Z. Wahrscheinlichkeitstheorie verw. Gebiete
68, 393–402 (1985)

Zeitschrift für
Wahrscheinlichkeitstheorie
und verwandte Gebiete

Quantum Central Limit Theorems for Strongly Mixing Random Variables

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1. Introduction

At the moment there are essentially two methods which allow to extend the central limit theorem from a classical to a quantum mechanical framework: one is based on a quantum analogue of the method of characteristic functions, the other one deals directly with the momenta. The former method, originated in a paper by Cushen and Hudson [2] and developed in several directions [1, 3, 7, 8], assumes little from the analytical point of view (essentially the existence of second moments), but needs rather strong assumptions on the algebraic structures involved. The latter, originated in a paper of Giri and von Waldenfels [4], requires more from the analytical point of view (existence of the momenta of all orders), but very little on the algebraic structure involved (only the existence of some kind of commutation relations) and, on the contrary, the C.C.R. and C.A.R. structures emerge themselves as a consequence of the central limit theorem [4, 10]. Therefore the latter method seems to be more suited to display the universality of the central limit phenomenon not only from the probabilistic, but also from the algebraic point of view.

In the present paper we extend the Giri-von Waldenfels method to the quantum analogue of sequences of dependent random variables satisfying a mixing condition (cf. condition (2.17) in Sect. 2). Our technique applies both to the Bose and to the Fermi case (cf. the Remark after the identity (2.1)), thus our results include those of [2, 5, 4, 10] and [6]. Moreover, since we deal with maps rather than linear functionals, also the recent results of [3, 8] are included in our ones up to the stronger analyticity assumption mentioned above. Finally since the ergodic quantum Markov chains are mixing at an exponential rate [0], our results imply the validity of the (boson) quantum central limit theorem for discrete, finite dimensional, ergodic generalized quantum Markov chains.

At the end of Sect. 2 we discuss the role of the exponent 1/2 in the normalized sums of the quantum central limit theorem, and show that, in our assumptions, for $\alpha > 1/2$ all the momenta converge to zero, while for $0 < \alpha < 1/2$, in the generic case they will converge either to zero or to infinity. It is however clear a priori that, since the convergence involved in our method consists in the convergence of the momenta of all orders, this method is not suited for an investigation of the stable distributions. In the case of product states quantum central limit theorems with a normalization different from $N^{1/2}$ have been studied by Schürmann [9]. For a detailed analysis of the relations between the classical and quantum central limit theorems we refer to [4, 6, 10]. The precise formulation of our quantum central limit theorem is contained in the following theorem.

Theorem (1.1). *In the notations introduced at the beginning of Sect. 2 let* E: $\mathscr{A} \to \mathscr{C}$ *be any* $\mathbb{R}$*-linear map satisfying conditions* (2.11) *(finiteness of all mixed moments) and* (2.17) *(faster than polynomial mixing). Then, both in the boson and the fermion case, for each* $k \in \mathbb{N}$ *and* $b_1, \ldots, b_k \in B$ *the limit in the semi-norm* $|\cdot|$:

$$\lim_N E\{[N^{1/2} S_N(b_1)] \cdot \ldots \cdot [N^{-1/2} S_N(b_k)]\} \tag{1.1}$$

exists and is equal to 0 *for* k*-odd. For* $k = 2n$ $(n \in \mathbb{N})$ *this limit exists in a topology weaker than the* $|\cdot|$*-topology if and only if in this topology the limit*

$$\lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} E(J_k(b \cdot b')) = C(b \cdot b') \tag{1.2}$$

exists for each $b, b' \in B$. *In this case the limit* (1.1) *is equal to*

$$\sum_{\text{p.p.}} \varepsilon(j_1, h_1, \ldots, j_n, h_n)\, C(b_{j_1} \cdot b_{h_1})\, C(b_{j_2} \cdot b_{h_2}) \cdot \ldots \cdot C(b_{j_n} \cdot b_{h_n}) \tag{1.3}$$

where here and in the following, p.p. *means that the sum is extended to all pair partitions* $(j_1, h_1; j_2, h_2; \ldots; j_n, h_n)$ *of the set* $\{1, \ldots, 2n\}$, *with* $j_\alpha < h_\alpha$, $j_\alpha < j_{\alpha+1}$ $(\alpha = 1, \ldots, n)$; *and* $\varepsilon(j_1, h_1, \ldots, j_n, h_n) = +1$ *in the boson case, and in the fermion case it is equal to the signum of the permutation* $\{1, \ldots, 2n\} \to \{j_1, h_1, \ldots, j_n, h_n\}$.

Remark. We recall that if $\mathscr{B}$ is a complex or real algebra and B a set of algebraic generators of $\mathscr{B}$ a linear functional γ: $\mathscr{B} \to \mathbb{C}$ such that $\forall n \in \mathbb{N}$, $\forall b_1, \ldots, b_{2n}, b_{2n+1} \in B$

$$\gamma(b_1 \cdot b_2 \cdot \ldots \cdot b_{2n+1}) = 0 \tag{1.4}$$

$$\gamma(b_1 \cdot \ldots \cdot b_{2n}) = \sum_{\text{p.p.}} \gamma(b_{j_1} b_{h_1})\, \gamma(b_{j_2} b_{h_2}) \cdot \ldots \cdot \gamma(b_{j_n} b_{h_n}) \tag{1.5}$$

is called (for obvious reasons) a *gaussian functional*, while a linear functional γ: $\mathscr{B} \to \mathbb{C}$, satisfying (1.4) and instead of (1.5):

$$\gamma(b_1 \cdot \ldots \cdot b_{2n}) = \sum_{\text{p.p.}} \operatorname{sgn}(j_1, h_1, \ldots, j_n, h_n)\, \gamma(b_{j_1} b_{h_1}) \cdot \ldots \cdot \gamma(b_{j_n} b_{h_n}) \tag{1.6}$$

is called a gaussian-Clifford functional. Gaussian and gaussian-Clifford functionals are a generalization of the quasifree states, commonly used in quantum field theory and quantum statistical mechanics. The result of our Theorem (1.1) suggests a similar generalization for the notion of a quasifree completely positive map.

2. Proof of the Main Theorem

Let $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}$ be associative real algebras. We assume $\mathscr{A}$ and $\mathscr{C}$ have an identity denoted, when no confusion can arise, with the same symbol 1. We assume that $\mathscr{C}$ acts on $\mathscr{A}$ on the left, (i.e. that for each $\lambda\in\mathscr{C}$ and $a\in\mathscr{A}$ one can define a unique element of $\mathscr{A}$, denoted $\lambda\cdot a$, and the map $(\lambda, a)\to\lambda\cdot a$ enjoys the usual algebraic properties). Let, for each $k\in\mathbb{N}$ be given an embedding J_k: $\mathscr{B}\to\mathscr{A}$ such that, for each $h\neq k$ $(h, k\in\mathbb{N})$ and for each $b, b'\in\mathscr{B}$ there exists an element $\varepsilon_{hk}(b, b')$ of $\mathscr{C}$ such that

$$J_h(b)\cdot J_k(b')=\varepsilon_{h,k}(b,b')\cdot J_k(b')\cdot J_h(b). \tag{2.1}$$

Remark. We will be mainly interested in the case in which, for any choice of $b, b'\in\mathscr{B}$, $\varepsilon_{h,k}(b,b')=+1$ (boson case) or $\varepsilon_{h,k}(b,b')=\pm 1$ (fermion case). In the fermion case the choice between $+1$ or -1 is performed according to the rules explained in [10].

In the following, if $p, k\in\mathbb{N}$ and $p\leqq k$, we denote $\mathscr{P}_{k,p}$ the family of all ordered partitions $(S_1, \ldots, S_p)$ of the set $\{1, 2, \ldots, k\}$ in exactly p non empty sub-sets. The partition $(S_1, \ldots, S_p)$ is ordered in the following sense: each set S_j has the natural order and the sets S_j themselves are ordered so that $1\in S_1$, and the smallest element of S_{j+1} is the smallest integer in $\{1, \ldots, k\}$ not belonging to $\coprod_{n=1}^{j} S_n$. For $(S_1, \ldots, S_p)\in\mathscr{P}_{k,p}$, and $N\in\mathbb{N}$, we denote $[S_1, \ldots, S_p]_N$ the set of all functions α: $\{1, \ldots, k\}\to\{1, \ldots, N\}$ such that:

(i) for each $j=1, \ldots, p$, α restricted to S_j has a constant value, denoted $\alpha(S_j)$.

(ii) if $j\neq k$ then $\alpha(S_j)\neq\alpha(S_k)$.

The sub-set of $[S_1, \ldots, S_p]_N$ of those α's such that

$$\alpha(S_1)<\alpha(S_2)<\ldots<\alpha(S_p) \tag{2.2}$$

will be denoted $I_N(S_1, \ldots, S_p)$.

Throughout the paper we will use the notation:

$$S_n(b)=\sum_{k=1}^{N} J_k(b); \qquad b\in\mathscr{B};\ N\in\mathbb{N}.$$

Lemma (2.1). *In the notations above, if $k\in\mathbb{N}$ and $b_1, \ldots, b_k\in\mathscr{B}$ one has:*

$$S_N(b_1)\cdot S_N(b_2)\cdot\ldots\cdot S_N(b_k) = \sum_{p=1}^{k}\ \sum_{(S_1,\ldots,S_p)\in\mathscr{P}_{k,p}}\ \sum_{\alpha\in I_N(S_1,\ldots,S_p)} \varepsilon(\alpha; b_1, \ldots, b_k)\cdot J_{\alpha(S_1)}(b_{S_1})\cdot\ldots\cdot J_{\alpha(S_p)}(b_{S_p}) \tag{2.3}$$

where $\varepsilon(\alpha; b_1, \ldots, b_k)$ is an element of $\mathscr{C}$ uniquely determined by $b_1, \ldots, b_k$, by $\alpha\in I_N(S_1, \ldots, S_p)$, and by the class of all $\beta\in[S_1, \ldots, S_p]_N$ which are obtained from α through a permutation of its values; and where if $S_j=(h_1<\ldots<h_{n_j})$ then

$$b_{S_j}=b_{h_1}b_{h_2}\ldots b_{h_{n_j}}. \tag{2.4}$$

Proof. Identifying $\{1, \ldots, N\}^k$ with the set of functions α: $\{1, \ldots, k\}\to\{1, \ldots, N\}$, one has the identity:

$$\{1, \ldots, N\}^k = \coprod_{p=1}^{k} \coprod_{(S_1, \ldots, S_p) \in \mathscr{P}_{k,p}} \coprod_{\alpha \in [S_1, \ldots, S_p]_N} \{\alpha\} \tag{2.5}$$

which implies

$$S_N(b_1) \cdot \ldots \cdot S_N(b_k) = \sum_{p=1}^{k} \sum_{(S_1, \ldots, S_p) \in \mathscr{P}_{k,p}} \sum_{\alpha \in [S_1, \ldots, S_p]_N} J_{\alpha_1}(b_1) \cdot \ldots \cdot J_{\alpha_k}(b_k). \tag{2.6}$$

Now, using (2.1), we can group together those α_h's with $h \in S_j$ ($j = 1, \ldots, p$), and each exchange will give rise to a factor $\varepsilon_{\alpha_i, \alpha_j}(b_i, b_j)$; subsequently we can arrange the indices so that (2.2) is fulfilled, and each exchange will give rise to a factor $\varepsilon_{\alpha(S_i), \alpha(S_j)}(b_{S_i}, b_{S_j})$. Eventually we obtain the right hand side of (2.3), with $\varepsilon(\alpha; b_1, \ldots, b_k)$ given by a multiple of the product of all the factors arosen in this way.

Remark 1. For our purpose it will be sufficient to determine the specific form of the ε-factor in (2.4) only for $\alpha \in [S_1, \ldots, S_p]_N$ with $k = 2p$. In the boson case ($\varepsilon_{h,k}(b; b') = +1$ in (2.1)) one easily verifies that

$$\varepsilon(\alpha; b_1, \ldots, b_k) = p! \qquad \forall p \leqq k \tag{2.7}$$

independently on α and on the b's.

Remark 2. Clearly, for each k, $N \in \mathbb{N}$, $p \in \{1, \ldots, k\}$, and $(S_1, \ldots, S_p) \in \mathscr{P}_{k,p}$ one has, denoting $|I|$ the cardinality of the set I:

$$|[S_1, \ldots, S_p]_N| = p! \binom{N}{p} \leqq C_p N^p, \tag{2.8}$$

$$|I_N(S_1, \ldots, S_p)| = \binom{N}{p} \leqq C_p N^p \tag{2.9}$$

where C_p is a constant independent on N.

Assume now that on $\mathscr{C}$ it is defined a semi-norm, denoted $|\cdot|$, and that E: $\mathscr{A} \to \mathscr{C}$ is a $\mathscr{C}$-linear map (if the $\varepsilon_{n,k}(b, b')$'s in (2.1) are real or complex, then only $\mathbb{R}$- or $\mathbb{C}$-linearity of E is required). Let B denote a sub-set of $\mathscr{B}$ with the following properties:

$$E(J_n(b)) = 0; \qquad \forall b \in B; \ \forall n \in \mathbb{N}. \tag{2.10}$$

For each $k \in \mathbb{N}$ and $b_1, \ldots, b_k \in B$, assume there exists a positive constant $v(b_1, \ldots, b_k) \in \mathbb{R}^+$ such that for any $p \in \{1, \ldots, k\}$, for any partition $(S_1, \ldots, S_p)$ of $\{1, \ldots, k\}$, for any $N \in \mathbb{N}$, and for any $\alpha \in [S_1, \ldots, S_p]_N$, one has

$$|E(J_{\alpha(S_1)}(b_{S_1}) \cdot \ldots \cdot J_{\alpha(S_p)}(b_{S_p}))| \leqq v(b_1, \ldots, b_k) \tag{2.11}$$

where b_{S_j} is given by (2.4).

Remark. If $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}$ are C^*-algebras and E is a positive map, then (2.11) is obviously satisfied. If $\mathscr{A}$, $\mathscr{B}$ are *-algebras, $\mathscr{C} \equiv \mathbb{C}$ and E is a state on $\mathscr{A}$ satisfying the following condition (2.14) and such that for each $k \in \mathbb{N}$

$$E \circ J_k = E \circ J_0 \underset{\text{df}}{=} E_0 \tag{2.12}$$

(stationarity) then one easily verifies that (2.11) is satisfied if the b's have finite mixed moments of all orders i.e. for each $n \in \mathbb{N}$ and $b_1, \ldots, b_n \in B$

$$|E_0(b_1 \cdot \ldots \cdot b_n)| < +\infty. \tag{2.13}$$

Finally we assume that there is a constant $v_0 \in \mathbb{R}^+$ such that:

$$|\varepsilon_{h,k}(b, b')| \leqq v_0(b, b') \tag{2.14}$$

uniformly in $h, k \in \mathbb{N}$, $b, b' \in B$, and that the map $E: \mathscr{A} \to \mathscr{C}$ fulfils the following mixing condition: there exist two functions $d, \delta: \mathbb{N} \to \mathbb{R}^+$ such that:

$$d(N) \to +\infty; \quad d(N)/N^{1/2} \to 0; \quad \text{as } N \to \infty, \tag{2.15}$$

$$N^q \delta(N) \to 0; \quad \text{as } N \to \infty; \ \forall q \in \mathbb{N} \tag{2.16}$$

and $\forall m, k \in \mathbb{N}$, $\forall b_1, \ldots, b_k \in B$, there exists a positive constant $v(b_1, \ldots, b_k) \in \mathbb{R}^+$ which we can assume to be the same as in (2.11), such that

$$|E(M_m \cdot N_{m+d(N)}) - E(M_m) \cdot E(N_{m+d(N)})| \leqq v(b_1, \ldots, b_k)\, \delta(N) \tag{2.17}$$

where M_m, $N_{m+d(N)}$ denote two monomials of the form:

$$M_m = J_{\alpha(S_1)}(b_{S_1}) \ldots J_{\alpha(S_q)}(b_{S_q}), \tag{2.18}$$

$$N_{m+d(N)} = J_{\alpha(S_{q+1})}(b_{S_{q+1}}) \ldots J_{\alpha(S_p)}(b_{S_p}) \tag{2.19}$$

with $p \in \{1, \ldots, k\}$, $(S_1, \ldots, S_p)$ – an arbitrary partition of $\{1, \ldots, k\}$, b_{S_j} – given by (2.4), and $\alpha_1, \ldots, \alpha_p$ such that

$$\alpha(S_j) \leqq m; \quad j = 1, \ldots, q; \quad \alpha(S_j) \geqq m + d(N); \quad j = q+1, \ldots, p.$$

In order to simplify the notations, in the following a map $E: \mathscr{A} \to \mathscr{C}$ satisfying these conditions will be called *mixing* (without further specification).

Remark 1. If E is exponentially mixing, i.e. if for N large enough and for some constants $c, \alpha > 0$ one has

$$|E(M_m \cdot N_{m+N}) - (E(M_m) \cdot E(N_{m+N})| \leqq v \cdot e^{-cN^\alpha}$$

then one can always find functions δ, d satisfying (2.15), (2.16), (2.17).

Remark 2. If $E: \mathscr{A} \to \mathscr{C}$ satisfies (2.17), $m_1, \ldots, m_h \in \mathbb{N}$ are such that $m_j \leqq m_{j+1} - d(N)$ $(j = 2, \ldots, h)$, and $M_{m_1}, M_{m_2}, \ldots, M_{m_h}$ have the form (2.18), (2.19) then one easily verifies that:

$$|E(M_{m_1} \cdot M_{m_2} \cdot \ldots \cdot M_{mh}) - E(M_{m_1}) \cdot E(M_{m_2}) \cdot \ldots \cdot E(M_{m_h})| \leqq \bar{v}\, \delta(N) \tag{2.20}$$

where

$$\bar{v} = \max_{1 \leqq h \leqq k} \frac{v^h - 1}{v - 1}; \quad v = v(b_1, \ldots, b_k). \tag{2.21}$$

Remark 3. Under assumption (2.14), $\forall k \in \mathbb{N}$, for each $p \leqq k$ $\forall (S_1, \ldots, S_p) \in \mathscr{P}_{k,p}$ for each $b_1, \ldots, b_k \in B$ and $\forall \alpha \in [S_1, \ldots, S_p]_N$ one has

$$|\varepsilon(\alpha; b_1, \dots, b_k)| \leqq v_0(b_1, \dots, b_k)$$

where $v_0(b_1, \dots, b_k)$ is a constant.

In the following unless otherwise specified, all the limits will be refered to the semi-norm $|\cdot|$ introduced above; k and $b_1, \dots, b_k$ will denote some fixed, arbitrarily chosen, positive integer and elements of B respectively, and we will use the notation (2.4).

A partition $(S_1, \dots, S_p)$ $(p \leqq k)$ of the set $\{1, \dots, k\}$ will be said to contain a singleton if, for some $j = 1, \dots, p$, S_j contains a single element.

Lemma (2.2). *In the notations above, let* $\varepsilon \geqq 1/2$, $(S_1, \dots, S_p)$ *be a partition of* $\{1, \dots, k\}$ *and assume that either of the following conditions is satisfied:*

i) $\varepsilon k > p$,

ii) $(S_1, \dots, S_p)$ *contains exactly* q *singletons, with* $q \geqq 1$. *Then:*

$$\lim_{N \to \infty} |N^{-\varepsilon k} \sum_{\alpha \in I_N(S_1, \dots, S_p)} \varepsilon(\alpha; b_1, \dots, b_k) E(J_{\alpha(S_1)}(b_{S_1}) \cdot \ldots \cdot J_{\alpha(S_p)}(b_{S_p})| = 0. \quad (2.22)$$

Proof. Because of (2.9), (2.11), (2.14), one has:

$$|N^{-\varepsilon k} \sum_{\alpha \in I_N(S_1, \dots, S_p)} \varepsilon(\alpha; b_1, \dots, b_k) E(J_{\alpha(S_1)}(b_{S_1}) \cdot \ldots \cdot J_{\alpha(S_p)}(b_{S_p})|$$
$$\leqq N^{-\varepsilon k} N^p C_p v_0^{(k)} \cdot v(b_1, \dots, b_k).$$

Hence, if $\varepsilon k > p$, (2.22) holds. Assume now that $(S_1, \dots, S_p)$ contains exactly q singletons and that they correspond to the sets $S_{j_1}, \dots, S_{j_q}$. Define the set:

$$I'_N = \{\alpha \in I_N(S_1, \dots, S_p): \quad \forall m = 1, \dots, q, \text{ either } \alpha_{j_m+1} - \alpha_{j_m} \leqq d(N) \text{ or } \alpha_{j_m} - \alpha_{j_m-1} \leqq d(N)$$

(where for simplicity we write α_j instead of $\alpha(S_j)$). Clearly one has:

$$|I'_N| \leqq N^{p-q} \cdot d(N)^q \quad (2.23)$$

because each α_{j_m} can be chosen in at most N ways and, having chosen α_{j_m}, then either for α_{j_m+1} or α_{j_m-1} one has at most $d(N)$ possibilities, while for each of the remaining $(\leqq p-2q)$ α_h's one has at most N choices. Define now the set

$$I''_N = I_N - I'_N = \{\alpha \in I_N(S_1, \dots, S_p): \ \alpha \notin I'_N\}.$$

If $\alpha \in I''_N$ then, in the notations of Remark 2 above, $E(J_{\alpha_1}(b_{S_1}) \cdot \ldots \cdot J_{\alpha_p}(b_{S_p}))$ has the form $E(M_{m_1} \cdot J_{\alpha_2}(b_{\alpha_2}) \cdot M_{\alpha_2 + d(N)})$, with $\alpha_2 \geqq m_1 + d(N)$ therefore, according to (2.11) and (2.20):

$$|E(J_{\alpha_1}(b_{S_1}) \cdot \ldots \cdot J_{\alpha_p}(b_{S_p}))| \leqq \bar{v}\,\delta(N) \quad (2.24)$$

with $\bar{v}$ given by (2.21). In particular:

$$|N^{-\varepsilon k} \sum_{\alpha \in I''_N} \varepsilon(\alpha; b_1, \dots, b_k) E(J_{\alpha_1}(b_{S_1}(b_{S_1}) \cdot \ldots \cdot J_{\alpha_p}(b_{S_p}))|$$
$$\leqq N^{-\varepsilon k} |I''_N| \cdot \delta(N) \cdot \bar{v}\,\bar{v}_0$$
$$\leqq N^{-\varepsilon k} |I_N| \cdot \delta(N) \cdot \bar{v}\,\bar{v}_0$$
$$\leqq N^{-\varepsilon k + p}\,\delta(N) \cdot \bar{v}\,\bar{v}_0 \quad (2.25)$$

and this tends to zero for any $p\in\mathbb{N}$ as $N\to\infty$, because of (2.16). Moreover, due to (2.23), one has:

$$\begin{aligned}&|N^{-\varepsilon k}\sum_{\alpha\in I'_N}\varepsilon(\alpha; b_1,\dots,b_k)\,E(J_{\alpha_1}(b_{S_1})\cdot\ldots\cdot J_{\alpha_p}(b_{S_p}))|\\&\leqq N^{p-\varepsilon k}[d(N)/N]^q\cdot(\text{const}).\end{aligned}\tag{2.26}$$

But, if $(S_1,\dots,S_p)$ has exactly q singletons, then:

$$k=\sum_{j=1}^{p}|S_j|\geqq q+(p-q)\cdot 2=2p-q\tag{2.27}$$

therefore in our assumptions

$$p\leqq k/2+q/2.\tag{2.28}$$

Thus the right hand side of (2.26) is majorized by:

$$\text{const}\,N^{\frac{k}{2}+\frac{q}{2}-k\varepsilon}[d(N)/N]^q=\text{const}\,N^{-k(\varepsilon-\frac{1}{2})}[d(N)/N^{1/2}]^q\tag{2.29}$$

which, in view of (2.15), tends to zero, as $N\to\infty$, since $q\geqq 1$. Since $\{I'_N, I''_N\}$ is a partition of $I_N(S_1,\dots,S_p)$, (2.22) is a consequence of (2.26), (2.29) and (2.25).

Corollary (2.3). *Assume that* (2.14) *is fulfilled and that* $E\colon \mathscr{A}\to\mathscr{C}$ *is a mixing linear map. Then* $\forall k\in\mathbb{N}$ *and for any* $b_1,\dots,b_k\in B$, *the limit*

$$\lim_{N\to\infty}N^{-k/2}E(S_N(b_1)\cdot\ldots\cdot S_N(b_k))\tag{2.30}$$

is equal to zero if k *is odd and, if* $k=2n$ *for some* $n\in\mathbb{N}$, *it is equal to*

$$\lim_{N\to\infty}N^{-k/2}\sum_{\text{p.p.}}\sum_{\alpha\in I_N(S_1,\dots,S_n)}\varepsilon(\alpha; b_1,\dots,b_{2n})\,E(J_{\alpha(S_1)}(b_{S_1})\cdot\ldots\cdot E(J_{\alpha(S_n)}(b_{S_n}))\tag{2.31}$$

where the first sum is extended to all the ordered pair partitions of the set $\{1,\dots,2n\}$.

Proof. Lemma (2.1) implies that

$$\begin{aligned}&N^{-k/2}E(S_N(b_1)\cdot\ldots\cdot S_N(b_k))\\&=N^{-k/2}\sum_{p=1}^{p}\sum_{(S_1,\dots,S_p)\in\mathscr{P}_{k,p}}\sum_{\alpha\in I_N(S_1,\dots,S_p)}\varepsilon(\alpha; b_1,\dots,b_k)\,E(J_{\alpha(S_1)}(b_{S_1})\cdot\ldots\cdot J_{\alpha(S_p)}(b_{S_p})).\end{aligned}\tag{2.32}$$

Because of Lemma (2.2) the only terms of the first two sums which might not vanish in the limit $N\to\infty$ are those corresponding to the partitions $(S_1,\dots,S_p)$ with $p\geqq k/2$ and with no singletons. But since in any case the number of singletons q of a partition $(S_1,\dots,S_p)$ satisfies $k\geqq 2p-q$, if $q=0$ we have also $k/2\geqq p$ and therefore $k=2p$, hence for k-odd the limit (2.30) vanishes.

If $k=2n$, the only partitions $(S_1,\dots,S_n)$ with no singletons are the pair partitions. Moreover, in the limit $N\to\infty$, one can restrict the α-summation in (2.32) to those α's satisfying

$$\alpha(S_{j+1})-\alpha(S_j)\geqq d(N),\quad j=1,\dots,n-1.\tag{2.33}$$

In fact the number of those α's for which (2.33) is not fulfilled is less or equal than $N^{n-1}d(N)$ hence, by the same argument as in Lemma (2.2), their contribution to the sum tends to zero as $N\to\infty$. But if $\alpha\in I_N(S_1,\dots,S_n)$ satisfies (2.33) then, using (2.20), one obtains:

$$|N^{-n}\sum_{\text{p}\cdot\text{p}\cdot}\sum_{\alpha\in I_N(S_1,\dots,S_n)}\varepsilon(\alpha;b_1,\dots,b_k)\,[E(J_{\alpha_1}(b_{S_1})\cdot\ldots\cdot J_{\alpha n}(b_{S_n})$$
$$-E(J_{\alpha_1}(b_{S_1}))\cdot\ldots\cdot E(J_{\alpha_n}(b_{S_n}))]|\leqq v_0\,\bar{v}\,\delta(N)\to 0\qquad(\text{as } N\to\infty)$$

and this proves the statement.

Now let us consider the limit (2.31) in the boson case, i.e.

$$J_h(b)\cdot J_k(b')=J_k(b')\,J_n(b);\qquad h\neq k,\ h,k\in\mathbb{N};\qquad b,b'\in B$$

In this case (cf. Remark 1 after Lemma (2.1)), $\varepsilon(\alpha;b_1,\dots,b_{2n})=n!$ and, since the number of non-injective maps $\{1,\dots,n\}\to\{1,\dots,N\}$ is of order N^{n-1}:

$$\lim_{N\to\infty}N^{-n}\sum_{\alpha\in I_N(S_1,\dots,S_n)}n!\,E(J_{\alpha_1}(b_{S_1}))\ldots E(J_{\alpha_n}(b_{S_n}))$$
$$=\lim_{N\to\infty}\left[\frac{1}{N}\sum_{\alpha_1=1}^{N}E(J_{\alpha_1}(b_{S_1}))\right]\cdot\ldots\cdot\left[\frac{1}{N}\sum_{\alpha_n=1}^{N}E(J_{\alpha_n}(b_{S_n}))\right]\tag{2.34}$$

for any pair partition $(S_1,\dots,S_n)$ of $\{1,\dots,2n\}$.

Therefore the limit (2.31) exists if and only if for each pair $b,b'\in B$ the limit

$$\lim_{N\to\infty}\frac{1}{N}\sum_{\alpha=1}^{N}E(J_\alpha(b\cdot b'))=C(b\cdot b')\tag{2.35}$$

exists in any topology weaker than the topology induced on $\mathscr{C}$ by the semi-norm $|\cdot|$. In such case the limit will exist in this topology and will be equal to:

$$\sum_{\text{p.p.}}C(b_{S_1})\cdot C(b_{S_2})\cdot\ldots\cdot C(b_{S_n})\tag{2.36}$$

the sum being extended to all pair ordered partitions $(S_1,\dots,S_n)$ of the set $\{1,\dots,2n\}$.

In the fermion case, we assume that for any pair $b,b'\in B$ the factor $\varepsilon_{h,k}(b,b')$ in (2.1) is always equal to -1. In this case the computation of the factors $\varepsilon(\alpha;b_1,\dots,b_{2n})$ with $n\in\mathbb{N}$, $b_1,\dots,b_{2n}\in B$, and $\alpha\in I_N(S_1,\dots,S_n)$ where $(S_1,\dots,S_n)$ is a pair partition of $(1,\dots,2n)$ is carried out as follows: first remark that the number of exchanges needed to put the product $J_{\alpha_1}(b_1)\cdot\ldots\cdot J_{\alpha_{2n}}(b_{2n})$ in the form $J_{\alpha(S_1)}(b_{S_1})\cdot\ldots\cdot J_{\alpha(S_n)}(b_{S_n})$ is equal to the number of exchanges of the permutation

$$\begin{pmatrix}1,\dots,2n\\(i_1,h_1),(i_2,h_2),\dots,(i_n,h_n)\end{pmatrix}=\begin{pmatrix}1,\dots,2n\\S_1,\dots,S_n\end{pmatrix}$$

where $S_j=(i_j,h_j)$, $i_j<h_j$. Remark, moreover, that since $|S_j|$ is even for each $j=1,\dots,n$, the signature of this permutation is the same if $S_{\pi(1)},\dots,S_{\pi(n)}$ is substituted for $S_1,\dots,S_n$ (π being an arbitrary permutation of $\{1,\dots,n\}$). Therefore also in the fermion case, the passage from summation over $[S_1,\dots,S_n]_N$ to summation over $I_N(S_1,\dots,S_n)(|S_j|=2;V_j)$ gives rise to the factor $n!$. More pre-

cisely if $(S_1, \dots, S_n)$ is a pair partition of $\{1, \dots, 2n\}$ and $\alpha \in I_N(S_1, \dots, S_n)$, then:

$$\varepsilon(\alpha; b_1, \dots, b_{2n}) = n! \operatorname{sgn} \begin{pmatrix} 1, \dots, 2n \\ S_1, \dots, S_n \end{pmatrix}$$

Therefore with the same argument used to establish (2.34) one obtains:

$$\lim_{N\to\infty} N^{-n} \sum_{\alpha \in I_N(S_1, \dots, S_n)} \varepsilon(\alpha; b_1, \dots, b_{2n}) E(J_{\alpha_1}(b_{S_1})) \cdot \dots \cdot E(J_{\alpha_n}(b_{S_n})) \tag{2.37}$$

$$= \operatorname{sgn} \begin{pmatrix} 1, \dots, 2n \\ S_1, \dots, S_n \end{pmatrix} \cdot \lim_{N\to\infty} \left[\frac{1}{N} \sum_{\alpha_1=1}^{N} E(J_{\alpha_1}(b_{S_1}))\right] \cdot \dots \cdot \left[\frac{1}{N} \sum_{\alpha_n=1}^{N} E(J_{\alpha_n}(b_{S_n}))\right] \tag{2.38}$$

for any pair partition $(S_1, \dots, S_n)$ of $\{1, \dots, 2n\}$.

Therefore also in the fermion case the existence of the limit (2.35) is a necessary and sufficient condition for the existence of the limit (2.31) which, in case of existence, turns out to be equal to:

$$\sum_{\text{p.p.}} \operatorname{sgn} \begin{pmatrix} 1, \dots, 2n \\ S_1, \dots, S_n \end{pmatrix} C(b_{S_1}) \cdot \dots \cdot C(b_{S_n}) \tag{2.39}$$

where again the sum is extended over all the pair partitions $(S_1, \dots, S_n)$ of the set $\{1, \dots, 2n\}$, and for any pair S_j such that $b_{S_j} = b \cdot b'$, $C(b_{S_j})$ is given by (2.35).

In order to complete the discussion of the limit (2.41) for the normalized sums $N^{-\varepsilon} S_N(b)$ $(b \in B)$ where ε is an arbitrary number greater than zero let us remark the following

Lemma (2.4). *Let $\varepsilon > 1/2$, then for any partition $(S_1, \dots, S_p) \in \mathcal{P}_{k,p}$ with no singletons:*

$$\lim_{N\to\infty} |N^{-\varepsilon k} \sum_{\alpha \in I'_N(S_1, \dots, S_p)} \varepsilon(\alpha; b_1, \dots, b_k) E(J_{\alpha_1}(b_{S_1}) \cdot \dots \cdot J_{\alpha_p}(b_{S_p}))| = 0. \tag{2.40}$$

Proof. Since $(S_1, \dots, S_p)$ has no singletons, then $\varepsilon k > k/2 \geqq p$ and, because of (2.11), (2.23), and Remark (3.) after formula (2.21):

$$|N^{-\varepsilon k} \sum_{\alpha \in I'_N(S_1, \dots, S_p)} \varepsilon(\alpha; b_1, \dots, b_k) E(J_{\alpha_1}(b_{S_1}) \cdot \dots \cdot J_{\alpha_p}(b_{S_p}))|$$

$$\leqq N^{-(\varepsilon k - p)} v_0 v \to 0, \quad \text{as } N \to \infty \tag{2.41}$$

and this proves (2.40).

Theorem (2.5). *If $\varepsilon > 1/2$ then $\forall b_1, \dots, b_k \in B$ $(k \in \mathbb{N})$*

$$\lim_{N\to\infty} [N^{-\varepsilon} S_N(b_1)] \cdot \dots \cdot [N^{-\varepsilon} S_N(b_k)] = 0. \tag{2.42}$$

Proof. Immediate from Lemma (2.2) and Lemma (2.4).

In the case $0 < \varepsilon < 1/2$ the following counterexample shows that even in the simplest circumstances the limit (2.42) can only be 0 or ∞. Let E be a stationary product state, i.e. $\mathcal{C} = \mathbb{C}$ and $E: \mathcal{A} \to \mathbb{C}$ is a positive map such that $E(1) = 1$ and, $\forall n \in \mathbb{N}$, $\forall h_1, \dots, h_n \in \mathbb{N}$ $h_i \neq h_j$ for $i \neq j$ and $\forall b_1, \dots, b_n \in \mathcal{B}$, one has:

$$E(J_{h_1}(b_1) \cdot \dots \cdot J_{h_n}(b_n)) = E_0(b_1) \cdot \dots \cdot E_0(b_n)$$

with $E_0 = E \circ J_0$. With the same arguments as in Lemma (2.2), and using the fact that $E(b) = 0$ for $b \in B$, one easily verifies that for $k = 2n$ or $k = 2n+1$:

$$\overline{\lim_{N \to \infty}} [N^{-\varepsilon} S_N(b_1)] \cdot \ldots \cdot [N^{-\varepsilon} S_N(b_k)] = \overline{\lim_{N \to \infty}} N^{(1-2\varepsilon)n} \sum_{\substack{(S_1, \ldots, S_n) \\ |S_j| \geqq 2}} E_0(b_{S_1}) \cdot \ldots \cdot E_0(b_{S_n}) \tag{2.43}$$

and that the corresponding identity holds for liminf. But for $0 < \varepsilon < 1/2$, this limit can only be 0 or ∞.

Acknowledgments. L.A. expresses his warmest thanks to Prof. K.R. Parthasarathy for the kind and stimulating hospitality at ISI.

A.B. acknowledges a grant from the DFG-CNR and is grateful for the hospitality at the Dipartimento di Matematica, Università di Roma II.

Both authors are very grateful to Prof. W. von Waldenfels for many illuminating discussions on the quantum central limit theorems.

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Received February 13, 1984