) \text{ i.o.}) = 0 (or\ 1) \tag{5.16}$$

and

$$P(S^*(n) > V_n^{\frac{1}{2}}\varphi(V_n)\text{ i.o.}) = 0 (or\ 1) \tag{5.17}$$

according as $I(\varphi) < \infty$ $(or = \infty)$, *where*

$$I(\varphi) = \int_1^\infty \frac{\varphi(t)}{t}\exp\left(-\frac{\varphi^2(t)}{2}\right)dt. \tag{5.18}$$

(b) *Let* $M_n = \max_{1 \leqq i \leqq n} |S(i)|$ *and* $M_n^* = \max_{1 \leqq i \leqq n} |S^*(i)|$. *Then, for every real function* $\varphi, 0 < \varphi \nearrow$,

$$P(M_n < V_n^{\frac{1}{2}}\{\varphi(V_n)\}^{-1}\text{ i.o.}) = 0 \quad (or\ 1) \tag{5.19}$$

and

$$P(M_n^* < V_n^{\frac{1}{2}}\{\varphi(V_n)\}^{-1}\text{ i.o.}) = 0 \quad (or\ 1) \tag{5.20}$$

according as $I_1(\varphi) < \infty$ $(or = \infty)$, *where*

$$I_1(\varphi) = \int_1^\infty \frac{\varphi^2(u)}{u}\exp\left(-\frac{8\varphi^2(u)}{\pi^2}\right)du. \tag{5.21}$$

The proofs of Theorems 4 and 5 need following lemmas.

Lemma 6. *Under the conditions of Theorem* 4 *we have that*

$$S^{(1)}(t) = \sigma W(t) + o(t^{\frac{1}{2}}(\log\log t)^{(1-\alpha)/2}) \quad a.s. \tag{5.22}$$

as $t \to \infty$, *where* $S^{(1)} = \{S^{(1)}(t), 0 \leqq t < \infty\}$ *is a random process defined by*

$$S^{(1)}(t) = S_k^{(1)} = \begin{cases} 0 & \text{if } k \leqq t < k+1, \quad 0 \leqq k \leqq m-1 \\ kU_k^{(1)} & \text{if } k \leqq t < k+1, \quad k \geqq m. \end{cases} \tag{5.23}$$

Proof. From (5.3)–(5.6), we have

$$S_k^{(1)} = \sum_{j=m}^{k} Y_j - Z_{k+1} + Z_m \tag{5.24}$$

and

$$\lim_{n\to\infty} (Z_{n+1} - Z_m)/n^{\frac{1}{3}} = 0 \quad a.s. \tag{5.25}$$

(cf. Lemma 8.4 in [5]). So, from Theorem 4.3 in [5], we have the lemma.

Lemma 7. *Under the conditions of Theorem* 4, *we have that as* $n \to \infty$

$$\sup_{k \geqq n}\left\{k\sum_{h=2}^{m}\binom{m}{h}U_k^{(h)}[k^{\frac{1}{2}}(\log\log k)^{(1-\alpha)/2}]^{-1}\right\} \xrightarrow{P} 0 \tag{5.26}$$

and

$$\sup_{k \geqq n}\left\{k\sum_{h=2}^{m}\binom{m}{h}U_k^{*(h)}[k^{\frac{1}{2}}(\log\log k)^{(1-\alpha)/2}]^{-1}\right\} \xrightarrow{P} 0. \tag{5.27}$$

Proof. We shall only prove (5.26). The proof of (5.27) is similar and so is omitted. Let

$$c_k = k\,\{k(\log\log k)\}^{-\frac{1}{2}}, \qquad k \geqq e^e.$$

To prove (5.26), it is enough to show that for any $\varepsilon > 0$

$$P\left(\sup_{k\geqq n} c_k \left|\sum_{h=2}^{m} \binom{m}{h} U_k^{(h)}\right| > \varepsilon\right) \leqq \sum_{h=2}^{m} P\left(\sup_{k\geqq n} c_k |U_k^{(h)}| > \frac{\varepsilon}{m}\right) \to 0 \tag{5.28}$$

as $n \to \infty$.

Since from Lemma 3, we have that

$$P\left(\sup_{k\geqq n} c_k |U_k^{(2)}| > \frac{\varepsilon}{m}\right) \leqq \sum_{k=n}^{\infty} P\left(c_k |U_k^{(2)}| > \frac{\varepsilon}{m}\right) \leqq M \sum_{k=n}^{\infty} c_k^4 k^{-3-\gamma'} = 0(n^{-\gamma'})$$

and for each $h\,(3 \leqq h \leqq m)$

$$P\left(\sup_{k\geqq n} c_k |U_k^{(h)}| > \frac{\varepsilon}{m}\right) \leqq \sum_{k=n}^{\infty} P\left(c_k |U_k^{(h)}| > \frac{\varepsilon}{m}\right) \leqq M \sum_{k=n}^{\infty} c_k^2 k^{-3} = 0(n^{-2}),$$

so we have (5.28) which, in turn, implies (5.26). Hence, the proof is completed.

Lemma 8. *Under the conditions of Theorem* 4, *we have that as* $n \to \infty$

$$\sup_{k\geqq n} \{k\,|\theta(F_k) - U_k|\,[k^{\frac{1}{2}} (\log\log k)^{(1-\alpha)/2}]^{-1}\} \xrightarrow{P} 0. \tag{5.29}$$

Proof. Since

$$|\theta(F_k) - U_k| \leqq \sum_{h=2}^{m} \{|U_k^{*(h)}| + |U_h^{(h)}|\},$$

so the proof of (5.29) is obtained by the same method as the one used in the proof of Lemma 7.

The proof of Theorem 4 is obtained from Theorem 4.3 in [5] and Lemmas 6, 7 and 8, and that of Theorem 5 follows from Theorems 5.2 and 6.3 in [5] and Lemmas 6 and 7.

Now, we shall consider a Doeblin process defined in [5]. Since Doeblin processes are ϕ-mixing with mixing coefficient $\phi(n) = 0(e^{-\rho n})$ $(\rho > 0)$ and to the processes Corollary 8.1 and Lemma 8.3 in [5] are applicable, so we have

$$|V_n - n\sigma^2| = O(n^{1-\varepsilon}) \qquad \text{for some } \varepsilon > 0. \tag{5.30}$$

Hence, from Theorems 4 and 5 we have the following theorem.

Theorem 6. *Let* $\{\xi_n\}$ *be a Doeblin process. If* (2.3) *and* (2.4) *hold with* $r = 4 + \delta$ *then the conclusions in Theorems* 4 *and* 5 *hold.*

6. Weak Convergence of Generalized U-Statistics

In this section, we shall consider generalized U-statistics (cf. [13]) and extend the results in Section 3.

Let $\{\xi_{ji},\ i=\cdots-1,0,1,\dots\}$ $(j=1,\dots,c)$ be $c(\geqq 2)$ independent sequences of strictly stationary stochastic vectors, defined on a probability space $(\Omega, \mathscr{A}, P)$, where ξ_{ji} has a *df* $F_j(x)$, $x\in R^p$, for $j=1,\dots,c$. We assume that for each $j(j=1,\dots,c)$ $\{\xi_{ji}\}$ is an absolutely regular process with coefficient $\beta_j(n)$. Let $g(\xi_{ji},\ i=1,\dots,m_j,\ j=1,\dots,c)$ be a Borel measurable kernel of degree $\mathbf{m}=(m_1,\dots,m_c)$, where we may assume (without any loss of generality) that g is symmetric in the $m_j(\geqq 1)$ arguments of the j-th set, for $j=1,\dots,c$. Let $m_0=m_1+\cdots+m_c$, $\mathbf{F}=(F_1,\dots,F_c)$ and consider a functional of $\mathbf{F}$

$$\theta(\mathbf{F})=\int_{R^{pm_0}}\!\!\cdots\!\int g(x_{11},\dots,x_{cm_c})\prod_{j=1}^{c}\prod_{i=1}^{m_j}dF_j(x_{ji}) \tag{6.1}$$

defined on $\mathfrak{F}=\{\mathbf{F}\colon |\theta(\mathbf{F})|<\infty\}$.

For a set of samples of sizes $\mathbf{n}=(n_1,\dots,n_c)$ with $n_j\geqq m_j$, $1\leqq j\leqq c$, the generalized U-statistics for $\theta(\mathbf{F})$ is defined by

$$U(\mathbf{n})=\prod_{j=1}^{c}\binom{n_j}{m_j}^{-1}\sum_{(\mathbf{n})}^{*}g(\xi_{j\alpha},\ \alpha=i_{j1},\dots,i_{jm_j},\ 1\leqq j\leqq c) \tag{6.2}$$

where the summation $\sum\limits_{(\mathbf{n})}^{*}$ extends over all $1\leqq i_{j1}<\cdots<i_{jm_j}\leqq n_j$, $1\leqq j\leqq c$. Now, we assume that

$$\lim_{n\to\infty} n_j/n=\lambda_j;\qquad 0<\lambda_j<1,\qquad j=1,\dots,c \tag{6.3}$$

where $n=n_1+\cdots+n_c$.

For each $d_j(0\leqq d_j\leqq m_j,\ 1\leqq j\leqq c)$, let

$$\begin{aligned} &g_{d_1\dots d_c}(x_{ji},\ i=1,\dots,d_j,\ 1\leqq j\leqq c)\\ &\quad=\int_{R_c^*}\!\!\cdots\!\int g(x_{j1},\dots,x_{jd_j},x_{jd_j+1},\dots,x_{jm_j},\ 1\leqq j\leqq c)\prod_{j=1}^{c}\prod_{v=d_j+1}^{m_j}dF_j(x_{jv}) \end{aligned} \tag{6.4}$$

so that $g_{00\dots0}=\theta(\mathbf{F})$ and $g_{m_1\dots m_c}(\cdot)=g(\cdot)$ where R_c^* is the $p(m_1-d_1)\dots(m_c-d_c)$-dimensional Euclidean space. Further, for each $j(1\leqq j\leqq c)$, let

$$\sigma_j^2=\{E(e_j(\xi_{j1}))^2-\theta^2(\mathbf{F})\}+2\sum_{k=1}^{\infty}\{Ee_j(\xi_{j1})\,e_j(\xi_{jk+1})-\theta^2(\mathbf{F})\} \tag{6.5}$$

where

$$e_j(x_{ji})=g_{\delta_{j1}\dots\delta_{jc}}(x_{ji}) \tag{6.6}$$

and $\delta_{ab}=1$ or 0 according as $a=b$ or not.

Let $E_c=[0,1]^c$ be the c-dimensional unit cube in R^c, $\mathbf{t}=(t_1,\dots,t_c)\in E_c$, and $[\mathbf{nt}]=([n_1t_1],\dots,[n_ct_c])$ where $[s]$ denotes the largest integer $\leqq s$. As in [13], let $X(\mathbf{n})=\{X(\mathbf{t}:\mathbf{n})\colon \mathbf{t}\in E_c\}$ be the process defined by

$$X(\mathbf{t}:\mathbf{n})=\begin{cases}\psi([\mathbf{nt}]:n)\,[U([\mathbf{nt}])-\theta(\mathbf{F})] & \text{for all } [\mathbf{nt}]\geqq\mathbf{m},\\ 0 \text{ otherwise,}\end{cases} \tag{6.7}$$

where for $\mathbf{k}=(k_1, \ldots, k_c)$ $(k_j>0, j=1, \ldots, c)$

$$\psi(\mathbf{k}:n)=n^{-\frac{1}{2}}\left(\sum_{j=1}^{c}\sigma_j\lambda_j^{\frac{1}{2}}\right)\left(\sum_{j=1}^{c}\sigma_j\lambda_j^{\frac{1}{2}}k_j^{-1}\right)^{-1} \tag{6.8}$$

and $\mathbf{a}\leqq\mathbf{b}$ means that $a_j\leqq b_j$ for all $1\leqq j\leqq c$.

Let $W_j=\{W_j(t)\colon 0\leqq t\leqq 1\}$ $(j=1, \ldots, c)$ be c independent copies of a standard Brownian motion on $[0, 1]$. Finally, let D_c be the space of all real functions on E_c with no discontinuities of the second kind with the extended Skorokhod J_1-topology defined as in [13]. The following theorem is both extensions of Theorem 2.1 in [13] and Theorem 2.

Theorem 7. *Suppose that $\{\xi_{ji}\}$ $(j=1, \ldots, c)$ are c independent sequences of strictly stationary, absolutely regular processes. Suppose that for some $\delta>0$, the following relations hold:*

$$\text{(i)}\ \int_{R^{pm_0}}\!\!\cdots\!\int |g(x_{11}, \ldots, x_{cm_c})|^{4+\delta}\prod_{j=1}^{c}\prod_{i=1}^{mj}dF_j(x_{ji})<\infty, \tag{6.9}$$

(ii) *for all integers $i_{j1}, \ldots, i_{jm_j}$*

$$E|g(\xi_{j\alpha}, \alpha=i_{j1}, \ldots, i_{jm_j}, 1\leqq j\leqq c)|^{4+\delta}\leqq M<\infty, \tag{6.10}$$

(iii) *for some $\delta'(0<\delta'<\delta)$*

$$\max_{1\leqq j\leqq c}\beta_j(n)=O(n^{-3(4+\delta')/(2+\delta')}). \tag{6.11}$$

Then, the series (6.5) converge absolutely; if $\max_{1\leqq j\leqq c}\sigma_j^2>0$, then $X(\mathbf{n})$ converges in law in the extended Skorokhod J_1-topology on D_c to a Gaussian function $W=\{W(\mathbf{t})\colon \mathbf{t}\in E_c\}$, where

$$W(\mathbf{t})=\left(\sum_{j=1}^{c}\sigma_j\lambda_j^{\frac{1}{2}}\right)\left(\sum_{j=1}^{c}\sigma_j\lambda_j^{-\frac{1}{2}}t_j^{-1}\right)\left[\sum_{j=1}^{c}\sigma_j\lambda_j^{-\frac{1}{2}}t_j^{-1}W_j(t_j)\right], \qquad t>0 \tag{6.12}$$

$=0$ *with probability one if $t_j=0$ for some $j(1\leqq j\leqq c)$.*

Next, let $W^*(\mathbf{n})=\{W^*(\mathbf{t}:\mathbf{n});\ \mathbf{t}\in E_c\}$ be the process defined by

$$W^*(\mathbf{t}:\mathbf{n})=r^{-1}(\mathbf{n})\,[U([\mathbf{n}/\mathbf{t}])-\theta(\mathbf{F})], \qquad \mathbf{t}\in E_c \tag{6.13}$$

where $r^2(\mathbf{n})=\operatorname{Var}(U(\mathbf{n}))$ and $[\mathbf{n}/\mathbf{t}]=([n_1/t_1], \ldots, [n_c/t_c])$. Further, let

$W^*=\{W^*(\mathbf{t})\colon \mathbf{t}\in E_c\}$ $(W^*(\mathbf{t})=\mathbf{w}'W(\mathbf{t}),\ \mathbf{t}\in E_c)$

be the process defined by

$$\mathbf{w}=(w_1, \ldots, w_c)'; \qquad w_j=\sigma_j\lambda_j^{-\frac{1}{2}}\left(\sum_{j=1}^{c}\sigma_j^2/\lambda_j\right)^{-\frac{1}{2}}, \qquad 1\leqq j\leqq c, \tag{6.14}$$

$$W(\mathbf{t})=(W_1(t_1), \ldots, W_c(t_c)), \qquad \mathbf{t}\in E_c. \tag{6.15}$$

Then, we can extend Theorem 2.2 in [13] as follows:

Theorem 8. *Under the conditions of Theorem 7, $W^*(\mathbf{n})$ converges in law in the extended Skorokhod J_1-topology on D_c to W^*.*

The proofs of these two theorems are obtained from the methods of the proofs of Theorems 2.1 and 2.2 in [13] using the technique used in the proof of Theorem 2, and so are omitted.

Remark. As in [13], we can prove analogous results to Theorems 7 and 8 for generalized von Mises' functionals.

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References

1. Billingsley, P.: Convergence of probability measures. New York: Wiley 1968
2. Heyde, C.C., Scott, D.J.: Invariance principles for the law of the iterated logarithm for martingales and processes with stationary increments. Ann. Probab. **1**, 428–436 (1973)
3. Ibragimov, I.A., Rozanov, Yu.A.: Gaussian stochastic processes (in Russian). Moscow: Izdatelistvo Nauka 1970
4. Ibragimov, I.A., Solev, V.N.: A condition for regularity of a Gaussian stationary process. Soviet. Math. Dokl. **10**, 371–375 (1969)
5. Jain, N.C., Jogdeo, K., Stout, W.F.: Upper and lower functions for martingales and mixing processes. Ann. Probab **3**, 119–145 (1975)
6. Miller, R.G. Jr., Sen, P.K.: Weak convergence of U-statistic and von Mises' differentiable statistical functions. Ann. Math. Statist. **43**, 31–41 (1972)
7. Oodaira, H., Yoshihara, K.: Note on the law of the iterated logarithm for stationary processes satisfying mixing conditions. Kōdai Math. Sem. Rep. **23**, 335–342 (1971)
8. Oodaira, H., Yoshihara, K.: Functional central limit theorems for strictly stationary processes satisfying the strong mixing condition. Kōdai Math. Sem. Rep. **24**, 259–269 (1971)
9. Phillip, W., Stout, W.: Almost sure invariance principles for partial sums of weakly dependent random variables. Mem. Amer. Math. Soc. No. 161 161 (1975)
10. Rozanov, Yu.A., Volkonski, V.A.: Some limit theorems for random functions I. Theor. Probab. Appl. **4**, 178–197 (1959)
11. Rozanov, Yu.A., Volkonskii, V.A.: Some limit theorems for random functions II. Theor. Probab. Appl. **6**, 186–198 (1961)
12. Sen, P.K.: Limiting behavior of regular functionals of empirical distributions for stationary $*$-mixing processes. Z. Wahrscheinlichkeitstheorie verw. Gebiete **25**, 71–82 (1972)
13. Sen, P.K.: Weak convergence of generalized U-statistics. Ann. Probab. **2**, 90–102 (1974)
14. Sen, P.K.: Almost sure behavior of U-statistics and von Mises' differentiable statistical functions. Ann. Statist. **2**, 387–395 (1974)
15. Strassen, V.: An invariance principle for the law of the iterated logarithm. Z. Wahrscheinlichkeitstheorie verw. Gebiete **3**, 211–226 (1964)
16. Strassen, V.: Almost sure behavior of sums of independent random variables and martingales. Proc. 5th Berkeley Sympos. Math. Statist. Probab., Univ. Calif. **2**, 315–343 (1967)

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