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On the Length of the Longest Excursion

E. Csáki, P. Erdős, and P. Révész

Mathematical Institute, Reáltanoda u. 13–15, H-1053 Budapest, Hungary

Summary. A lower limit of the length of the longest excursion of a symmetric random walk is given. Certain related problems are also discussed. It is shown e.g. that for any $\varepsilon>0$ and all sufficiently large n there are $c(\varepsilon)$ $\log\log n$ excursions in the interval $(0, n)$ with total length greater than $n(1-\varepsilon)$, with probability 1.

1. Introduction

Let $X_1, X_2, \ldots$ be a sequence of i.i.d.r.v.'s with

$$\mathbb{P}(X_i=+1)=\mathbb{P}(X_i=-1)=\tfrac{1}{2} \qquad (i=1,2,\ldots)$$

and consider the random walk $S(0)=0$, $S(n)=X_1+X_2+\ldots+X_n$ $(n=1,2,\ldots)$. Introduce the following notations:

$$\begin{aligned}\mathscr{P}(n)&=\mathscr{N}o.\{k\colon 0<k\leqq n,\ S(k)>0\},\\ \mathscr{R}(n)&=\mathscr{N}o.\{k\colon 0<k\leqq n,\ S(k)=0\},\end{aligned}$$

($\mathscr{N}o.\{\ldots\}$ stands for cardinality of the set in brackets).

$$\begin{aligned}&\rho_0=0,\\ &\rho_1=\inf\{k\colon k>0,\ S(k)=0\},\\ &\rho_2=\inf\{k\colon k>\rho_1,\ S(k)=0\},\\ &\ldots\ldots\ldots\ldots\ldots\ldots\\ &\rho_{n+1}=\inf\{k\colon k>\rho_n,\ S(k)=0\},\\ &\ldots\ldots\ldots\ldots\ldots\ldots\\ &\mathscr{T}(n)=\max\{\rho_1,\rho_2-\rho_1,\ldots,\rho_{\mathscr{R}(n)}-\rho_{\mathscr{R}(n)-1},n-\rho_{\mathscr{R}(n)}\}.\end{aligned}$$

Here $\mathscr{T}(n)$ is the length of the longest excursion of the random walk $S(0)$, $S(1), \ldots, S(n)$. The main goal of the present paper is to study the properties of $\mathscr{T}(n)$.

The properties of $\mathscr{R}(n)$ and $\mathscr{P}(n)$ were studied by Chung and Hunt (1949) and Chung and Erdős (1952) resp. Here we recall the Chung-Erdős theorem.

Theorem A. *Let $f(x)$ be a non-decreasing function for which $\lim_{x\to\infty} f(x)=\infty$ and put*

$$I(f)=\int_1^\infty \frac{dx}{xf^{\frac{1}{2}}(x)}.$$

Then

$$\mathbb{P}\left\{\mathscr{P}(n)\geqq n\left(1-\frac{1}{f(n)}\right) \text{ i.o.}\right\}=\begin{cases}1 & \text{if } I(f)=\infty,\\ 0 & \text{if } I(f)<\infty\end{cases} \tag{1.1}$$

and

$$\mathbb{P}\left\{\mathscr{P}(n)\leqq \frac{n}{f(n)} \text{ i.o.}\right\}=\begin{cases}1 & \text{if } I(f)=\infty,\\ 0 & \text{if } I(f)<\infty.\end{cases} \tag{1.2}$$

Studying the proof of Theorem A we can realize that the following stronger statement is also proved by Chung and Erdős:

Theorem B

$$\mathbb{P}\left\{\mathscr{T}(n)\geqq n\left(1-\frac{1}{f(n)}\right) \text{ i.o.}\right\}=\begin{cases}1 & \text{if } I(f)=\infty,\\ 0 & \text{if } I(f)<\infty,\end{cases} \tag{1.3}$$

provided that $f(x)\nearrow\infty$.

(1.3) gives the best possible upper bound for $\mathscr{T}(n)$. For example it implies that for any $\varepsilon>0$

$$\mathscr{T}(n)\leqq n-\frac{n}{(\log n)^{2+\varepsilon}}$$

except finitely many n with probability one and

$$\mathscr{T}(n)\geqq n-\frac{n}{(\log n)^2}$$

infinitely often with probability one. We are interested to find a lower bound for $\mathscr{T}(n)$. Our main result is

Theorem 1. *Let $f(x)$ be a non-decreasing function for which*

$$f(x)\nearrow\infty, \quad \frac{x}{f(x)}\nearrow\infty \quad \text{as} \quad x\to\infty$$

and let

$$\mathscr{I}(f)=\sum_{n=1}^\infty \frac{f(n)}{n}e^{-f(n)}.$$

Then

$$\mathbb{P}\left\{\mathscr{T}(n)\leqq \beta\frac{n}{f(n)} \text{ i.o.}\right\}=\begin{cases}0 & \text{if } \mathscr{I}(f)<\infty,\\ 1 & \text{if } \mathscr{I}(f)=\infty\end{cases} \tag{1.4}$$

where $\beta=0{,}85403\ldots$ is the root of the equation

$$\sum_{k=1}^\infty \frac{\beta^k}{k!(2k-1)}=1. \tag{1.5}$$

(1.4) says for example that

$$\liminf_{n\to\infty} \frac{\log\log n}{n} \mathscr{T}(n) = \beta \quad \text{a.s.} \tag{1.6}$$

Remark. Equation (1.5) emerges in a paper by Shepp (1967) (see also Greenwood and Perkins (1983)).

Beside of studying the properties of the length of the longest excursion, it looks interesting to say something about the second, third ... etc. longest excursion. Consider the sample $\rho_1, \rho_2-\rho_1, \ldots, \rho_{\mathscr{R}(n)}-\rho_{\mathscr{R}(n)-1}, n-\rho_{\mathscr{R}(n)}$ (the lengths of the excursions) and the corresponding ordered sample $\mathscr{T}_1(n) = \mathscr{T}(n) \geqq T_2(n) \geqq \ldots \geqq \mathscr{T}_{\mathscr{R}(n)+1}(n)$. Now we present our

Theorem 2. *For any fixed* $k=1,2,\ldots$ *we have*

$$\liminf_{n\to\infty} \frac{\log\log n}{n} \sum_{j=1}^{k} \mathscr{T}_j(n) = k\beta \quad \textit{a.s.}$$

This Theorem, in some sense, answers the question "How small can be $\mathscr{T}_2(n), \mathscr{T}_3(n), \ldots$?" In order to obtain a more complete description of these r.v.'s we present the following:

Problem 1. Characterize the set of those non-decreasing functions $f(n)$ $(n=1,2,\ldots)$ for which

$$\mathbb{P}\left\{\mathscr{T}_2(n) \geqq \frac{n}{2}\left(1-\frac{1}{f(n)}\right) \text{ i.o.}\right\} = 1.$$

(1.3) says that for some n nearly the whole random walk $S(0), S(1), \ldots, S(n)$ is one excursion. (1.4) and (1.6) say that for some n the random walk consists of at least $\beta^{-1} \log\log n$ excursions. These results suggest the question: For what value of $k=k(n)$ will the sum $\sum_{j=1}^{k} \mathscr{T}_j(n)$ be nearly equal to n? In fact we formulate two questions:

Question 1. For any $\varepsilon>0$ let $\mathscr{F}(\varepsilon)$ be the set of those functions $f(n)$ $(n=1,2,\ldots)$ for which

$$\sum_{j=1}^{f(n)} \mathscr{T}_j(n) \geqq n(1-\varepsilon)$$

with probability one except finitely many n. How can we characterize $\mathscr{F}(\varepsilon)$ for some $\varepsilon>0$?

Question 2. Let $\mathscr{F}(o)$ be the set of those functions $f(n)$ $(n=1,2,\ldots)$ for which

$$\lim_{n\to\infty} n^{-1} \sum_{j=1}^{f(n)} \mathscr{T}_j(n) = 1 \quad \text{a.s.}$$

How can we characterize $\mathscr{F}(o)$?

Studying our first question we have

Theorem 3. *For any* $\varepsilon>0$ *there exists a* $C=C(\varepsilon)>0$ *such that*

$$C \log\log n \in \mathscr{F}(\varepsilon).$$

Concerning our Question 2, we have the following result:

Theorem 4. *For any* $C>0$

$$f(n)=C \log\log n \notin \mathscr{F}(o) \tag{1.7}$$

and for any $\omega(n)\nearrow\infty$ $(n\to\infty)$

$$\omega(n) \log\log n \in \mathscr{F}(o). \tag{1.8}$$

2. Proof of Theorem 1

We recall the following well-known

Theorem C

$$b_k=\mathbb{P}(\rho_1=2k)=\frac{1}{k2^{k-1}}\binom{2k-2}{k-1} \qquad (k=1,2,\ldots).$$

Consequently

$$b_k=\frac{1}{2\sqrt{\pi}}\,k^{-\frac{3}{2}}\exp(\vartheta_k/k) \tag{2.1}$$

where $|\vartheta_k|\leqq 1$.

By (2.1) we easily obtain

Lemma 1

$$\sum_{j=1}^{a} b_j \exp(j\beta/a)=1+\mathcal{O}(a^{-\frac{3}{2}}) \qquad (a\to\infty) \tag{2.2}$$

where β *is the root of Eq.* (1.5).

Proof. Clearly we have

$$\sum_{j=1}^{a} b_j \exp\left(\frac{j\beta}{a}\right)=\sum_{j=1}^{a} b_j+\sum_{j=1}^{a} b_j\left(\exp\left(\frac{j\beta}{a}\right)-1\right)$$
$$=\sum_{j=1}^{a} b_j+\frac{1}{2\sqrt{\pi}}\sum_{j=1}^{a} j^{-\frac{3}{2}}\left(\exp\left(\frac{j\beta}{a}\right)-1\right)+\frac{1}{2\sqrt{\pi}}\sum_{j=1}^{a} j^{-\frac{3}{2}}\left(\exp\left(\frac{\vartheta_j}{j}\right)-1\right)$$
$$\times\left(\exp\left(\frac{j\beta}{a}\right)-1\right)=A_1+A_2+A_3$$

and

$$A_1=1-(\pi a)^{-\frac{1}{2}}+\mathcal{O}(a^{-\frac{3}{2}})$$
$$A_2=\frac{1}{2\sqrt{\pi}}\sum_{k=1}^{\infty}\frac{\beta^k}{a^k\,k!}\sum_{j=1}^{a} j^{k-\frac{3}{2}}$$
$$=(\pi a)^{-\frac{1}{2}}\sum_{k=1}^{\infty}\frac{\beta^k}{k!(2k-1)}+\mathcal{O}(a^{-\frac{3}{2}})=(\pi a)^{-\frac{1}{2}}+\mathcal{O}(a^{-\frac{3}{2}})$$

$$A_3 = \mathcal{O}(a^{-\frac{3}{2}})$$

what proves (2.2).

Let

$$p_n = p_n(a) = \begin{cases} \mathbb{P}(\mathcal{T}(2n) \leqq 2a) & \text{if } n \geqq a, \\ 1 & \text{if } n < a. \end{cases}$$

Then we clearly have

Lemma 2.

$$p_n = \sum_{j=1}^{a} p_{n-j} b_j \qquad (n = a, a+1, \ldots). \tag{2.3}$$

Now we are looking for the solution of (2.3) satisfying the initial condition $p_n = 1$ if $n = 1, 2, \ldots, a-1$. We obtain

Lemma 3. *There exist positive constants* $0 < C_1 \leqq C_2 < \infty$ *such that*

$$p_n(a) = C(n, a) \exp\left(-\beta \frac{n}{a}\right) \tag{2.4}$$

and

$$C_1 \leqq C(n, a) \leqq C_2$$

provided that $n \leqq a^{\frac{3}{2}}$.

From now on C (with or without index) will stand for an absolute constant whose actual value may change from line to line.

Proof. Replacing (2.4) in (2.3) we get

$$C(n, a) = \sum_{j=1}^{a} C(n-j, a) \exp\left(\beta \frac{j}{a}\right) b_j. \tag{2.5}$$

In case $n \leqq 2a$ our statement is trivial. For $n > 2a$ the statement follows from (2.5) by induction.

Lemma 4. *Let* $f(n)$ $(n = 1, 2, \ldots)$ *be a non-decreasing positive function for which*

$$f(n) < \tfrac{1}{4} \log\log n \quad i.o. \tag{2.6}$$

and put

$$\mathcal{J} = \mathcal{J}(f) = \sum_{n=1}^{\infty} \frac{f(n)}{n} e^{-f(n)}, \qquad \bar{\mathcal{J}} = \bar{\mathcal{J}}(f) = \sum_{k=2}^{\infty} e^{-f(n_k)}$$

$$n_k = \left[\exp \frac{k}{\log k}\right] \qquad (k = 2, 3, \ldots).$$

Then $\mathcal{J} = \bar{\mathcal{J}} = \infty$.

Proof. Suppose that $f(N) < \frac{1}{4} \log\log N$ for a fixed N. Then a simple calculation gives

$$\mathcal{J} \geqq \sum_{n=1}^{N} \frac{f(n)}{n} e^{-f(n)} \geqq C e^{-f(N)} \sum_{n=1}^{N} \frac{1}{n} \geqq C (\log N)^{\frac{3}{4}}. \tag{2.7}$$

Thus $\mathscr{J} \geqq C(\log N)^{\frac{3}{4}}$ for infinitely many N, we have $\mathscr{J} = \infty$. One can see similarly that $\bar{\mathscr{J}} = \infty$ by observing that condition (2.6) implies that $f(n_k) < \frac{1}{2} \log\log n_k$ i.o.

Lemma 5. *Let $f(n)$ $(n=1, 2, \ldots)$ be a non-decreasing, positive function. Then $\mathscr{J} = \infty$ if and only if $\bar{\mathscr{J}} = \infty$.*

Such a lemma like this and the previous lemma is frequently used in the proofs of theorems like our Theorem 1, hence its proof is routine. For the convenience of the reader we present it.

Proof

$$\mathscr{J} = \sum_{k=2}^{\infty} \sum_{j=n_k+1}^{n_{k+1}} \frac{f(j)}{j} e^{-f(j)} \leqq C \sum_k \frac{n_{k+1} - n_k}{n_k} f(n_k) e^{-f(n_k)}$$
$$\leqq C \sum_k \frac{f(n_k)}{\log k} e^{-f(n_k)} = C \mathscr{J}^*.$$

Similarly one can obtain that

$$\mathscr{J}^* \leqq C \mathscr{J}.$$

By Lemma 4 one can assume that $f(n) > \frac{1}{4} \log\log n$ $(n=3, 4, \ldots)$. Hence we have

$$\mathscr{J}^* \geqq C \bar{\mathscr{J}}$$

that is $\bar{\mathscr{J}} = \infty$ implies $\mathscr{J} = \infty$. In order to see the converse statement let

$$A = \{k : f(n_k) < 2 \log\log n_k\}.$$

Then

$$\mathscr{J}^* = \sum_{k \in A} \frac{f(n_k)}{\log k} e^{-f(n_k)} + \sum_{k \notin A} \frac{f(n_k)}{\log k} e^{-f(n_k)}$$
$$\leqq C \sum_{k \in A} e^{-f(n_k)} + \sum_{k \notin A} \frac{f(n_k)}{\log k} e^{-f(n_k)} \leqq C \bar{\mathscr{J}} + C$$

what proves the implication: if $\mathscr{J} = \infty$ then $\bar{\mathscr{J}} = \infty$.

The following lemma is trivial, we give it without proof.

Lemma 6. *Let $\{a_k\}$ be a non-increasing sequence of positive numbers for which $\sum_{n=1}^{\infty} a_n < \infty$. Then*

$$\sum_{n=1}^{\infty} (a_n)^{1 - \frac{1}{\log n}} < \infty.$$

Lemma 7. *Let*

$$A_k = \{\mathscr{T}(n_k) \leqq a_k\} \qquad k = 2, 3, \ldots$$

where

$$a_k = \frac{\beta n_k}{f(n_k)}$$

and $f(n)$ is a non-decreasing positive function such that $f(n) \leqq \beta n^{\frac{1}{3}}$. Then

$$\mathbb{P}(A_k A_l) \leqq C \mathbb{P}(A_k) \exp\left(-\beta \frac{n_l - n_k}{a_l}\right) \qquad (1 \leqq k < l < \infty). \tag{2.8}$$

Proof. Let $\mathscr{T}(a, b)$ $(0 \leqq a < b < \infty)$ be the length of the longest excursion of the random walk $S(a), S(a+1), \ldots, S(b)$. Then

$$\begin{aligned}
\mathbb{P}(A_k A_l) &\leqq \mathbb{P}(\mathscr{T}(n_k) \leqq a_k, \mathscr{T}(n_k, n_l) \leqq a_l)\\
&= \sum_j \mathbb{P}(\mathscr{T}(n_k) \leqq a_k \mid S(n_k) = j)\, \mathbb{P}(\mathscr{T}(n_k, n_l) \leqq a_l \mid S(n_k) = j)\, \mathbb{P}(S(n_k) = j)\\
&\leqq \sum_j \mathbb{P}(\mathscr{T}(n_k) \leqq a_k \mid S(n_k) = j)\, \mathbb{P}(\mathscr{T}(n_l - n_k) \leqq a_l)\, \mathbb{P}(S(n_k) = j)\\
&= \mathbb{P}(\mathscr{T}(n_l - n_k) \leqq a_l)\, \mathbb{P}(A_k) \leqq C \exp\left(-\beta \frac{n_l - n_k}{a_l}\right) \mathbb{P}(A_k)
\end{aligned}$$

(the last inequality follows from Lemma 3). Hence we have (2.8)

Lemma 8. *Let $f(n)$ be a non-decreasing function for which $\mathscr{I}(f) = \infty$. Then for any $0 < \varepsilon < 1$ there exists a non-decreasing function $\bar{f}$ such that*

(i) $\bar{f}(n) \geqq f(n)$ $(n = 1, 2, \ldots)$,
(ii) $\mathscr{I}(\bar{f}) = \infty$,
(iii) $\bar{f}(n) \geqq \varepsilon \operatorname{loglog} n$.

Proof of this lemma is based on the same idea as that of Lemma 4 and will be omitted.

Lemma 9. *Let*

$$B_n = \{\mathscr{T}(n) \leqq b_n\}$$

where

$$b_n = \frac{\beta n}{f(n)}$$

and $f(n)$ is a non-decreasing function for which $\mathscr{I} < \infty$. Then

$$\mathbb{P}(B_n \ i.o.) = 0. \tag{2.9}$$

Proof. Let

$$\tilde{f}(n_k) = \frac{n_k}{n_{k+1}} f(n_{k+1}).$$

Then by Lemmas 5 and 6

$$\sum_{k=2}^{\infty} e^{-\tilde{f}(n_k)} < \infty$$

and by Lemma 3

$$\mathbb{P}\left(\mathscr{T}(n_k) \leqq \frac{\beta n_k}{\tilde{f}(n_k)} \text{ i.o.}\right) = 0,$$

provided that $f(n) \leqq \beta n^{\frac{1}{3}}$.

Now let $n_k \leqq n \leqq n_{k+1}$ then

$$\mathscr{T}(n) \geqq \mathscr{T}(n_k) \geqq \frac{\beta n_k}{\tilde{f}(n_k)} = \beta \frac{n_{k+1}}{f(n_{k+1})} \geqq \beta \frac{n}{f(n)}$$

with probability one except finitely many k. Hence we have (2.9), if $f(n) \leqq \beta n^{\frac{1}{3}}$ $(n \geqq n_0)$.

In the case when this condition does not hold, define $f_1(n)=\min(f(n), \beta n^{\frac{1}{3}})$. $f_1(n)$ is non-decreasing with $\mathscr{I}(f_1)<\infty$ and $f_1(n)\leqq\beta n^{\frac{1}{3}}$, hence (2.9) holds for $f(n)$ replaced by $f_1(n)$. Since $f_1(n)\leqq f(n)$, we have also (2.9) with the original $f(n)$. This proves the first part of Theorem 1.

To show the second part, assume that

$$\tfrac{1}{5}\log\log n\leqq f(n)\leqq 2\log\log n \qquad (n=3,4,\ldots) \tag{2.10}$$

The lower inequality can be assumed by Lemma 8, while if the upper inequality does not hold for all n large enough, then by eliminating those n's for which $f(n)>2\log\log n$, the whole procedure below can be done for the remaining subsequence and still conclude the second part of (1.4).

Defining n_k as in Lemma 4, for large enough k and $k<l$ we have

$$\begin{aligned}\log\frac{n_l}{n_k}&=\frac{l-k}{\log l}-\frac{k(\log l-\log k)}{\log l\log k}\\&\geqq\frac{l-k}{\log l}-\frac{l-k}{(\log l)(\log k)}\geqq\frac{1}{2}\frac{l-k}{\log l}.\end{aligned} \tag{2.11}$$

Now for k fixed, split the indices l $(k<l\leqq n)$ into three parts:

$$\begin{aligned}&L_1=\{l\colon 0<l-k\leqq\log l\}\\&L_2=\{l\colon \log l<l-k\leqq\log^2 l\}\\&L_3=\{l\colon \log^2 l<l-k\}.\end{aligned}$$

For $l\in L_1$ we have from (2.10) and (2.11)

$$\begin{aligned}\frac{n_l-n_k}{n_l}f(n_l)&\geqq\left(1-\exp\left(-\frac{1}{2}\frac{l-k}{\log l}\right)\right)\frac{1}{5}\log\log n_l\\&\geqq c\frac{l-k}{\log l}\log\frac{l}{\log l}\geqq c(l-k).\end{aligned} \tag{2.12}$$

For $l\in L_1$ we have from (2.11)

$$\frac{n_l-n_k}{n_l}\geqq 1-\exp\left(-\frac{1}{2}\frac{l-k}{\log l}\right)\geqq 1-e^{-\frac{1}{2}}=c'>0.$$

Hence by Lemma 7 and (2.10)

$$\begin{aligned}P(A_kA_l)&\leqq cP(A_k)e^{-c'f(n_l)}\leqq cP(A_k)\left(\frac{\log l}{l}\right)^{c_1'}\\&\leqq cP(A_k)\left(\frac{\log k}{k}\right)^{c_1'}.\end{aligned} \tag{2.13}$$

For $l\in L_3$ we have from (2.10) and (2.11)

$$\frac{n_k}{n_l}f(n_l)\leqq f(n_l)\exp\left(-\frac{1}{2}\frac{l-k}{\log l}\right)\leqq f(n_l)\exp\left(-\frac{1}{2}\log l\right)\leqq c. \tag{2.14}$$

Hence
$$\sum_{l\in L_1} P(A_k A_l) \leqq c P(A_k) \sum_{l\in L_1} e^{-c'(l-k)} \leqq c P(A_k), \tag{2.15}$$

$$\sum_{l\in L_2} P(A_k A_l) \leqq c P(A_k) \left(\frac{\log k}{k}\right)^{c_1'} \sum_{l\in L_2} 1 \leqq c P(A_k), \tag{2.16}$$

since $\sum_{l\in L_2} 1 \leqq c(\log k)^2$.

By Lemmas 3 and 7, (2.14), (2.15), (2.16)

$$\sum_k P(A_k) = \infty$$

and

$$\sum_{l=1}^{n} \sum_{k=1}^{n} \mathbb{P}(A_k A_l) \leqq C \left(\sum_{k=1}^{n} \mathbb{P}(A_k)\right)^2 + C \sum_{k=1}^{n} \mathbb{P}(A_k)$$

consequently

$$\liminf_{n\to\infty} \frac{\sum_{l=1}^{n} \sum_{k=1}^{n} \mathbb{P}(A_k A_l)}{\left(\sum_{k=1}^{n} \mathbb{P}(A_k)\right)^2} \leqq C$$

and by the Borel-Cantelli lemma (cf. Spitzer (1964)) we have

$$\mathbb{P}(A_k \text{ i.o.}) \geqq C^{-1} > 0.$$

Hence we have our Theorem 1 by the 0–1 law.

3. Proof of Theorem 2

We give the following analogue of Lemma 3.

Lemma 10. *Let*

$$\tilde{p}_n = \tilde{p}_n(a) = \begin{cases} \mathbb{P}(\mathscr{T}(2n) \leqq 2a,\ S(2n) = 0) & \text{if } n \geqq a, \\ \mathbb{P}(S(2n) = 0) & \text{if } n < a. \end{cases}$$

Then there exist positive constants $0 < C_1 \leqq C_2 < \infty$ *such that*

$$\tilde{p}_n(a) = C(n,a) \min((n+1)^{-\frac{1}{2}}, a^{-\frac{1}{2}}) \exp\left(-\beta \frac{n}{a}\right)$$

and

$$C_1 \leqq C(n,a) \leqq C_2$$

provided that $0 \leqq n \leqq a^{\frac{3}{2}}$.

Proof. Observe that the statement is trivial if $0 \leqq n \leqq 2a$ and we have

$$\tilde{p}_n = \sum_{j=1}^{a} \tilde{p}_{n-j} b_j \qquad (n \geqq 2a).$$

Now we obtain our Lemma 10 using the method of proof of Lemma 3.

Let $a_1, a_2, \ldots, a_k$ be a sequence of positive integers and $m = \min(a_1, a_2, \ldots, a_k)$. Further let $d_1, d_2, \ldots, d_{k+1}$ be a sequence of non-negative integers such that

$$\begin{gathered} d_1 \geqq 0, d_2 \geqq 0, d_3 \geqq 0, \ldots, d_k \geqq 0, d_{k+1} \geqq 0 \\ d_1 + d_2 + \ldots + d_{k+1} + a_1 + a_2 + \ldots + a_k = n \end{gathered} \tag{3.1}$$

Introduce the following notations

$$\begin{aligned} &B_1 = \{S(d_1) = 0,\ \mathcal{T}(0, d_1) \leqq m\}, \\ &A_1 = \{S(d_1 + i) \neq 0\ (i = 1, 2, \ldots, a_1 - 1), S(d_1 + a_1) = 0\}, \\ &B_2 = \{S(d_1 + a_1 + d_2) = 0,\ \mathcal{T}(d_1 + a_1, d_1 + a_1 + d_2) \leqq m\}, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &A_k = \{S(d_1 + a_1 + \ldots + a_{k-1} + d_k + i) \neq 0\ (i = 1, 2, \ldots, a_k - 1), \\ &\qquad S(d_1 + a_1 + \ldots + d_k + a_k) = 0\} \\ &B_{k+1} = \{S(d_1 + a_1 + \ldots + d_k + a_k) = 0,\ \mathcal{T}(d_1 + a_1 + \ldots + d_k + a_k, n) \leqq m\}, \\ &\quad A = A_1 A_2 \ldots A_k B_1 B_2 \ldots B_{k+1}, \\ &\quad A^* = A^*(a_1, a_2, \ldots, a_k) = \Sigma^* A \end{aligned}$$

where in the sum $\sum^*$ the indices $d_1, d_2, \ldots, d_{k+1}$ run over all $(k+1)$-tuples of integers which satisfy (3.1). Clearly A^* is the event that the random walk $S(1), S(2), \ldots, S(n)$ consists of excursions of size $a_1, a_2, \ldots, a_k$ in this order but all other excursions are shorter than m.

Lemma 11. *Let $n \leqq m^{\frac{3}{2}}$. Then*

$$\mathbb{P}(A^*) \leqq C m^{-\frac{3k}{2}} (n m^{-\frac{1}{2}} + m^{\frac{1}{2}})^k \exp\left(-\beta m^{-1}\left(n - \sum_{i=1}^{k} a_i\right)\right) \tag{3.2}$$

where the constant C may depend on k.

Proof. By Theorem C we have

$$P(A_i / A_1 \ldots A_{i-1} B_1 \ldots B_i) \leqq C a_i^{-\frac{3}{2}} \qquad (i = 1, \ldots, k).$$

Since $d_i \leqq n \leqq m^{\frac{3}{2}}$, by Lemma 10

$$P(B_i / A_1 \ldots A_{i-1} B_1 \ldots B_{i-1}) \leqq C(m^{-\frac{1}{2}} + (d_i + 1)^{-\frac{1}{2}}) \exp(-\beta m^{-1} d_i) \qquad (i = 1, \ldots, k)$$

and by Lemma 3

$$P(B_{k+1} / A_1 \ldots A_k B_1 \ldots B_k) \leqq C \exp(-\beta m^{-1} d_{k+1}).$$

Hence

$$P(A) \leqq C \exp\left(-\beta m^{-1}\left(n - \sum_{i=1}^{k} a_i\right)\right) \prod_{i=1}^{k} (m^{-\frac{1}{2}} + (d_i + 1)^{-\frac{1}{2}}).$$

Now

$$\sum_{d_i}(m^{-\frac{1}{2}}+(d_i+1)^{-\frac{1}{2}})\leqq c(nm^{-\frac{1}{2}}+m^{\frac{1}{2}})$$

and (3.2) follows.

A trivial consequence of Lemma 11 is

Lemma 12. *Let* $a_1 \geqq a_2 \geqq \ldots \geqq a_k \geqq n^{\frac{2}{3}}$ *be a sequence of integers. Then*

$$P(T_1(n)=a_1, \ldots, T_k(n)=a_k)\leqq c\, a_k^{-\frac{3k}{2}}(n a_k^{-\frac{1}{2}}+a_k^{\frac{1}{2}})^k \exp\left(-\beta a_k^{-1}\left(n-\sum_{i=1}^{k} a_i\right)\right). \tag{3.3}$$

Lemma 13. *Let* $kn^{\frac{2}{3}} \leqq u < n.$

$$P\left(\sum_{j=1}^{k} T_j(n)\leqq u,\ T_k(n)\geqq n^{\frac{2}{3}}\right)\leqq c u^k u^{-\frac{3k}{2}}(nu^{-\frac{1}{2}}+u^{\frac{1}{2}})^k \exp\left(-\beta k\frac{n-u}{u}\right). \tag{3.4}$$

Proof. (3.4) follows from (3.3) by summation for the possible a_i $(i=1,2,\ldots,k)$ observing that $n^{\frac{2}{3}}\leqq a_k \leqq u/k$ and the fact that a sequence $a_1\geqq a_2\geqq \ldots \geqq a_k \geqq n^{\frac{2}{3}}$ of integers for which $\sum_{i=1}^{k} a_i \leqq u$ can be chosen at most u^k different ways.

Lemma 14. *For large enough n we have*

$$P\left(\sum_{j=1}^{k} T_j(n)\leqq \beta(1-\varepsilon)k\frac{n}{\log\log n}\right)\leqq c(\log n)^{-\left(1+\frac{\varepsilon}{2}\right)}. \tag{3.5}$$

Proof. By letting

$$u=u_n=\beta(1-\varepsilon)k\frac{n}{\log\log n}$$

we obtain from (3.4) that

$$P\left(\sum_{j=1}^{k} T_j(n)\leqq u_n,\ T_k(n)\geqq n^{\frac{2}{3}}\right)\leqq c(\log n)^{-\left(1+\frac{\varepsilon}{2}\right)}. \tag{3.6}$$

Furthermore

$$P\left(\sum_{j=1}^{k} T_j(n)\leqq u_n/k,\ T_k(n)< n^{\frac{2}{3}}\right)\leqq P(T_1(n)\leqq u_n/k)\leqq c(\log n)^{-\left(1+\frac{\varepsilon}{2}\right)}.$$

Finally, if $u_n/k\leqq a_1+\ldots+a_k\leqq u_n$ and $a_k<n^{\frac{2}{3}}$, then $\max_{1\leqq i\leqq k+1} d_i \geqq c_2 n$ with some constant c_2, i.e. there exists an interval of length $\geqq c_2 n$ such that longest excursion within this interval is shorter than $n^{\frac{2}{3}}$, the probability of which is less than $c_1 e^{-c_3 n^{\frac{1}{3}}}$. The number of possible choices of $a_1 \ldots a_k, d_1 \ldots d_{k+1}$ is obviously at most n^{2k+1}, hence

$$P\left(u_n/k\leqq \sum_{j=1}^{k} T_j(n)\leqq u_n,\ T_k(n)<n^{\frac{2}{3}}\right)\leqq c_1 n^{2k+1} e^{-c_3 n^{\frac{1}{3}}}. \tag{3.8}$$

Since for large n the upper bound in (3.8) is less than the upper bound in (3.5), we have Lemma 14 by combining (3.6), (3.7) and (3.8) with some constant c (different from that in (3.6) and (3.7)).

(3.5) by well-known methods implies

$$\liminf_{n\to\infty} \frac{\log\log n}{n} \sum_{j=1}^{k} \mathcal{T}_j(n) \geqq k\beta \text{ a.s.} \tag{3.9}$$

Now Theorem 2 follows from (1.6) and (3.9).

4. Proof of Theorem 3

Instead of proving Theorem 3 we prove the analogue statement (Theorem 3*) for a Wiener process $\{W(t), t \geqq 0\}$. Theorem 3 can be obtained from Theorem 3* constructing the sequence $\{X_i\}$ from $W(t)$ by the Skorohod stopping rule.

Theorem 3*. *Let $\{W(t), t \geqq 0\}$ be a Wiener process. Then for any $\varepsilon > 0$ there exist $\alpha(T) = [C_1 \varepsilon^{-1} \log\log T]$ excursions $\mathscr{E}_1, \mathscr{E}_2, \ldots, \mathscr{E}_{\alpha(T)}$ of W in $[0, T]$ such that*

$$\sum_{i=1}^{\alpha(T)} |\mathscr{E}_i| \geqq (1-\varepsilon) T$$

if T is large enough with probability one where $|\mathscr{E}_i|$ is the length of the excursion $\mathscr{E}_i$ and C_1 is an absolute constant.

Introduce the following notations: Let $a_T > 0$ be a function of T and

$$Y_0 = 0,$$

$$Y_i = Y_i(T) = \inf\{s: s > Y_{i-1} + a_T, W(s) = 0\} \qquad (i = 1, 2, \ldots),$$

$$\nu_T = \max\{k: Y_k \leqq T\},$$

$$Z_i = Z_i(T) = Y_i - (Y_{i-1} + a_T),$$

$$M_i = Z_i / a_T.$$

The following lemma is well-known.

Lemma 15. (i) *$\{Z_i\}$ is a sequence of i.i.d.r.v.'s,*

(ii) *$\{U_i\} = \{Z_i(W(Y_{i-1} + a_T))^{-2}\}$ is a sequence of i.i.d.r.v.'s*

(iii) $\mathbb{P}(U_i < x) = \mathbb{P}(U_i < x \mid W(Y_{i-1} + a_T) = w) = (2\pi)^{-\frac{1}{2}} \int_0^x v^{-\frac{3}{2}} e^{-\frac{1}{2v}} dv,$

(iv) $E(\exp\{-t U_i\}) = \exp\{-t^{\frac{1}{2}}\}$, $t > 0$.

The next lemma is an easy consequence of a theorem of Steinebach (1978).

Lemma 16

$$\lim_{m\to\infty} (P(M_1 + \ldots + M_m \leqq \alpha m))^{\frac{1}{m}} = \rho(\alpha),$$

where $\rho(\alpha) = \inf_t (\lambda(t) e^{\alpha t})$, $\lambda(t) = E(e^{-t M_1})$.

Lemma 17. *Let*

$$a_T = \varepsilon^2 T(\log\log T)^{-1},$$

$$m_T = [(3\pi)^{\frac{1}{2}} \varepsilon^{-1} \log\log T].$$

Then

$$\mathbb{P}(\nu_T > m_T) = O((\log T)^{-\frac{3}{2}}) \quad as \quad T \to \infty.$$

Proof

$$\mathbb{P}(\nu_T > m_T) = \mathbb{P}(Z_1 + Z_2 + \ldots + Z_{m_T} + m_T a_T \leqq T)$$
$$\leqq \mathbb{P}(Z_1 + \ldots + Z_{m_T} \leqq T) = \mathbb{P}\left(M_1 + \ldots + M_{m_T} \leqq \frac{T}{a_T m_T} m_T\right).$$

It is easy to check that

$$\lambda(t) = E(e^{-tM_1}) = 2e^{\frac{t}{2}}(1 - \phi(t^{\frac{1}{2}})) \leqq \exp\{-(2t/\pi)^{\frac{1}{2}}\}$$

where $\phi(x)$ is the standard normal distribution function and hence

$$\rho(\alpha) < \exp\{-(2\pi\alpha)^{-1}\}.$$

Lemma 17 now follows from Lemma 16.

Considering the excursions $\mathscr{E}_i$ around the points $Y_i + a_T$ $(i = 1, 2, \ldots, \nu_T)$ the non-covered part of the interval $[0, T]$ will be less than $\nu_T a_T$. Hence Lemma 17 implies

Lemma 18. *With* $C_1 = (3\pi)^{\frac{1}{2}}$ *we have*

$$\mathbb{P}\left(\sum_{i=1}^{\alpha(T)} |\mathscr{E}_i| < (1 - \varepsilon C_1) T\right) = \mathcal{O}((\log T)^{-\frac{3}{2}}) \quad (T \to \infty).$$

Lemma 18 via standard methods implies Theorem 3* with $C_1 = (3\pi)^{\frac{1}{2}}$.

5. Proof of Theorem 4

It is easy to see that (1.8) is a simple consequence of Theorem 3. Instead of proving (1.7), we present again the proof of the analogue statement for a Wiener process. In fact we prove our

Theorem 4*. *Let* $\{W(t), t \geqq 0\}$ *be a Wiener process and let* $\mathscr{T}_1(T) \geqq \mathscr{T}_2(T) \geqq \ldots$ *be the lengths of the longest, second longest excursions of* W *up to* T. *Then for any* $D > 0$ *there exists an* $\varepsilon = \varepsilon(D) > 0$ *such that*

$$\mathbb{P}\{\mathscr{T}_1(T) + \mathscr{T}_2(T) + \ldots + \mathscr{T}_{b(T)} \leqq T(1 - \varepsilon) \ i.o.\} = 1 \tag{5.1}$$

where $b(T) = [D \log\log T]$.

Introduce the following notations:

$$a_T = a(T) = \delta \frac{T}{\log\log T},$$
$$Y_0 = 0,$$
$$V_1 = \sup\{s\colon s < a_T,\ W(s) = 0\},$$
$$Y_1 = \inf\{s\colon s > a_T,\ W(s) = 0\},$$
$$\dots\dots\dots\dots\dots\dots\dots\dots$$
$$V_{i+1} = \sup\{s\colon s < Y_i + a_T,\ W(s) = 0\},$$
$$Y_{i+1} = \inf\{s\colon s > Y_i + a_T,\ W(s) = 0\},$$
$$\dots\dots\dots\dots\dots\dots\dots\dots$$
$$\Delta_i = Y_i - V_i, \qquad Z_i = Y_i - (Y_{i-1} + a_T),$$
$$U_i = (W(Y_{i-1} + a_T))^{-2} Z_i, \qquad N_i = a_T^{-\frac{1}{2}} W(Y_{i-1} + a_T),$$
$$\nu_T = \min\{i\colon Y_i \geqq T\},$$
$$R_i = V_i - Y_{i-1}.$$

The next lemma is an easy consequence of Lemma 16.

Lemma 19

$$\mathbb{P}\left(m^{-1} \sum_{i=1}^{m} U_i N_i^2 < \alpha\right) \geqq C e^{-m/\alpha} \tag{5.2}$$

for any $\alpha > 0$ *and* m *big enough.*

Lemma 20

$$\mathbb{P}(Z_1 + Z_2 + \dots + Z_{b(T)} + a(T) b(T) \leqq T) \geqq C (\log T)^{-1} \tag{5.3}$$

if $\delta = (D^2 + D)^{-1}$.

Proof

$$\begin{aligned}
&\mathbb{P}(Z_1 + Z_2 + \dots + Z_{b(T)} + a(T) b(T) \leqq T)\\
&= \mathbb{P}(Z_1 + Z_2 + \dots + Z_{b(T)} \leqq (1 - \delta D) T)\\
&= \mathbb{P}(b^{-1}(T) a^{-1}(T)(Z_1 + Z_2 + \dots + Z_{b(T)}) < (1 - \delta D) \delta^{-1} D^{-1})\\
&= \mathbb{P}\left(b^{-1}(T) \sum_{i=1}^{b(T)} U_i N_i^2 < D\right).
\end{aligned}$$

Hence we have (5.3) by (5.2).

Lemma 21. $\mathbb{P}(Z_1 + Z_2 + \dots + Z_{b(T)} + a(T) b(T) \leqq T$ *i.o.*$) = 1$ *equivalently*

$$\mathbb{P}\{\nu_T \geqq b_T \text{ i.o.}\} = 1 \quad \text{or} \quad \mathbb{P}\{Y_{b_T} \leqq T \text{ i.o.}\} = 1.$$

Proof. Let $T_k = k^k$ and let the events A_k, A_k^* be defined by

$$A_k = \{Z_1 + Z_2 + \ldots + Z_{b(T_k)} + a(T_k)\, b(T_k) \leqq T_k\},$$
$$A_k^* = \{Z_2 + \ldots + Z_{b(T_k)} + a(T_k)\, b(T_k) \leqq T_k\}.$$

By Lemma 20, we have

$$\sum_k \mathbb{P}(A_k) = \infty.$$

Furthermore, since for large k, $T_k < a(T_{k+1})$, the events A_k and A_l^* are independent for $k < l$. Hence

$$\begin{aligned}\mathbb{P}(A_k A_l) &\leqq \mathbb{P}(A_k A_l^*) = \mathbb{P}(A_k)\,\mathbb{P}(A_l^*)\\ &\leqq (1+\varepsilon)\,\mathbb{P}(A_k)\,\mathbb{P}(A_l),\end{aligned}$$

for any $\varepsilon > 0$ provided $k < l$ and k is large enough, where the last step follows from Lemma 16. One easily verifies that

$$\liminf_{n\to\infty} \frac{\sum_{l=1}^{n} \sum_{k=1}^{n} \mathbb{P}(A_k A_l)}{\left(\sum_{k=1}^{n} \mathbb{P}(A_k)\right)^2} \leqq 1$$

and by the already quoted Borel-Cantelli lemma (cf. Spitzer, 1964) we have

$$\mathbb{P}(A_k \text{ i.o.}) = 1$$

which proves Lemma 21.

By the above procedure we have chosen b_T excursions (V_i, Y_i) $(i = 1, 2, \ldots, b_T)$ which however are not necessarily the b_T largest ones. It is possible that some of them can be replaced by larger from the intervals (Y_i, V_{i+1}) $(i = 0, \ldots, b_T - 1)$. But it is readily seen that even the largest b_T excursions in $(0, Y_{b_T})$ can not cover more than $\sum_{i=1}^{b_T} (Z_i + \max(R_i, a_T - R_i))$. Hence the non-covered part of $(0, Y_{b_T})$ is at least $\sum_{i=1}^{b_T} \min(R_i, a_T - R_i)$. Since R_i/a_T $(i = 1, \ldots, b_T)$ are i.i.d. random variables having arc sine distribution and

$$E\left(\min\left(\frac{R_i}{a_T}, 1 - \frac{R_i}{a_T}\right)\right) = \int_0^{\frac{1}{2}} \frac{2}{\pi} \sqrt{\frac{v}{1-v}}\, dv = \frac{1}{2} - \frac{1}{\pi},$$

we have by the law of the large numbers

Lemma 22

$$\lim_{T\to\infty} (a_T b_T)^{-1} \sum_{i=1}^{b_T} \min(R_i, a_T - R_i) = \frac{1}{2} - \frac{1}{\pi} \qquad a.s.$$

It follows that for large enough T

$$\sum_{i=1}^{b_T} \min(R_i, a_T - R_i) \geqq \frac{1}{6} a_T b_T \geqq \frac{T}{6(D+1)} \qquad \text{a.s.},$$

which together with Lemma 21 proves Theorem 4.

6. A Consequence and some Problems

Introduce the following notations:

$$M_1(n)=\max_{0\leqq k\leqq n} S(k), \qquad M_2(n)=\max_{0\leqq k\leqq n} |S(k)|,$$

$$\begin{aligned}
&\alpha_0=\alpha_0(j)=0, \qquad \beta_0=\beta_0(j)=\max\{i\colon i\geqq 0, M_j(i)=0\},\\
&\alpha_1=\alpha_1(j)=\min\{i\colon i>\beta_0, M_j(i)=M_j(i+1)\},\\
&\beta_1=\beta_1(j)=\max\{i\colon M_j(i)=M_j(\alpha_1(j))\},\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&\alpha_k=\alpha_k(j)=\min\{i\colon i>\beta_{k-1}, M_j(i)=M_i(i+1)\},\\
&\beta_k=\beta_k(j)=\max\{i\colon M_j(i)=M_j(\alpha_k(j))\},\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&\bar{\mathscr{R}}(n)=\bar{\mathscr{R}}_j(n)=\max\{k\colon \alpha_k(j)\leqq n\},\\
&\bar{\mathscr{T}}(n)=\bar{\mathscr{T}}^{(j)}(n)=\max\{\beta_0-\alpha_0, \beta_1-\alpha_1, \ldots, \beta_{\bar{\mathscr{R}}(n)-1}-\alpha_{\bar{\mathscr{R}}(n)-1}, n-\alpha_{\bar{\mathscr{R}}(n)}\}\\
&\qquad\qquad (j=1,2).
\end{aligned}$$

Here $\bar{\mathscr{T}}^{(j)}(n)$ is the length of the longest flat interval of $M_j(i)$ $(0\leqq i\leqq n; j=1,2)$. A famous theorem of Lévy (see e.g. Knight (1981) p. 130 and Csáki and Révész (1983)) says that the limit behaviour of $M_1(n)$ is the same as that of $\bar{\mathscr{R}}(n)$. Applying this result and Theorems B and 1 one has

Consequence. *Let $f(x)$ be a non-decreasing function for which*

$$f(x)\nearrow\infty, \qquad \frac{x}{f(x)}\nearrow\infty, \qquad \text{as } x\to\infty.$$

Then

$$\mathbb{P}\left\{\bar{\mathscr{T}}^{(1)}(n)\geqq n\left(1-\frac{1}{f(n)}\right) \ i.o.\right\}=\begin{cases}1 & \text{if } I(f)=\infty,\\ 0 & \text{if } I(f)<\infty\end{cases}$$

and

$$\mathbb{P}\left\{\bar{\mathscr{T}}^{(1)}(n)\leqq\beta\,\frac{n}{f(n)} \ i.o.\right\}=\begin{cases}1 & \text{if } \mathscr{I}(f)=\infty,\\ 0 & \text{if } \mathscr{I}(f)<\infty,\end{cases}$$

where β is defined by (1.5) *and I resp. $\mathscr{I}$ are defined in Theorems A resp. 1.*

This Consequence gives a complete characterization of $\bar{\mathscr{T}}^{(1)}(n)$ and suggests our

Problem 2. Characterize the sequence $\bar{\mathscr{T}}^{(2)}(n)$.

Let $\{a_n\}$ be a non-decreasing sequence of positive integers and consider the process

$$m(n)=m(n,a_n)=\min_{0\leqq k\leqq n-a_n}(\mathscr{R}(k+a_n)-\mathscr{R}(k)).$$

Theorems B and 1 imply

$$\limsup m(n, a_n) = 0 \text{ a.s.} \quad \text{if } a_n \leqq \beta \frac{n}{f(n)} \quad \text{and} \quad \mathscr{I}(f) < \infty,$$

$$\liminf m(n, a_n) = 0 \text{ a.s.} \quad \text{if } a_n \leqq n\left(1 - \frac{1}{f(n)}\right) \quad \text{and} \quad I(f) = \infty.$$

Problem 3. Characterize those sequences $\{a_n\}$ for which

$$\limsup m(n, a_n) = K \quad \text{a.s.}$$

where K is a given positive integer.

Problem 4. For a given sequence $\{a_n\}$ find the normalizing factors $i(n) = i(n, a_n)$ $\left(a_n > n\left(1 - \frac{1}{f(n)}\right) \text{ whenever } I(f) = \infty\right)$ and $s(n) = s(n, a_n)$ $(a_n > \beta n/f(n)$ whenever $\mathscr{I}(f) < \infty)$ such that

$$\limsup \frac{m(n, a_n)}{s(n)} = 1 \quad \text{a.s.}$$

and

$$\liminf \frac{m(n, a_n)}{i(n)} = 1 \quad \text{a.s.}$$

Remarks. 1. The properties of

$$\max_{0 \leqq k \leqq n - a_n} (\mathscr{R}(k + a_n) - \mathscr{R}(k))$$

were studied by Csáki et al. (1983) and by Csáki and Földes (1984).

2. Our Theorems were formulated originally for random walks. In order to get a simpler proof we reformulated some of them for Wiener processes and noted that the reformulated versions imply the original ones by invariance principle. Here we wish to mention that Theorems 1 and 2 can be reformulated for Wiener process as well.

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