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Invariance Principles for Dependent Variables

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A weak invariance principle for dependent random variables is proved under conditions restricting the size of the conditional expectation of random variables with respect to their distant predecessors. The conditions are shown to be satisfied by martingale differences, sequences satisfying weaker versions of the φ-mixing and strong mixing condition, functions of mixing processes, and known conditions for convergence to normality of these are improved under a variety of moment restrictions. Stationarity is not required, although there are restrictions on the growth of the variance of partial sums.

1. Introduction

Billingsley (1968) proves various invariance principles (cf. Theorems 20.1, 20.2, and 21.1) for φ-mixing and functionals of φ-mixing sequences of random variables when the sequence is strictly stationary and second moments are assumed finite. Davydov (1968, 1970) has extended these theorems to stationary sequences for which higher moments than the second are assumed finite, and slower rates of mixing are required. He also proves analogues for strong mixing random variables. Invariance principles have also been proved by Brown (1971) for martingales, and Loynes (1969) for reverse martingales.

In Section 2 of this paper we give an invariance principle similar to the central limit theorem of Serfling (1968) under assumptions on the conditional expectations of variables with respect to the distant past. Tightness is proved under an "asymptotic martingale" type condition, and when conditional variances of the partial sums are asymptotically constant, the limit is shown to be Brownian motion. In Section 3, this invariance principle is used to show results for φ-mixing and strong mixing sequences of random variables. The stationarity assumption of Billingsley and Davydov is dropped, and the rates at which sequences need to be mixing weakened (for example Billingsley's condition $\sum_n \varphi_n^{\frac{1}{2}} < \infty$ may be replaced by $\varphi_n = 0[1/n(\log n)^{2+\varepsilon}]$, $\varepsilon > 0$ and a variety of moment conditions are considered. In Section 4, we treat non-stationary sequences which are functions either of φ-mixing or strong mixing variables in the sense of Billingsley's section 21. Again, rates at which mixing must occur are weakened over Billingsley and Davydov's results. In Section 5 we summarize a number of additional corollaries to our invariance principles, such as invariance principles for martingales, and random sum versions of all of the preceding theorems are given.

Proofs of all of the theorems of this paper, as well as two interesting lemmas, are given in Section 6.

Several authors have recently and independently obtained results of a similar nature to those of this paper, but generally under somewhat stronger dependence restrictions. Scott [16], and Heyde [9] prove central limit theorems, invariance principles, and iterated logarithm results for stationary sequences under similar but stronger conditions, and W. Phillips and Stout have imbedded an analogue of our "mixingales" in Brownian motion under suitable restrictions on the mixing coefficients. In [12] the author proves a strong law of large numbers, also under "mixingale" conditions.

Let $\{X_i; i=1,2,3,\ldots\}$ be a sequence of square integrable random variables on the probability triple $(\Omega, \mathfrak{F}, P)$ and put $S_n=\sum_{i=1}^n X_i$ for all n. We will denote the various types of convergence; almost sure, in probability, in L_p, and weak by $\to_{\text{a.s.}}, \to_p, \to_{L_p}$, and $\to_w$ respectively. We denote $E^{1/p}|U|^p$ by $\|U\|_p$ and $\operatorname{ess\,sup}|U|$ by $\|U\|_\infty$. We also assume unless otherwise indicated, that

(1.1) $EX_i=0$ for all i, and $E\frac{S_n^2}{n}\to\sigma^2$, some positive contant. Consider the space $D=D[0,\infty)$, the set of all functions on the interval $[0,\infty)$ which have left hand limits and are continuous from the right at every point. We endow this space with Stone's (1963) extension of the Skorohod J_1 topology. (Since we are dealing with $_{\text{a.s.}}$ continuous limits, we could use instead the topology of uniform convergence on compacta for Sections 1–4.) In order to demonstrate weak convergence of random elements of D with Stone's topology to an a.s. continuous process, it is sufficient to demonstrate weak convergence in the J_1 (or uniform) topology of the elements restricted to each compact interval of the form $[0, K]$.

Let $\mathfrak{B}$ be the Borel sigma algebra in D and define a random function by:

(1.2) $$W_n(t)=\frac{S_{[nt]}}{n^{\frac{1}{2}}\sigma}$$

where $[x]$ is the "greatest integer contained in x". This is a measurable map from $(\Omega,\mathfrak{F})$ into $(D,\mathfrak{B})$, and we will demonstrate weak convergence of W_n to standard Brownian motion process on D.

(1.3) *Definition.* A sequence W_n of random elements of a metric space is said to be Renyi-mixing (R-mixing) with limiting process W if

(1.4) $P\{W_n\in\cdot\,|F\}$ converge weakly to the measure $P\{W\in\cdot\}$ for every $F\in\mathfrak{F}$ such that $P(F)>0$.

R-mixing is a useful concept when passing from non-random to random invariance principles (cf. Billingsley, T 17.2) and it follows from Theorem (4.5) of this book that we need only verify (1.4) in the definition for all F of positive measure in some algebra of sets generating $\sigma(W_1, W_2, \ldots)$.

2. The Invariance Principle

The following concept, as will be seen later, generalizes under moment restrictions the notions of martingale differences, φ-mixing and strong mixing sequences, and various functions of mixing processes.

Let $\{\mathfrak{F}^i\colon \infty<i<\infty\}$ be a nondecreasing sequence of sigma-algebras, $\mathfrak{F}^{-\infty}$ the largest sigma algebra contained in all $\mathfrak{F}^i$ and $\mathfrak{F}^\infty$ the smallest sigma algebra which contains all $\mathfrak{F}^i$. For brevity, we introduce the notation $E_k U = E(U \mid \mathfrak{F}^k)$.

(2.1) *Definition.* The sequence $\{(X_n, \mathfrak{F}^n)\}$ will be called a mixingale if there exists a positive sequence $\psi_k \to 0$ as $k \to \infty$ such that for all $i \geqq 1$, $k \geqq 0$,

(2.2) $$\|E_{i-k} X_i\|_2 \leqq \psi_k$$

and

(2.3) $$\|X_i - E_{i+k} X_i\|_2 \leqq \psi_{k+1}.$$

For fixed i, Lemma 1, p. 184 of Billingsley implies that the left hand side of (2.3) is non-increasing in k, and by the conditional Jensen's inequality, this holds as well for the left hand side of (2.2). Thus we may (and do) assume that the ψ_k are non-increasing. Observe further that these conditions imply $E_\infty X_i = X_i$ and $E_{-\infty} X_i = 0$ a.s. for all i. Moreover, if each X_i is $\mathfrak{F}^i$ measurable as will frequently be the case, condition (2.3) will hold trivially. Finally, it is immaterial whether the sequences are infinite ($n = 1, 2, \ldots$) or doubly infinite ($n = \ldots -1, 0, 1, 2, \ldots$) for in the former case, we can define $\mathfrak{F}^k$ to be the trivial σ-field for $k \leqq 0$.

(2.4) *Definition.* We will call the sequence $\{\psi_k\}$ of size $-p$ if there exists a positive sequence $\{L(k)\}$ such that

(a) $\sum_n \frac{1}{n L(n)} < \infty$,

(b) $L_n - L_{n-1} = 0\left(\frac{L(n)}{n}\right)$,

(c) L_n is eventually non-decreasing and

(d) $\psi_n = o\left[\frac{1}{n^{\frac{1}{2}} L(n)}\right]^{2p}$.

Remark. Observe that condition (b) will follow for any sequence L_n such that $L_n - L_{n-1}$ is regularly varying with exponent -1 (c.f. Feller (1971), p. 280). For example any sequence which is $0[n^{\frac{1}{2}} \log n (\log\log n)^{1+\delta}]^{-p}$ with $\delta > 0$ is of size $-p/2$. Also summability conditions such as $\sum_{n=1}^\infty \varphi_n^\theta < \infty$ imply, for monotone sequences φ_n, that $\varphi_n = o\left(\frac{1}{n^{1/\theta}}\right)$ and hence that φ_n is of size $-q$ for any $q < \frac{1}{\theta}$.[1]

(2.5) **Theorem.** *Let $\{(X_i, \mathfrak{F}^i)\}$ be a mixingale satisfying 1.1 with ψ_k of size $-\frac{1}{2}$. If $\{X_i^2; i = 1, 2, \ldots\}$ is uniformly integrable, then $\{W_n\}$ is tight in D and any limit process is a.s. continuous.*

We now state our main invariance principle:

(2.6) **Theorem.** *Suppose, in addition to the conditions of* (2.5),

$$E\left\{\frac{(S_{k+n} - S_k)^2}{n} \middle| \mathfrak{F}^{k-m}\right\} \to \sigma^2 \quad \text{in } L_1(\Omega)$$

norm as $\min(m, k, n) \to \infty$.

Then W_n is R-mixing with limit W, a standard Brownian motion process on D.

[1] Knopp (1946), p. 124: I am indebted to P. Billingsley for both the information and reference.

3. Mixing

We now apply the concept of mixingale to prove invariance principles under strong and φ-mixing conditions. Define two measures of dependence between sigma algebras $\mathfrak{F}$ and $\mathfrak{A}$ by

$$\varphi(\mathfrak{F}, \mathfrak{A}) = \sup_{\{F \in \mathfrak{F},\, G \in \mathfrak{A},\, P(F) > 0\}} |P(G|F) - P(G)|$$

and

$$\alpha(\mathfrak{F}, \mathfrak{A}) = \sup_{F \in \mathfrak{F},\, G \in \mathfrak{A}} |P(FG) - P(F)\,P(G)|.$$

In this section we consider a doubly infinite sequence of random variables $\{X_i;\ -\infty < i < \infty\}$ defined on $(\Omega, \mathfrak{F}, P)$ and put $\mathfrak{F}_n^m = \sigma(X_i;\ n \leqq i \leqq m), \mathfrak{R}_n^m = \sigma(S_m - S_{n-1})$ for all $n \leqq m$.

(A one-sided sequence may be handled within this framework by defining $X_j = 0$ for all negative j.) For each $m \geqq 0$, define

$$\hat{\varphi}_m = \sup_n \varphi(\mathfrak{F}_{-\infty}^n, \mathfrak{F}_{n+m}^\infty), \tag{3.1}$$

$$\hat{\alpha}_m = \sup_n \alpha(\mathfrak{F}_{-\infty}^n, \mathfrak{F}_{n+m}^\infty), \tag{3.2}$$

$$\varphi_m = \sup_n \sup_{j \geqq n+m} \varphi(\mathfrak{F}_{-\infty}^n, \mathfrak{R}_{n+m}^j) \tag{3.3}$$

and

$$\alpha_m = \sup_n \sup_{j \geqq n+m} \alpha(\mathfrak{F}_{-\infty}^n, \mathfrak{R}_{n+m}^j). \tag{3.4}$$

Observe that the variables are φ-mixing if $\hat{\varphi}_m \to 0$, strong mixing if $\hat{\alpha}_m \to 0$ as $m \to \infty$. It is clear in general that $\hat{\alpha}_m \leqq \hat{\varphi}_m$, and $\alpha_m \leqq \varphi_m \leqq \hat{\varphi}_m$, and so in this section we will treat only the two weaker conditions $\varphi_m \to 0$ and $\alpha_m \to 0$. The following lemma relates the concept of mixing to that of a mixingale. (3.6) is due to Serfling (1968).

(3.5) **Lemma.** *Suppose X is a random variable measurable with respect to $\mathfrak{A}$, and $1 \leqq p \leqq r \leqq \infty$. Then*

$$\|E(X|\mathfrak{F}) - EX\|_p \leqq 2\{\varphi(\mathfrak{F}, \mathfrak{A})\}^{1-1/r} \|X\|_r \tag{3.6}$$

and

$$\|E(X|\mathfrak{F}) - EX\|_p \leqq 2(2^{1/p} + 1)\{\alpha(\mathfrak{F}, \mathfrak{A})\}^{1/p - 1/r} \|X\|_r. \tag{3.7}$$

(3.8) **Theorem.** *Let $\{X_i\}$ be a sequence satisfying* (1.1) *and*

(a) *$\{X_i^2\}$ is uniformly integrable,*

(b) *$\sup_i \|X_i\|_\beta < \infty$ for some $2 \leqq \beta \leqq \infty$,*

(c) *$\dfrac{E(S_{k+n} - S_k)^2}{n} \to \sigma^2 > 0$ as $\min(k, n) \to \infty$ and*

(d) *$\{\varphi_n\}$ is of size $\dfrac{-\beta}{2\beta - 2}$ or*

(d′) *$\beta > 2$ and $\{\alpha_n\}$ is of size $\dfrac{-\beta}{\beta - 2}$.*

Then W_n is R-mixing with limit a standard Brownian motion process on D.

Remarks. Observe that this theorem with (*d*) and $\beta=2$ improves on the condition $\sum_{n=1}^{\infty} \varphi_n^{\frac{1}{2}} < \infty$ of Billingsley's theorem 20.1 and drops the assumption of stationarity at the expense only of the additional requirement (c). The theorem also improves on the stationarity assumption and the condition $\sum_n \alpha_n^{(\beta-2)/2\beta} < \infty$ of Davydov's (1968) Theorem 4.2.

For uniformly bounded stationary random variables, $d(d')$. Becomes "$\varphi_n(\alpha_n)$ is of size $-\frac{1}{2}(-1)$" which improves on Davydov's condition

$$\sum_{n=1}^{\infty} \varphi_n^{\frac{1}{2}} < \infty \left(\sum_{n=1}^{\infty} \alpha_n^{\frac{1}{2}} < \infty\right).$$

When the sequence is weakly stationary, the method of Billingsley's Lemma 3 (p. 172) shows that under the stronger conditions

$$\sum_{n=1}^{\infty} \varphi_n^{1-1/\beta} < \infty \left(\text{or} \sum_{n=1}^{\infty} \alpha_n^{1-2/\beta} < \infty\right),$$

the sequence $E\frac{S_n^2}{n}$ converges to some $\sigma^2 \geqq 0$. This combined with (3.8) leads to the following corollary.

(3.9) **Corollary.** *Let $\{X_i\}$ be a stationary sequence centred at expectations satisfying the conditions:*

(a) $\|X_1\|_\beta < \infty$ *for some* $2 \leqq \beta < \infty$ *and*

(b) $\sum_{n=1}^{\infty} \varphi_n^{1-1/\beta} < \infty$ *or*

(b′) $\beta > 2$ *and* $\sum_{n=1}^{\infty} \alpha_n^{1-2/\beta} < \infty$.

Then if $\sigma^2 > 0$, W_n is R-mixing with limit a standard Brownian motion process on D.

In the case $\sigma^2 = 0$, $\frac{S_{[nt]}}{n^{\frac{1}{2}}}$ converges in probability (hence weakly) to the zero function.

4. Functions of Mixing Processes

This section provides some improvement over the results of Section 21 of Billingsley and of Davydov (1970) for functions of strong and φ-mixing sequences. The stationarity assumptions have been weakened and the conditions on the mixing coefficients relaxed.

We will suppose $\{\xi_i;\ -\infty < i < \infty\}$ is a process satisfying the φ or strong mixing conditions, $\mathfrak{F}_m^n = \sigma(\xi_m, \xi_{m+1}, \dots, \xi_n)$, and $X_i = X_i(\xi_i;\ -\infty < i < \infty)$ is a function of these variables satisfying;

(4.1) $\|E(X_i \mid \mathfrak{F}_{i-m}^{i+m}) - X_i\|_2 \leqq \gamma_m$ for all i, m, where $\{\gamma_m\}$ is a sequence of size $-\frac{1}{2}$.

All other notation conforms to that of previous sections

(4.2) **Theorem.** *Let* $\{X_i\}$ *be centred at expectations, satisfying* (4.1) *and:*

(a) $\{X_i^2\}$ *is uniformly integrable,*

(b) $\sup_i \|X_i\|_{\beta<\infty}$ *for some* $2 \leqq \beta \leqq \infty$,

(c) $E\frac{(S_{k+n}-S_k)^2}{n} \to \sigma^2 > 0$ *as* $\min(k, n) \to \infty$, *and*

(d) $\{\xi_i\}$ *is* φ*-mixing with* $\{\hat{\varphi}_n\}$ *of size* $\frac{-\beta}{2\beta-2}$ *or*

(d′) $\beta > 2$ *and* $\{\xi_i\}$ *is strong mixing with* $\hat{\alpha}_n$ *of size* $\frac{-\beta}{\beta-2}$.

Then W_n *is R-mixing with limit a standard Brownian motion on D.*

5. Corollaries

The simplest application of Theorem (2.5) is to sequences of square integrable martingale differences (square integrable sequences for which each X_i is $\mathfrak{F}^i$-measurable, and $E_{i-1}X_i = 0$ a.s.). In this case Theorem 2.5 becomes:

(5.1) **Theorem.** *Let* X_i *be a sequence of martingale differences such that:*

(a) *The set* $\{X_1^2;\ -\infty < i < \infty\}$ *is uniformly integrable, and*

(b) $\frac{1}{n}\sum_{n=1}^{n} E_{k-m} X_{k+i}^2 \to \sigma^2 > 0$ *in* $L_1(\Omega)$ *norm as* $\min(m, k, n) \to \infty$.

Then W_n *is R-mixing with limit a standard Brownian motion process.*

It is obvious that (b) can be replaced by either of the following conditions;

(b′) $E_k X_{k+n}^2 \to \sigma^2$ in $L_1(\Omega)$ norm as $\min(k, n) \to \infty$,

(b″) $\frac{1}{n}\sum_{i=1}^{n} X_{k+i}^2 \to \sigma^2$ in $L_1(\Omega)$ norm as $\min(k, n) \to \infty$.

It is also clear in view of Lemma 2.15 of [10] that (b″) may be replaced by the two conditions:

(b‴) (i) There exists a function $r(n) \to 1$ as $n \to \infty$ such that $EX_k^2 X_{k+n}^2 \leqq r(n) EX_k^2 EX_{k+n}^2$ for all k, and

(ii) $E\frac{(S_{k+n}-S_k)^2}{n} \to \sigma^2$ as $\min(n, k) \to \infty$.

If we are interested only in the central limit theorem, we may apply Lemma 3.1 of Dvoretsky (1972) to weaken the assumptions of Theorem (2.5). This lemma allows us to look only at sequences of partial sums which are Markovian, in which case we may replace $\mathfrak{F}^m$ both in the definition of mixingale and in the condition of (2.6) by $\mathfrak{R}_1^m$ for each m, and conclude that the limiting distribution of $S_n/n^{\frac{1}{2}}\sigma$ is standard normal. This remark is valid also for the theorems of Section 3, allowing replacement of the mixing coefficients in (3.8) with:

$$\bar{\varphi}_m = \sup_n \sup_{j \geqq n+m} \varphi(\mathfrak{R}_1^n, \mathfrak{R}_{n+m}^j)$$

and

$$\bar{\alpha}_m = \sup_n \sup_{j \geqq n+m} \alpha(\mathfrak{N}_1^n, \mathfrak{N}_{n+m}^j),$$

and retain the conclusion of limiting normality for $S_n/n^{\frac{1}{2}}$.

The many results corollary to an invariance principle such as the arc-sin law, and the distribution of various functionals need no further exposure here, due to their excellent treatment in Billingsley (1968). We will state a random sum invariance principle which follows from any invariance principle in which we have R-mixing, and refer to Billingsley, Theorem (17.2) for the proof.

(5.4) **Theorem.** *If $\{X_i\}$ satisfies the conditions of any one of the preceding invariance principles, and if v_n is a sequence of random variables such that for some sequence of positive numbers $a_n \uparrow \infty$, we have $\frac{v_n}{a_n} \xrightarrow{P} v$, some strictly positive random variable, then the process $\frac{S_{[v_n t]}}{v_n^{\frac{1}{2}}\sigma}$ converges weakly to the Brownian motion process on D.*

It should be observed that all of the invariance principles of this paper remain in force when P is replaced by any probability measure absolutely continuous with respect to P. This is true of any R-mixing limit theorem (for example see Billingsley, Theorem 20.2).

6. Proofs

(6.1) **Lemma.** *Let x_i be complex numbers, a_i non-negative real numbers, and $\gamma > 1$. Then:*

$$\left|\sum_{i=-\infty}^{\infty} x_i\right|^\gamma \leqq \left(\sum_{i=-\infty}^{\infty} a_i\right)^{\gamma-1} \sum_{i=-\infty}^{\infty} \frac{|x_i|^\gamma}{a_i^{\gamma-1}}.{}^2$$

Proof. Assume without loss of generality that $x_i = 0$ whenever $a_i = 0$, and $K = (\sum_{i=-\infty}^{\infty} a_i) < \infty$. Then by Jensen's inequality,

$$\left|\sum_i x_i\right|^\gamma = K^\gamma \left|\sum_i \frac{x_i}{a_i} \frac{a_i}{K}\right|^\gamma \leqq K^{\gamma-1} \sum_i \frac{|x_i|^\gamma}{a_i^{\gamma-1}}$$

where all summations are over $\{i;\, a_i \neq 0\}$.

(6.2) **Lemma.** *Let $\{(X_i, \mathfrak{F}^i)\}$ be a mixingale, and put $Y_{j,k} = \sum_{i=1}^{j} (E_{i+k} X_i - E_{i+k-1} X_i)$. Then for any $\gamma > 1$, and non-negative sequence $\{a_i\}$, we have:*

$$E\{\max_{j \leqq n} |S_j|^\gamma\} \leqq \left\{\frac{\gamma}{\gamma-1}\right\}^\gamma \left(\sum_{i=-\infty}^{\infty} a_i\right)^{\gamma-1} \sum_{k=-\infty}^{\infty} a_k^{1-\gamma} E|Y_{n,k}|^\gamma.$$

Proof. We show first that

$$X_i = \sum_{k=-\infty}^{\infty} (E_{i+k} X_i - E_{i+k-1} X_i) \quad \text{a.s.}$$

Now

$$\sum_{k=-m}^{n} (E_{i+k} X_i - E_{i+k-1} X_i) = E_{i+n} X_i - E_{i-m-1} X_i.$$

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[2] Henceforth, 0/0 will be taken to be 0 in formulae.

The first term on the right hand side forms, for fixed i, a martingale sequence, and hence, by Doob (1953), Theorem 4.3 (p. 331) converges as $n\to\infty$ to $E(X_i|\mathfrak{F}^\infty)=X_i$ by the remarks following (2.3). Similarly, the second term forms a backwards martingale sequence, and converges to 0 almost surely by the same authority.

Therefore

$$S_j=\sum_{k=-\infty}^{\infty} Y_{j,k} \quad \text{a.s.}$$

Put $K=\sum_{i=-\infty}^{\infty} a_i$ and observe that by Lemma (6.1)

$$\max_{j\leqq n}|S_j|^\gamma \leqq K^{\gamma-1}\sum_{k=-\infty}^{\infty} a_k^{1-\gamma}\max_{j\leqq n}|Y_{j,k}|^\gamma \quad \text{a.s.}$$

If we now note that, for each k, $\{(Y_{j,k},\mathfrak{F}^{j+k});\ j=1,2,\ldots,n\}$ is a martingale, we may take expectations on both sides and apply Doob's inequality[3] to get the result. Q.E.D.

(6.3) **Lemma.** *Let $(X_i,\mathfrak{F}^i)$ be a mixingale of size $-\frac{1}{2}$ with $\psi_k\leqq\dfrac{b}{k^{\frac{1}{2}}L(k)}$ for all $k>m$ where L is a function satisfying* 2.4 a)–c). *Then there are finite constants K_1 and K_2 independent of m, b, ψ_0, and n such that for all $n\geqq 1$, $m\geqq 1$,*

$$E\{\max_{j\leqq n} S_j^2\}\leqq nK_1\left\{\psi_0^2 m L(m)+K_2 b^2\sum_{k=m+1}^{\infty}\frac{1}{kL(k)}\right\}$$

Proof. Put $Z_{i,k}=X_i-E_{i+k}X_i$. Then by (6.2)

$$\begin{aligned}E\{\max_{j\leqq n} S_j^2\}&\leqq 4\left(\sum_{i=-\infty}^{\infty}a_i\right)\sum_{k=-\infty}^{\infty}\frac{1}{a_k}\left\{\sum_{i=1}^{n}EE_{i+k}^2X_i-EE_{i+k-1}^2X_i\right\}\\&\leqq 4\left(\sum_{i=-\infty}^{\infty}a_i\right)\sum_{i=1}^{n}\left\{\frac{EZ_{i,0}^2}{a_1}+\frac{EE_i^2X_i}{a_0}+\sum_{k=1}^{\infty}EZ_{i,k}^2(a_{k+1}^{-1}-a_k^{-1})\right.\\&\qquad\left.+\sum_{k=1}^{\infty}EE_{i-k}^2X_i(a_k^{-1}-a_{k-1}^{-1})\right\}.\end{aligned}$$

$$(6.4)\qquad \leqq 4n\left(\sum_{i=-\infty}^{\infty}a_i\right)\left\{\frac{\psi_0^2+\psi_1^2}{a_0}+2\sum_{k=1}^{\infty}\psi_k^2(a_k^{-1}-a_{k-1}^{-1})\right\}.$$

Now let

$$a_i=\begin{cases}\dfrac{1}{mL(m)} & \text{for } 0\leqq|i|\leqq m,\\[2ex] \min\limits_{m\leqq j\leqq|i|}\dfrac{1}{jL(j)} & \text{for } |i|>m\end{cases}$$

Let $K_3, K_4, \ldots$ represent generic constants independent of m, b, ψ_0 and n. Observe that

$$\sum_{i=-\infty}^{\infty}a_i\leqq\frac{2m+1}{mL(m)}+2\sum_{j=m+1}^{\infty}\frac{1}{jL(j)}\leqq K_3.$$

[3] Doob (1953), p. 317.

Since $\dfrac{1}{kL(k)}$ is eventually monotone, $a_k = \dfrac{1}{kL(k)}$ for $k > K_5$ so

$$a_k^{-1} - a_{k-1}^{-1} \leqq K_6 \{kL(k) - (k-1)L(k-1)\}$$
$$\leqq K_7 L(k) \quad \text{for } k > K_5 \text{ by 2.4 (b).}$$

Therefore,

$$\sum_{k=m+1}^{\infty} \psi_k^2 (a_k^{-1} - a_{k-1}^{-1}) \leqq K_8 b^2 \sum_{k=m+1}^{\infty} \frac{1}{kL(k)}$$

and substituting for this and the values of a_i in (6.4) give the result.

(6.5) **Lemma.** *Let $(X_i, \mathfrak{F}^i)$ be a mixingale with ψ_i of size $-\frac{1}{2}$ such that $\{X_i^2; i=1,2,\ldots\}$ is uniformly integrable.*

Then the set

$$\left\{ \max_{j \leqq n} \frac{(S_{j+k} - S_k)^2}{n};\ k=1,2,\ldots,n=1,2,\ldots \right\}$$

is uniformly integrable.

Proof. For positive c and m to be determined later, put

$$X_i^c = X_i I[|X_i| \leqq c], \qquad Y_i = E_{i+m} X_i^c - E_{i-m} X_i^c,$$
$$U_i = X_i - E_{i+m} X_i + E_{i-m} X_i, \qquad Z_i = E_{i+m}(X_i - X_i^c) - E_{i-m}(X_i - X_i^c)$$

and note that $X_i = Y_i + Z_i + U_i$. We will approach the proof in a similar way to that of Theorem 23.1 of Billingsley. Let us use the notation $\in_y U = \int_{[U>y]} U\,dP$,

$$\bar{Y}_j = \sum_{i=1}^{j} Y_i, \quad \bar{Z}_j = \sum_{i=1}^{j} Z_i, \quad \text{and} \quad \bar{U}_j = \sum_{i=1}^{j} U_i.$$

Then $S_j^2 \leqq 3(\bar{U}_j^2 + \bar{Y}_j^2 + \bar{Z}_j^2)$ and hence

$$\in_y \left(\max_{j \leqq n} \frac{S_j^2}{n} \right) \leqq 9(\mathrm{I} + \mathrm{II} + \mathrm{III})$$

where

$$\mathrm{I} = \in_{y/3} \left(\max_{j \leqq n} \frac{\bar{Y}_j^2}{n} \right), \qquad \mathrm{II} = E\left(\max_{j \leqq n} \frac{\bar{Z}_j^2}{n} \right)$$

and

$$\mathrm{III} = E\left(\max_{j \leqq n} \frac{\bar{U}_j^2}{n} \right).$$

Note that $E(U_i - E_{i+k} U_i)^2$ is less than ψ_m^2 for $k \leqq m$, and ψ_k^2 for $k > m$, and similarly, $EE_{i-k}^2 U_i$ is less than ψ_m^2 for $k \leqq m$, and ψ_k^2 for $k > m$. Therefore, by (6.3),

$$\mathrm{III} \leqq K_1 \left\{ \psi_m^2 m L(m) + K_2 \sum_{k=m+1}^{\infty} \frac{1}{kL(k)} \right\}$$

and for arbitrary $\varepsilon > 0$, we may choose and fix m sufficiently large that $\mathrm{III} \leqq \varepsilon/27$.

Similarly, each of the terms $E(Z_i - E_{i+k} Z_i)^2$ and $EE_{i-k}^2 Z_i$ are less than

$$EZ_i^2 \leqq EE_{i+m}^2 (X_i - X_i^c)$$
$$\leqq \max_i \in_c X_i^2 = g(c),$$

say, for $k \leqq m$, and for $k > m$, each of these terms is equal to 0. Therefore, putting $a_i = 1$ for $|i| \leqq m+1, 0$ otherwise, in (6.4);

$\mathrm{II} \leqq 4(2m+3)\,2g(c)$ and for our now fixed value of m, we may choose and fix c such that this is less than $\varepsilon/27$.

Finally, with these fixed values of m and c, we apply Lemma (6.2) to the sequence $\{Y_i\}$ with $\gamma = 4$, and

$$a_i = \begin{cases} 1 & \text{for } |i| \leqq m \\ 0 & \text{for } |i| > m \end{cases},$$

obtaining

$$E\big(\max_{j \leqq n} \bar{Y}_j^4\big) \leqq (\tfrac{4}{3})^4 (2m+1)^3 \sum_{|k| \leqq m} E|Y_{n,k}|^4$$

where

$$Y_{n,k} = \sum_{i=1}^{n} (E_{i+k} Y_i - E_{i+k-1} Y_i),$$

where each term in this summand is bounded absolutely and with probability 1 by $4c$. It therefore follows from inequality 23.7 of Billingsley that $E|Y_{n,k}|^4 \leqq 6n^2(4c)^4$. Substituting this above gives

$$E\left(\max_{j \leqq n} \frac{\bar{Y}_j^4}{n^2}\right) \leqq \left(\frac{2^{17}}{3^3}\right)(2m+1)^4 c^4 = K_3,$$

say.

Therefore,

$$\epsilon_{y/3}\left(\max_{j \leqq n} \frac{\bar{Y}_j^2}{n}\right) \leqq \frac{3K_3}{y},$$

which, for our fixed values of m and c, may be made less than $\varepsilon/27$ by choosing y sufficiently large. Then, for this value of y,

$$\epsilon_y\left(\max_{j \leqq n} \frac{S_j^2}{n}\right) \leqq \varepsilon.$$

Clearly y was chosen idependently of our location in the sequence or the value of n, so

$$\left\{\frac{(S_{k+n} - S_k)^2}{n},\ n = 1, 2, \ldots\ k = 1, 2, \ldots\right\}$$

is U.I.

Proof of Theorem (2.5). Clearly we may restrict ourselves to functions on the closed unit interval; functions on the interval $[0, K]$ for arbitrary finite K are treated in the same way. Put $S_{k,n} = S_{k+n} - S_k$. For tightness we use (14.9) and Theorem 8.4 of Billingsley by which it is sufficient to show;

$$\limsup_{\lambda \to \infty} \lambda^2 P\{\max_{j \leqq n} |S_{k,j}| > \lambda n^{\frac{1}{2}}\} = 0$$

uniformly in (n, k). Clearly this follows from the uniform integrability of the set

$$\left\{\max_{j \leqq n} \frac{S_{k,j}^2}{n};\ k = 1, 2, \ldots;\ n = 1, 2, \ldots\right\}$$

which is shown in Lemma (6.5). Theorem 15.5 of Billingsley also shows that thightness in the modulus $w_{x'}$ also implies that any weak limit process of W_n must be a.s. concentrated on the continuous functions.

Proof of Theorem (2.6). Since $\bigcup_{n=1}^{\infty} \mathfrak{F}^n$ is an algebra generating $\sigma(W_1, W_2, \ldots)$ we need only verify for arbitrary m and $F \in \mathfrak{F}^m$ with $P(F) > 0$ that (1.4) holds where W is standard Brownian motion. Tightness of the measures $P(W_n \in \cdot | F)$ and continuity of their limits follows directly from Theorem (2.5). We will verify the conditions 1° a, 2°, and 3° a of Billingsley's theorem 19.4 with $\rho(t) = 0$ and $\sigma^2(t) = t$, and with $P(\cdot)$ replaced by $\hat{P}(\cdot) = P(\cdot | F)$, $E(\cdot)$ by $\hat{E}(\cdot) = E(\cdot | F)$.

Observe first that (6.5) implies

(6.6) the set $\left\{\frac{(W_n(t+h) - W_n(t))^2}{h};\ 0 \leqq t < t+h < \infty,\ n = 1, 2, \ldots, h \geqq \frac{1}{n}\right\}$ is uniformly integrable.

Consequently, both 2°, viz.: $\sup\limits_{t \leqq K} \limsup\limits_{n \to \infty} E\left(W_n^2(t) | F\right) < \infty$ for all $K < \infty$, and 3° a follow. We can easily show that 1° a can be replaced in our case by the condition;

(6.7) For arbitrary k, real $u_1, u_2, \ldots, u_{k-1}$, and $0 \leqq t_1 < \cdots < t_k < t_{k+1}$,

$$\text{(a)}\quad \hat{E}\left[\exp\left\{\sum_{j=1}^{k-1} i u_j W_n(t_j)\right\}\{W_n(t_{k+1}) - W_n(t_k)\}\right] \to 0 \quad \text{as } n \to \infty,$$

and

$$\text{(b)}\quad \hat{E}\left[\exp\left\{\sum_{j=1}^{k-1} i u_j W_n(t_j)\right\}\{(W_n(t_{k+1}) - W_n(t_k))^2 - (t_{k+1} - t_k)\}\right]$$

converge to 0 as $n \to \infty$.

These conditions differ from Billingsley's 1° a only in that we replace $t_{k-1} \leqq t_k$ by $t_{k-1} < t_k$.

Define

$$U_n = E_{[nt_{k-1}]}\left(\sum_{j=1}^{k-1} u_j W_n(t_j)\right)$$

and $V_n = W_n(t_{k+1}) - W_n(t_k)$. Then

$$\left\| U_n - \sum_{j=1}^{k-1} u_j W_n(t_j) \right\|_2 \leqq \frac{\left(\max\limits_{j \leqq k-1} |u_j|\right)}{n^{\frac{1}{2}}\sigma} \sum_{j=1}^{k-1} \sum_{i=1}^{[nt_j]} \|E_{[nt_{k-1}]} X_i - X_i\|_2$$

$$= 0\left(n^{-\frac{1}{2}} \sum_{i=1}^{[nt_{k-1}]} \psi_i\right)$$

and thus converges to 0 by Kronecker's lemma and (2.4). This and the uniform integrability of V_n imply

$$\text{(6.8)}\qquad \hat{E}\left|\left\{\exp \sum_{j=1}^{k-1} i u_j W_n(t_j) - \exp i U_n\right\} V_n\right| \to 0.$$

But for sufficiently large n, $F \subset \mathfrak{F}^{[nt_k-1]}$ and

$$
\begin{aligned}
|\hat{E}(\exp iU_n)V_n| &= \frac{1}{P(F)} \left| \int_F (\exp iU_n) E_{[nt_k-1]} V_n \, dP \right| \\
&\leqq \frac{1}{P(F)} \|E_{[nt_k-1]} V_n\|_2 \\
&= 0\left(\frac{1}{n^{\frac{1}{2}}} \sum_{i=1}^{[nt_{k+1}]} \psi_i\right)
\end{aligned}
$$

and this converges to 0 by (2.4). This, with (6.8), verifies (6.7. a).

For (6.7. b), observe that the uniform integrability of V_n^2 implies

$$
\hat{E}\left|\left\{\exp \sum_{j=1}^{k-1} iu_j W_n(t_j) - \exp iU_n\right\}\{V_n^2 - (t_{k+1} - t_k)\}\right| \to 0. \tag{6.9}
$$

But

$$
|\hat{E}(\exp iU_n)\{V_n^2 - (t_{k+1} - t_k)\}| \leqq \frac{1}{P(F)} \|E_{[nt_k-1]} V_n^2 - (t_{k+1} - t_k)\|_1 \to 0 \quad \text{by (2.5).}
$$

This with (6.9) verifies (6.7. b)..

Proof of (3.5). (3.6) is proved in Serfling (1968) for $r < \infty$. For the case $p, r = \infty$, use the continuity of the L_p norms at $p = \infty$.

For (3.7), put $\alpha = \alpha(\mathfrak{F}, \mathfrak{A})$, $c = \alpha^{-1/r} \|X\|_r$, and $X_1 = X\, I(|X| \leqq c)$, where $I(A)$ is the indicator function of the set A, and $X_2 = X - X_1$. Here we have neglected the trivial independent case and assumed $\alpha > 0$. Then

$$
\begin{aligned}
\|E(X \mid \mathfrak{F}) - EX\|_p &\leqq \|E(X_1 \mid \mathfrak{F}) - EX_1\|_p + \|E(X_2 \mid \mathfrak{F}) - EX_2\|_p \\
&\leqq (2c)^{(p-1)/p} E^{1/p} |E(X_1 \mid \mathfrak{F}) - EX_1| + 2\|X_2\|_p \\
&\leqq (2c)^{(p-1)/p} (4\alpha c)^{1/p} + 2 \frac{\|X_2\|_r^{r/p}}{c^{(r-p)/p}},
\end{aligned}
$$

where the first term in the last step follows from Lemma 5.2 of Dvoretsky (1972), and the second from the standard inequality

$$
E|X|^p I(|X| > c) \leqq \frac{1}{c^{r-p}} E|X|^r I(|X| > c).
$$

Substituting for c and using the fact that $\|X_2\|_r \leqq \|X\|_r$, this bound becomes

$$
2(2^{1/p} + 1)\alpha^{1/p - 1/r} \|X\|_r. \quad \text{Q.E.D.}
$$

Proof of (3.8). If we apply Lemma (3.5) to the sequence,

$$
\|E_n X_{n+m}\|_2 \leqq \min(2\varphi_m^{1-1/\beta}, 5\alpha_m^{\frac{1}{2}-1/\beta}) \|X_{n+m}\|_\beta
$$

so by (d) or (d′), $\{(X_i, \mathfrak{F}^i)\}$ is a mixingale of size $-\frac{1}{2}$.

We now verify (2.5).

Put $S_{k,n} = S_{k+n} - S_k$ and for some c to be determined later,

$$
\bar{S}_{k,n} = S_{k,n} I\left(\frac{S_{k,n}^2}{n} \leqq c\right).
$$

Observe that

$$g(c) \doteq \sup_{k,n} \left\| \frac{S_{k,n}^2}{n} - \frac{\bar{S}_{k,n}^2}{n} \right\|_1 \to 0 \quad \text{as } c \to \infty$$

by Lemma (6.5). Then,

$$\begin{aligned}\left\| E_{k-m} \frac{S_{k,n}^2}{n} - \sigma^2 \right\|_1 &\leqq 2 \left\| \frac{S_{k,n}^2}{n} - \frac{\bar{S}_{k,n}^2}{n} \right\|_1 \\ &\quad + \left\| E_{k-m} \frac{\bar{S}_{k,n}^2}{n} - E \frac{\bar{S}_{k,n}^2}{n} \right\|_1 \\ &\quad + \left\| E \frac{S_{k,n}^2}{n} - \sigma^2 \right\| \\ &\leqq 2 g(c) + \min(2\varphi_m, 5\alpha_m^{\frac{1}{2}}) c + \left| E \frac{S_{k,n}^2}{n} - \sigma^2 \right|.\end{aligned}$$

We choose and fix, c sufficiently large that the first term is less than $\varepsilon/3$. Clearly the second and third terms can each be made $< \varepsilon/3$ for $\min(m, k, n)$ sufficiently large.

Proof of (4.2). We first show that the sequence $\{X_i\}$ is a mixingale with ψ_n of size $-\frac{1}{2}$. Note that by Theorem (4.1) and Lemma (3.5),

$$\begin{aligned}\|E_{i-2m} X_i\|_2 &\leqq \|E_{i-2m} E(X_i | \mathfrak{F}_{i-m}^{i+m})\|_2 + \|E(X_i | \mathfrak{F}_{i-m}^{i+m}) - X_i\|_2 \\ &\leqq \min 2\hat{\varphi}_m^{1-1/\beta}, 5\hat{\alpha}_m^{1/2-1/\beta}) \|E(X_i | \mathfrak{F}_{i-m}^{i+m})\|_\beta + \gamma_m\end{aligned}$$

and this is of size $-1/2$ by assumption. Also, by Lemma 1, p. 184 of Billingsley,

$$\|E_{i+m} X_i - X_i\|_2 \leqq \|E(X_i | \mathfrak{F}_{i-m}^{i+m}) - X_i\|_2 \leqq \gamma_m.$$

Therefore, Theorem (2.5) is in force. We now use Billingsley's theorem 19.1 to characterize the possible limits of the sequence $\hat{P}\{W_n \in \cdot\} \doteq P\{W_n \in \cdot | F\}$ where m is arbitrary, and F any member of $\mathfrak{F}^m$ of positive measure. Again, put $\hat{E}(\cdot) = E(\cdot | F)$. For national simplicity we consider only the intervals $(0, t_1)$ and (t_2, t_3) with $0 < t_1 < t_2 < t_3$ and put

$$U_n = E\{W_n(t_1) | \mathfrak{F}_{-\infty}^{[nt_1]}\} \quad \text{and} \quad V_n = E\{W_n(t_3) - W_n(t_2) | \mathfrak{F}_{[nt_2]}^{\infty}\}.$$

Then by (4.1) and Lemma 1, p. 184 of Billingsley,

$$\begin{aligned}\|U_n - W_n(t_1)\| &\leqq \frac{1}{\sigma n^{\frac{1}{2}}} \sum_{i=1}^{[nt_1]} \|X_i - E(X_i | \mathfrak{F}_{-\infty}^{[nt_1]})\|_2 \\ &= 0 \left(n^{-\frac{1}{2}} \sum_{i=1}^{[nt_1]} \gamma_i \right) \to 0\end{aligned} \tag{6.10}$$

by Kronecker's lemma. Similarly,

(6.11) $\|V_n - [W_n(t_3) - W_n(t_2)]\|_2 \to 0$ as $n \to \infty$. But by the mixing assumption, for any Borel sets A_1 and A_2,

$$P(U_n \in A_1) - P(V_n \in A_2 | F) \to 0 \quad \text{as } n \to 0.$$

Thus U_n and V_n are asymptotically $(\hat{P})$ independent and by (6.10) and (6.11), so are $W_n(t_1)$ and $W_n(t_3) - W_n(t_2)$. The remaining moment and uniform integrability conditions of Billingsley's theorem 19.2 are satisfied by assumption and Lemma (6.5).

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