Z. Wahrscheinlichkeitstheorie verw. Gebiete
35, 75–86 (1976)

Zeitschrift für
Wahrscheinlichkeitstheorie
und verwandte Gebiete

On the Uniqueness of Certain Interacting Particle Systems

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0. Introduction

We discuss the existence, uniqueness and ergodicity of certain configuration-valued stochastic processes $(\xi_t)_{t\in\mathbb{R}^+}$. Let V be a countable set of vertices, or *sites*, and S a compact metric space. At each time t our process assumes a value $\xi_t \in \Xi = S^V$; ξ_t is to be thought of as a configuration of values from S on the sites of V, $\xi_t(x)$ being the value at x. Let $\mathbb{D} = \mathbb{D}(\mathbb{R}^+, \Xi)$ be the path space of right continuous functions with left limits from $[0, \infty)$ to Ξ, define $\xi_t\colon \mathbb{D} \to \Xi$ as the evaluation map $\omega \mapsto \omega(t)$, $\omega = (\omega_t(x)) \in \mathbb{D}$, and let $\mathscr{B} = \sigma\langle(\xi_t)_{t\in\mathbb{R}^+}\rangle$ be the σ-algebra generated by the ξ_t. Also, put $\mathscr{B}_0^t = \sigma\langle(\xi_r)_{0\leq r\leq t}\rangle$. We view the desired stochastic system as the coordinate process (ξ_t) on $(\mathbb{D}, \mathscr{B})$ governed by any of a collection $(P_\xi)_{\xi\in\Xi}$ of probability measures such that

$$P_\xi(\xi_0 = \xi) = 1, \qquad \xi \in \Xi. \tag{1}$$

E_ξ will denote the expectation operator corresponding to P_ξ.

Three additional properties characterize the transition mechanism for the processes (ξ_t) which we will consider. First, the value $\xi_t(x)$ at each site x changes, or "flips", only finitely often in any given interval. Next, the expected number of flips at x from time s to time t, given $\mathscr{B}_0^s$, is determined by certain *flip rates* c_x. And finally, the values at two different sites never change simultaneously. More precisely, let $c_x = \{c_x(\xi, \cdot)\}_{\xi\in\Xi}$ be a weakly continuous collection of finite non-negative Borel measures on S, modified for convenience so that $c_x(\xi, \xi(x)) = 0$. (Here and below, $\{a\}$ will often be abbreviated as a.) Let $\mathbb{D}_0$ comprise those paths in $\mathbb{D}$ assuming finitely many values at each site in any finite interval. Write $\omega_{t-}(x) = \lim_{u\uparrow t} \omega_u(x)$, and define $\xi_{t-}(x)$ analogously. For $\omega \in \mathbb{D}_0$, E a Borel set of S, $x \in V$, and $0 \leq s \leq t$, set

$$N_E^x(s,t)(\omega) = |\{u \in (s,t]\colon \omega_{u-}(x) \neq \omega_u(x) \text{ and } \omega_u(x) \in E\}|.$$

$N_E^x(s,t)$ is the number of flips to the set E at x between times s and t. The three additional properties we require of (ξ_t) may now be stated as

$$P_\xi(\mathbb{D}_0)=1, \tag{2}$$

$$E_\xi(N_E^x(s,t)\,|\,\mathscr{B}_0^s)=\int_s^t E_\xi(c_x(\xi_u,E)\,|\,\mathscr{B}_0^s)\,du, \tag{3}$$

and

$$P_\xi(\xi_{u-}(x)\neq\xi_u(x),\ \xi_{u-}(y)\neq\xi_u(y) \text{ for some } u\geqq 0)=0, \tag{4}$$

$\xi\in\Xi$, E Borel in S, $x\neq y\in V$, $0\leqq s\leqq t$. A process satisfying (1)–(4) will be called a *spin system with rates* $c=\{c_x\}_{x\in V}$. Property (3) says that the expected number of flips at x in $(s,t]$, given $\mathscr{B}_0^s$, is the integral from s to t of the expected flip rate given $\mathscr{B}_0^s$. In the leading case, when $S=\{-1,1\}$, our formulation is often called the *spin-flip* model.

Weaker versions of (3) and (4), stressed in previous papers on spin systems, are

$$P_\xi(\xi_{t+h}(x)\in E\setminus\xi_t(x)\,|\,\mathscr{B}_0^t)=c_x(\xi_t,E)\,h+o(h), \tag{3'}$$

and

$$P_\xi(\xi_{t+h}(x)\neq\xi_t(x),\ \xi_{t+h}(y)\neq\xi_t(y)\,|\,\mathscr{B}_0^t)=o(h), \qquad x\neq y. \tag{4'}$$

There are, however, systems which satisfy (1), (2), (3′) and (4′), but not (3), and which are not worthy of membership in the class of spin systems. We illustrate this point with an example due to S. Kalikow (private communication).

Example. Let $V=\mathbb{Z}^+=\{1,2,\ldots\}$, $S=\{-1,1\}$. Define

$$c_n(\xi,-\xi(n))=\begin{cases}1 & \text{if } \xi(n)=-1 \text{ and } n\geqq 2\\ 0 & \text{otherwise.}\end{cases}$$

There is a very simple (strong Feller) spin system with the above flip rates, namely the process for which the value at site 1 does not change, while a -1 at site $n\geqq 2$ flips to $+1$ after an exponential time with mean 1, independently of all other sites, and a $+1$ does not change. But consider the following process: starting from $\xi^-=$ "all -1's" the value at site $n\geqq 2$ flips as above, while the value at site 1 changes from -1 to $+1$ deterministically at time 1. Such a process still satisfies (1), (2), (3′) and (4′) for the given rates. Moreover, it is strong Markov, being the independent product of two strong Markov processes. One cannot eliminate such behavior by requiring time homogeneity, because the deterministic flip can be made compatible with a homogeneous Markov semigroup. The idea is to view the values at sites $n\geqq 2$ as a clock which is used "to tell what time it is." The process will then be time homogeneous for the same reason that the space-time modification of any nonhomogeneous process is. The fact that the values at sites $n\geqq 2$ can be used as a clock follows from the strong law of large numbers, which implies that

$$P_{\xi^-}\left(\lim_{n\to\infty}\frac{1}{n}\sum_{k=2}^{n}\xi_t(k)=1-2e^{-t} \text{ for all } t\geqq 0\right)=1.$$

Although spin systems have been studied actively for several years, their theory is just beginning to come into focus. The most important problems are threefold:

(i) *Existence.* For which rates c is there a process (ξ_t) satisfying (1)–(4)?;

(ii) *Uniqueness.* What conditions on c guarantee a process which is, in some sense, uniquely determined by the rates?;

(iii) *Ergodicity.* When is there a unique equilibrium probability measure μ on Ξ such that the distribution of ξ_t converges weakly to μ from any initial state ξ?

Let us review briefly some of the known results regarding (i)–(iii). In [8], Liggett proved the existence of a "unique" process with rates c provided that

$$\sup_{x\in V,\,\xi\in\Xi} c_x(\xi, S) < \infty, \tag{5}$$

and

$$\sup_{x\in V} \sum_{y\in V-x} \sup_{s\in S,\,\xi\in\Xi} \|c_x(\xi,\cdot) - c_x(\xi_{ys},\cdot)\| < \infty, \tag{6}$$

where ξ_{ys} is the modification of ξ defined by $\xi_{ys}(x) = \begin{cases} s & \text{if } x=y \\ \xi(x) & \text{otherwise} \end{cases}$, and $\|\ \|$ = variation norm. To phrase Liggett's result precisely we need more notation. Let $\mathscr{C} = \mathscr{C}(\Xi)$ be the Banach space of continuous real-valued functions with the supremum norm, $\mathscr{F}^A$ the subspace of functions depending only on sites in the finite set $A \subset V$, and $\mathscr{F} = \bigcup_{\text{finite } A} \mathscr{F}^A$. For $x\in V$, $s\in S$, define the operator $\Delta_x^s: \mathscr{C}\to\mathscr{C}$ by $(\Delta_x^s f)(\xi) = f(\xi_{xs}) - f(\xi)$, and $G_x: \mathscr{C}\to\mathscr{C}$ by $(G_x f)(\xi) = \int_{s\in S} c_x(\xi, ds)\,\Delta_x^s f(\xi)$. Finally, define $G: \mathscr{F}\to\mathscr{C}$, the *pregenerator with rates c*, as

$$G = \sum_{x\in V} G_x. \tag{7}$$

If $f\in\mathscr{F}^A$, then $Gf = \sum_{x\in A} G_x f$, since $\Delta_x^s f \equiv 0$ when $x\notin A$. Continuity of $G_x f$ follows from the hypothesis on the $c_x(\xi,\cdot)$, so G is well-defined. Now under conditions (5) and (6), the methods of [9] yield a strong Feller spin system whose infinitesimal generator extends G. Moreover, the Markov semigroup for (ξ_t) is the *unique* such semigroup whose generator extends G; this is the content of the uniqueness assertion.

Dobrushin [1] also proved a uniqueness theorem, and gave the first general ergodicity criterion. Sullivan [9] unified and extended the results of Liggett and Dobrushin; his theorem applies in some cases when the rates are not uniformly bounded (i.e. when (5) fails).

Recently, Holley and Stroock [6] have made important contributions to the understanding of all three of the above mentioned questions. The treatment in [6] is based on solutions to the "martingale problem" for c. While presented for the case $S=\{-1,1\}$, their results presumably have straightforward generalizations to our setting. A family $(P_\xi)_{\xi\in\Xi}$ of measures on $(\mathbb{D}, \mathscr{B})$ forms a *solution to the martingale problem for c* if (1) and (2) hold, and in addition,

$$f(\xi_t) - \int_0^t Gf(\xi_s)\,ds \quad \text{is a } P_\xi\text{-martingale for all } \xi\in\Xi,\ f\in\mathscr{F}. \tag{8}$$

Among the main results of [6] are the following: (a) *Always* when $S=\{-1, 1\}$ (and even in some cases when the rates c_x are not continuous) there is a process with rates c which solves the martingale problem; (b) There are rates c such that there are *two* distinct solutions to the martingale problem for c; (c) If there is a *unique* solution $(P_\xi)_{\xi\in\Xi}$ to the martingale problem for c, then (ξ_t) is a strong Feller process with respect to (P_ξ); (d) If (5) and (6) hold then there is a unique solution to the martingale problem for c. In addition, they give a combined uniqueness and ergodic theorem which neither implies nor follows from the theorems in [1, 8] and [9].

It can be shown that the class of solutions to the martingale problem for c *coincides* with the class of spin systems with rates c. In other words, (1), (2) and (8) are equivalent to (1)–(4). We leave the proof of this fact to the interested reader. Thus the existence problem for spin systems is completely solved, at least when $S=\{-1, 1\}$. The corresponding problem for *strong Markov* spin systems remains open, however; this is because there are non-Markovian processes which satisfy (1)–(4). These considerations also imply that a standard Markov process on $(\mathbb{D}_0, \mathscr{B})$ is a spin system if and only if its generator extends a pregenerator of the form (7). The strong Markov process of Example 1 does not have $\mathscr{F}$ in the domain of its generator, and hence is not a spin system.

The main purpose of this paper is to present improved uniqueness criteria for spin systems. Our approach, like that used in [8, 9], and parts of [6], is based on the Hille-Yosida theorem, but the space of cylinder functions is exploited more methodically. The processes we will obtain satisfy the *strong extension property (s.e.p.)*: the pregenerator G with rates c has well-defined closure $\bar{G}$ which generates a Feller process. In this case (ξ_t) is the unique spin system with rates c, a consequence of Theorem 4.2(b) in [6] and the equivalence of spin systems and solutions to the martingale problem. Section 1 contains notation necessary in the sequel, a version of the Hille-Yosida theorem most suited for our purposes, and some preliminary results. Section 2 contains our main result (Theorem 3), which may be viewed as an improved version of Sullivan's uniqueness and ergodic theorem. Indeed, certain estimates from [9] play an integral part in the formulation and proof. Our extension has the advantage that it applies to a much wider class of pregenerators; for example, a new result which is a corollary of Theorem 3 states that G has the s.e.p. whenever (6) holds. Finally, Section 3 contains a very simple example of nonuniqueness, for which one can construct two distinct *Feller* processes having the same pregenerator G.

1. Preliminaries

If $\beta=(\beta_x)_{x\in V}$ is a (non-negative) density on V, set

$$l^1(\beta)=\{u=(u_x)_{x\in V}\in\mathbb{R}^V: \sum_{x\in V}|u_x|\beta_x<\infty\},$$

and $l^0=\{u=(u_x)_{x\in V}\in\mathbb{R}^V: u_x=0$ for all but finitely many $x\}$. Note that $l^0\subset l^1(\beta)$ for any density β.

Throughout the discussion, the term "operator" will always mean either a *densely defined* linear operator on $\mathscr{C}$ or a linear operator on $l^1(\beta)$ or l^0. The norm symbol $\| \ \|$ denotes

$$\|f\| = \sup_{\xi \in \Xi} |f(\xi)|, \qquad f \in \mathscr{C};$$

$$\|u\| = \sum_{x \in V} |u_x| \beta_x, \qquad u \in l^1(\beta);$$

$$\|\mu\| = \text{variation norm of } \mu, \qquad \mu \text{ a signed Borel measure};$$

$$\|L\| = \text{the usual operator norm of } L, \qquad L \text{ an operator};$$

the appropriate meaning will be clear from the context. When $f_n, f \in \mathscr{C}$, we often write $f_n \xrightarrow{s} f$ to mean $\|f - f_n\| \to 0$ as $n \to \infty$. Also, if μ and ν are two measures on S, then $\mu \wedge \nu = \mu - (\mu - \nu)^+$.

The symbols A and B will *always* denote *finite* subsets of V, even when this is not mentioned explicitly. Fix a reference state $s_0 \in S$, and for given $\xi \in \Xi$, $A \subset V$, define the modification

$$\xi^A(x) = \begin{cases} \xi(x), & x \in A, \\ s_0 & \text{otherwise.} \end{cases}$$

For $f \in \mathscr{C}$, the modification $f^A \in \mathscr{F}^A$ is given by $f^A(\xi) = f(\xi^A)$. Note that $f^A \xrightarrow{s} f$ as $A \uparrow V$; this shows that $\mathscr{F}$ is *dense* in $\mathscr{C}$. Also, set $\delta f = ((\delta f)_x)_{x \in V}$, where

$$(\delta f)_x = \sup_{s \in S,\, \xi \in \Xi} |(\Delta_x^s f)(\xi)|.$$

We remark that $\delta f \in l^0$ whenever $f \in \mathscr{F}$.

The domain and range of an operator L are denoted by $\mathscr{D}(L)$ and $\mathscr{R}(L)$ respectively. Recall that an operator L on $\mathscr{C}$ has closure $\bar{L}$ iff the closure of the set $\{(f, Lf)\}_{f \in \mathscr{D}(L)}$ in $\mathscr{C} \times \mathscr{C}$ is the graph of a well-defined operator $\bar{L}$. If $\bar{L}$ exists, then given any $f \in \mathscr{D}(\bar{L})$ we can find $f_n \in \mathscr{D}(L)$ such that $f_n \xrightarrow{s} f$ *and* $Lf_n \xrightarrow{s} \bar{L}f$. Also, L satisfies the *maximum property* iff whenever $f \in \mathscr{D}(L)$ and $f(\bar{\xi}) = \max_{\xi \in \Xi} f(\xi)$, then $(Lf)(\bar{\xi}) \leq 0$. Note that if $G: \mathscr{F} \to \mathscr{C}$ is a pregenerator of the form (7), then the maximum property is self-evident.

Our main objective in this paper is to derive uniqueness criteria for spin systems with given rates c, or equivalently, with given pregenerator G. We will make use of Theorems 1 and 2 below, which constitute a probabilistic version of the Hille-Yosida theorem. Proofs of these results may be found in [2, Theorem 2.8] and [7, Theorem 2.1].

Theorem 1. *An operator L on $\mathscr{C}$ is the generator of a strongly continuous semigroup T_t of positive conservative operators on $\mathscr{C}$ if and only if*

(i) *L satisfies the maximum property, and*

(ii) *$\mathscr{R}(\lambda - L) = \mathscr{C}$ for some (all) $\lambda > 0$.*

Moreover, (i) *and* (ii) *imply that $R_\lambda^L = (\lambda - L)^{-1}$ exists, and $\|R_\lambda^L\| \leq \frac{1}{\lambda}$ for all $\lambda > 0$.*

(Note that (ii) always holds when L is a bounded operator on $\mathscr{C}$ satisfying (i).)

We say that an operator L on $\mathscr{C}$ has the *strong extension property (s.e.p.)* iff L has closure $\bar{L}$ which generates a semigroup T_t of the above type. When L has the s.e.p. we put $R_\lambda^L = (\lambda - \bar{L})^{-1}$, which exists on all of $\mathscr{C}$ by Theorem 1.

Theorem 2. *A pregenerator G has the s.e.p. if and only if*

$\overline{\mathscr{R}(\lambda - L)} = \mathscr{C}$ *for some (all)* $\lambda > 0$.

We conclude this section with four elementary results which will be useful later. The proof of the first consists of a straightforward computation.

Proposition 1. *Let* L_1 *and* L_2 *be operators on* $\mathscr{C}$ *with a common domain (or operators on* $l^1(\beta)$*). Suppose* $\mathscr{R}(\lambda - L_1) = \mathscr{C}$ *(or* $l^1(\beta)$*),* $\lambda - L_1$ *is invertible, and* $\|(L_2 - L_1)(\lambda - L_1)^{-1}\| < 1$*. Then* $\lambda - L_2$ *is invertible, and*

$$(\lambda - L_2)^{-1} = (\lambda - L_1)^{-1}(1 - (L_2 - L_1)(\lambda - L_1)^{-1})^{-1}.$$

Proposition 2. *Suppose* G_1 *and* G_2 *are pregenerators such that* $G_0 = G_2 - G_1$ *is bounded. If* G_1 *has the s.e.p., then so does* G_2.

Proof. $\bar{G}_2 = \bar{G}_1 + \bar{G}_0|_{\mathscr{D}(\bar{G}_1)}$. For $\lambda > \|\bar{G}_0\|$, $\|\bar{G}_0 R_\lambda^{G_1}\| < 1$, and hence $(\lambda - \bar{G}_2)^{-1}$ is defined on $\mathscr{C}$ by Proposition 1 and Theorem 1 applied to $\bar{G}_1$. Since $\overline{\mathscr{R}(\lambda - G_2)} \supset \mathscr{R}(\lambda - \bar{G}_2)$, the claim now follows from Theorem 2.

When G has rates c, set $c_x^A(\xi, ds) = c_x(\xi^A, ds)$; $(G_x^A f)(\xi) = \int_S c_x^A(\xi, ds)\, \Delta_x^s f(\xi)$, and $G^A = \sum_{x \in A} G_x^A$, $A \subset V$.

Proposition 3. *Let G be a pregenerator with the s.e.p. Then*

$$(\bar{G} f)^A = G^A(f^A), \quad f \in \mathscr{D}(\bar{G}),\ A \subset V.$$

Proof. Choose $f_n \in \mathscr{F}$ such that $f_n \xrightarrow{s} f$ and $G f_n \xrightarrow{s} \bar{G} f$. Then clearly $(G f_n)^A \xrightarrow{s} (\bar{G} f)^A$. But also $(G f_n)^A = G^A(f_n^A) \xrightarrow{s} G^A(f^A)$, this last since G^A is bounded.

A pregenerator G is called a *box pregenerator* iff there is an increasing sequence of finite "boxes" $B(n) \subset V$, $n \geqq 0$, such that $B(n) \uparrow V$ and $G_x = G_x^{B(n)}$ whenever $x \in B(n)$.

Proposition 4. *Let G be a box pregenerator, with boxes* $B(n)$*. Then*

$$G(f^{B(n)}) = G^{B(n)}(f^{B(n)}), \quad f \in \mathscr{C},\ n \geqq 0,$$

and G has the s.e.p.

Proof.

$$G(f^{B(n)}) = \sum_{x \in B(n)} G_x(f^{B(n)}) = \sum_{x \in B(n)} G_x^{B(n)}(f^{B(n)}) = G^{B(n)}(f^{B(n)}).$$

Since $G^{B(n)}(f^{B(n)}) \in \mathscr{F}^{B(n)}$, G maps $\mathscr{F}^{B(n)}$ into itself for each n. Thus $G|_{\mathscr{F}^{B(n)}}$ may be thought of as a bounded operator with the maximum property on $\mathscr{C}(S^{B(n)})$. By Theorem 1 and the remark which follows, $(\lambda - G)(\mathscr{F}^{B(n)}) = \mathscr{F}^{B(n)}$ for all $\lambda > 0$, $n \geqq 0$. Hence $\mathscr{R}(\lambda - G) = \mathscr{F}$, and G has the s.e.p. by Theorem 2.

A spin process with a box pregenerator is called a *box process*. The next section contains our main result, which states that if a pregenerator is "sufficiently close" to a box pregenerator, then it has the s.e.p. As will be seen, box processes play an important role.

2. An Improved Uniqueness and Ergodic Theorem

We present here a uniqueness and ergodic theorem which extends results of Liggett [8], Dobrushin [1] and Sullivan [9]. Our theorem makes use of the *comparison matrix* $C^G=(C^G_{xy})_{x,y\in V}$ for the pregenerator G, i.e. certain quantities introduced in [1] and [8], and refined in [9]. Following Sullivan, C^G is given by

$$\begin{aligned} C^G_{xx}&=-\inf_{\substack{\xi\in\Xi,\, s\in S:\\ s\neq\xi(x)}}\{(c_x(\xi,\cdot)\wedge c_x(\xi_{xs},\cdot))(S-\{\xi(x),s\})+c_x(\xi_{xs},\xi(x))+c_x(\xi,s)\},\\ C^G_{xy}&=\tfrac{1}{2}\sup_{\xi\in\Xi,\, s\in S}\{\|c_y(\xi,\cdot)-c_y(\xi_{xs},\cdot)\|+|c_y(\xi,S)-c_y(\xi_{xs},S)|\},\qquad x\neq y,\end{aligned}\tag{9}$$

and is to be viewed as an operator on l^0. Note that our convention that $c_x(\xi,\xi(x))=0$ simplifies C^G somewhat from [9]. The intuitive meaning of the comparison matrix is perhaps best explained in terms of a certain Markovian "coupling", for which C^G_{xy} represents the maximal rate at which two dependent copies of (ξ_t) grow apart at site y when their configurations differ only at x, and $-C^G_{xx}$ represents the minimal rate at which the two copies assume the same value at x when they differ only there. For the details of this interpretation in a special case, the reader is referred to [3]. Actually, a somewhat different coupling than the one in [3] is required in order to obtain Sullivan's ergodic theorem.

Roughly, though, the components of C^G measure the amount of influence transmitted by the rates from one site to another. The formal properties needed for our uniqueness and ergodic theorem are given in

Lemma 1. *Let G be a pregenerator with comparison matrix C^G.*

(i) *If $f\in\mathscr{C}$ and $\Delta^{\bar{s}}_y f(\bar{\xi})=(\delta f)_y$, then*

$$C^G_{xy}(\delta f)_y\geqq\begin{cases}\Delta^{\bar{s}}_x G_x f(\bar{\xi}), & x=y,\\ |[\Delta^{\bar{s}}_x, G_y]f(\bar{\xi})|, & x\neq y,\end{cases}$$

where $[\Delta^{\bar{s}}_x, G_y]=\Delta^{\bar{s}}_x G_y-G_y\Delta^{\bar{s}}_x$.

(ii) *Suppose there is a (non-negative) density $\beta=(\beta_x)_{x\in V}$ such that*

$$\sum_{x\in V,\, x\neq y}\beta_x C^G_{xy}\leqq\rho(\alpha-C^G_{yy})\beta_y,\qquad y\in V,\tag{10}$$

for some $\rho\in[0,1)$, $\alpha\in\mathbb{R}$. Then for each $\lambda>\alpha$ and $A\subset V$, $\lambda-C^{GA}$ maps l^0 into itself. Also, $(\lambda-C^G)^{-1}$ and $(\lambda-C^{GA})^{-1}$ exist, and are positive bounded operators on $l^1(\beta)$. Finally, if $0\leqq u\in l^1(\beta)$, then

$$((\lambda-C^{GA})^{-1}u)_x\leqq((\lambda-C^G)^{-1}u)_x\qquad\textit{for all } x\in A.$$

Remark. The conclusions of part (ii) hold under conditions (1) and (3) of Theorem 3.4 in [9]. But, as noted by Sullivan, his (3) is difficult to check unless one assumes our (10).

Proof. (i) See Lemmas 4.4 and 4.5 in [9]. (ii) The argument is similar to parts of the proofs of Lemmas 4.6–4.8 in [9]. However, the claim made there that $B_A-\alpha$ is dissipative cannot be justified for $\alpha<0$, and we therefore sketch a slightly different approach in which C^{GA} takes the role of B_A. The first fact is obvious.

For the rest, write $C^G = -D^G + E^G$, where D^G is diagonal and E^G has all zeroes on the diagonal. Then whenever $u \in l^1(\beta)$,

$$\|E^G(\lambda+D^G)^{-1}u\| \leqq \sum_{y\in V}\sum_{x\neq y} \frac{\beta_x C^G_{xy}}{\lambda - C^G_{yy}}|u_y| \leqq \sum_{y\in V}\frac{\rho(\alpha - C^G_{yy})}{(\lambda - C^G_{yy})}\beta_y|u_y| < \rho\|u\|,$$

and so by Proposition 1, $(\lambda - C^G)^{-1} = (\lambda + D^G)^{-1}(1 - E^G(\lambda + D^G)^{-1})^{-1}$, which is evidently a positive operator. The same argument clearly applies to the comparison matrix C^{G^A} for the pregenerator G^A. Finally, if $0 \leqq u \in l^1(\beta)$, one checks that for $x \in A$,

$$\begin{aligned}[(\lambda - C^{G^A})^{-1}u]_x &= [(\lambda+D^G)^{-1}(1 - E^{G^A}(\lambda+D^G)^{-1}u]_x\\ &\leqq [(\lambda+D^G)^{-1}(1 - E^G(\lambda+D^G)^{-1}u]_x,\end{aligned}$$

since $E^G - E^{G^A} \geqq 0$.

We are now prepared to state and prove the main result. It should be mentioned that our real contribution is to the question of uniqueness. Sullivan's ergodicity criterion depends on his uniqueness conditions, so that our extension also gives ergodicity for a wider class of rates. But the method of proving ergodicity, once uniqueness is established, follows [9] precisely. In fact, the ergodic theorem presented here is still essentially that of Dobrushin [1], with somewhat better estimates. Similar ergodicity criteria, at least in special cases, are discussed in [3, 4] and [5].

Theorem 3. *A pregenerator G has the s.e.p. if there is a density β on V such that* (10) *holds and, in addition,*

there is a positive constant γ and a sequence of boxes $B(n)\uparrow V$ such that for all $x\in V$,

$$\sup_{\xi\in\Xi}\|c_x(\xi,\cdot) - c_x(\xi^{B(n_x)},\cdot)\| \leqq \gamma\beta_x, \tag{11}$$

where n_x is the first n such that $x\in B(n)$.

Moreover, if $\inf_{x\in V}\beta_x > 0$, and $\alpha < 0$ in (10), *then the Feller spin system (ξ_t) with pregenerator G is ergodic.*

Proof. Set $G' = \sum_{x\in V} G_x^{B(n_x)}$, noting that G' is a box pregenerator with boxes $B(n)$. Introduce "approximate" pregenerators $G^{(n)}$, $n \geqq 0$, defined by

$$G^{(n)} = \sum_{x\in B(n)} G_x + \sum_{x\in V-B(n)} G_x^{B(n_x)}.$$

Then $L^{(n)} = G^{(n)} - G' = \sum_{x\in B(n)} (G_x - G_x^{B(n_x)})$ is a bounded operator. Hence $G^{(n)}$ has the s.e.p. by Propositions 2 and 4. *Fix* $\lambda > \alpha$, and choose $g\in\mathscr{F}$. Propositions 3 and 4 show that for any $f\in\mathscr{D}(\bar{G}')$,

$$(\lambda - \bar{G}')f^{B(m)} = ((\lambda - \bar{G}')f)^{B(m)} \xrightarrow{s} (\lambda - \bar{G}')f \quad \text{as } m\to\infty,$$

while $L^{(n)}f^{B(m)} \xrightarrow{s} L^{(n)}f$ as $m\to\infty$ by boundedness. It follows that we can take $f_n = (R_\lambda^{G^{(n)}}g)^{B(m)} \in \mathscr{F}^{B(m)}$ for some $m \geqq n$ with

$$\begin{aligned}\|(\lambda - G)f_n - g\| &\leqq \|(\lambda - G)f_n - (\lambda - G^{(n)})f_n\| + \|(\lambda - G^{(n)})f_n - g\|\\ &\leqq \|(G - G^{(n)})f_n\| + \frac{1}{n}.\end{aligned}$$

Now

$$\begin{aligned}\|(G-G^{(n)})f_n\| &= \|\sum_{x\in B(m)-B(n)} (G_x - G_x^{B(n_x)})f_n\| \\ &\leqq \sum_{x\in V-B(n)} \{\sup_{\xi\in\Xi} \|c_x(\xi,\cdot)-c_x^{B(n_x)}(\xi,\cdot)\|\}(\delta f_n)_x \\ &\leqq \gamma \sum_{x\in V-B(n)} \beta_x(\delta(R_\lambda^{G^{(n)}}g))_x.\end{aligned}$$

Put $v^g=\sup_n\{\delta(R_\lambda^{G^{(n)}}g)\}$. If $v^g\in l^1(\beta)$, then evidently $(\lambda-G)f_n\xrightarrow{s} g$ as $n\to\infty$. To verify the condition of Theorem 2, thereby proving uniqueness, it remains only to show that

$$v^g\in l^1(\beta) \quad \text{for all } g\in\mathscr{F}. \tag{12}$$

To this end, choose $g\in\mathscr{F}$, $n\geqq 0$, and write $h_N=(R_\lambda^{G^{(n)}}g)^{B(N)}$, $N\geqq 0$. For all $N>n$ which are large enough that $g\in\mathscr{F}^{B(N)}$, Proposition 3 yields

$$\begin{aligned} g &= [(\lambda-G^{(n)})(R_\lambda^{G^{(n)}}g)]^{B(N)} \\ &= \Big(\lambda-\sum_{y\in B(n)} G_y^{B(N)} - \sum_{y\in B(N)-B(n)} G_y^{B(n_y)}\Big)h_N.\end{aligned}$$

If $x\in B(n)$, then algebraic manipulations show that

$$\begin{aligned}\Delta_x^s g &= \lambda\Delta_x^s h_N - \Delta_x^s G_x^{B(N)}h_N - \sum_{y\in B(n)-x} G_y^{B(N)}\Delta_x^s h_N - \sum_{y\in B(n)-x}[\Delta_x^s, G_y^{B(N)}]h_N \\ &\quad - \sum_{y\in B(N)-B(n)} G_y^{B(n_y)}\Delta_x^s h_N - \sum_{y\in B(N)-B(n)}[\Delta_x^s, G_y^{B(n_y)}]h_N.\end{aligned}$$

Choose $\bar{s}$ and $\bar{\xi}$ so that $\Delta_x^{\bar{s}}h_N(\bar{\xi})=(\delta h_N)_x$, and evaluate at $\bar{s}$ and $\bar{\xi}$. By part (i) of Lemma 1, and the maximum property, we obtain

$$\begin{aligned}(\delta g)_x &\geqq \lambda(\delta h_N)_x - C_{xx}^{GB(N)}(\delta h_N)_x - \sum_{y\in B(n)-x} C_{xy}^{GB(N)}(\delta h_N)_y \\ &\quad - \sum_{y\in B(N)-B(n)} C_{xy}^{GB(n_y)}(\delta h_N)_y.\end{aligned}$$

The explicit form (9) of the comparison matrix shows that $C_{xy}^{GB(n_y)}$ is dominated by $C_{xy}^{GB(N)}$ whenever $y\in B(N)$. Hence

$$[(\lambda-C^{GB(N)})(\delta h_N)]_x \leqq (\delta g)_x. \tag{13}$$

An analogous computation establishes (13) for $x\in B(N)-B(n)$, while both sides vanish when $x\notin B(N)$. By part (ii) of Lemma 1, both sides of (13) are in $l^1(\beta)$ when considered as vectors, and we may apply the positive operator $(\lambda-C^{GB(N)})^{-1}$ to obtain

$$\delta h_N \leqq (\lambda-C^{GB(N)})^{-1}(\delta g) \leqq (\lambda-C^G)^{-1}(\delta g)\in l^1(\beta).$$

Letting $N\to\infty$, we see that $\delta(R_\lambda^{G^{(n)}}g)$ is bounded by an $l^1(\beta)$ vector which is independent of n, so (12) holds. This completes the proof of uniqueness. The proof that ergodicity obtains when $\inf\beta_x>0$ and $\alpha<0$ mimics [9] exactly. Actually the methods there show exponential convergence to the unique equilibrium measure μ, in the sense that

$$\sup_{\xi_0 \in \Xi} \|P_{\xi_0}(\xi_t|_A \in \cdot) - \mu(\xi|_A \in \cdot)\| \leqq K_A e^{-\alpha t},$$

finite $A \subset V$, for some $K_A \geqq 0$.

An easy consequence of Theorem 3 is the following useful uniqueness criterion, which includes the result mentioned in the introduction.

Corollary 1. *Let* $M^G_{yx} = \sup_{s \in S, \xi \in \Xi} \|c_x(\xi, \cdot) - c_x(\xi_{ys}, \cdot)\|$, $y \neq x$. *G has the s.e.p. whenever there is a density* β *with* $\inf_{x \in V} \beta_x = \varepsilon > 0$ *such that*

$$\sup_{x \in V} \sum_{y \in V-x} \left(\frac{\beta_y}{\beta_x}\right) M^G_{yx} = M < \infty. \tag{14}$$

In particular, G has the s.e.p. whenever (6) *holds.*

Proof. Weak continuity of the c_x and the triangle inequality show that

$$\|c_x(\xi, \cdot) - c_x(\xi^{B(n_x)}, \cdot)\| \leqq \sum_{y \in V-x} M^G_{yx} \quad (\text{any } B(n) \uparrow V).$$

Assuming (14), this implies (11) with $\gamma = M/\varepsilon$, while (10) follows immediately from the fact that $C^G_{yx} \leqq M^G_{yx}$. Theorem 3 now yields the first claim; the second is the special case $\beta_x \equiv 1$.

We remark that Corollary 1 does not follow from Sullivan's methods, since the analogue of (11) in [9] is the more restrictive condition

$$\sup_{x \in V, \xi \in \Xi} \frac{c_x(\xi, S)}{\beta_x} < \infty \tag{15}$$

and (15) is equivalent to (5) when β is bounded away from 0.

The next two corollaries are useful in comparing Theorem 3 to Sullivan's existence and uniqueness theorem.

Corollary 2. *Let G be a pregenerator satisfying* (10) *and* (15) *for some density* $\beta \geqq 0$ *and constants* α_1 *and* ρ_1. *Let* G^0 *be a box pregenerator with boxes* $B(n)$ *satisfying* (10) *for the same density* β *and (possibly different) constants* α_2, ρ_2. *Then* $G + G^0$ *has the s.e.p.*

Proof. $G + G^0$ satisfies conditions (10) and (11) with density β, $\alpha = \alpha_1 + \alpha_2$, $\rho = \max\{\rho_1, \rho_2\}$ and boxes $B(n)$.

We remark that if S is a finite set *every* pregenerator G' which satisfies the hypotheses of Theorem 3 can be written as a sum $G + G^0$, with G and G^0 as in Corollary 2. This is not true when S is infinite.

A special case of Corollary 2 occurs when G^0 is a pregenerator for a *product process* (i.e. a process for which the flip rate $c_x(\xi, \cdot)$ depends only on $\xi(x)$). We call such an operator G^0 a *product pregenerator.*

Corollary 3. *Let G be a pregenerator satisfying* (10) *and* (15) (*or* (10) *and* (11)), *and let* G^0 *be a product pregenerator. Then* $G + G^0$ *has the s.e.p. Moreover, if G satisfies the ergodicity criteria of Theorem* 3, *then so does* $G + G^0$.

Proof. A product pregenerator satisfies (10) for any density $\beta \geqq 0$, with $\rho = \alpha = 0$.

It appears to be an open question whether or not the sum of an arbitrary pregenerator with the s.e.p. and a product pregenerator always inherits the s.e.p.

3. An Example of Nonuniqueness; Open Problems

We give a simple construction of two different Feller processes with the same rates, by using the duality theory developed by Holley and Liggett in [5]. Holley and Stroock [6] suggested this method of producing nonuniqueness examples. Let $V=\mathbb{Z}^+\cup\{\infty\}$, where ∞ is an adjoined point "at infinity." We describe a certain process $\{A_t\}$ called a branching process with interference (b.p.i.); for details see [5]. A_t can be thought of as a finite collection of particles at time t positioned on distinct vertices of V; thus the state space of $\{A_t\}$ is $\mathcal{T}=\{$all finite subsets of $V\}$. The particles jump at random times, independently of one another, with rates which depend on their positions: a particle at site $n\in\mathbb{Z}^+$ waits an exponential time with mean $1/n^2$ and then jumps to site $n+1$. If two particles attempt to occupy one site, then they merge into one. A particle at site ∞ does not jump at all. Finally, if there are infinitely many jumps in a finite time (this will happen with probability one), we write $T_n=$the time of the n'th jump, $T=\lim_{n\to\infty} T_n$, and set $A_T=(\lim_{t\uparrow T}\bigcap_{t<s<T} A_s)\cup\{\infty\}$. After time T, the process continues as before. T can be thought of as an explosion time, and since $|A_t-\{\infty\}|$ is diminished by one at each explosion, the number of explosions is bounded by $|A_0|$. Although Holley and Liggett only consider b.p.i.'s without explosion, it is easy to use their methods to construct explicitly the above process, and to verify that it is Markov. Let $\{P_F\}_{F\in\mathcal{T}}$ be the collection of probability measures governing $\{A_t\}$, where $P_F(A_0=F)=1$. We now use the duality formula of [5] to define a transition semigroup for a Feller spin system on S^V, where $S=\{-1,1\}$. For $F\in\mathcal{T}$, write $I_F(\xi)=\begin{cases}1 & \text{if } \xi(x)=-1 \text{ for all } x\in F\\ 0 & \text{otherwise.}\end{cases}$ When $t\geqq 0$, define

$$T_t I_F(\xi)=P_F(A_t\cap C(\xi)=\emptyset),\qquad F\in\mathcal{T},\tag{16}$$

where $C(\xi)=\{x\in V\colon \xi(x)=1\}$. Extend T_t to all of $\mathcal{F}$ linearly. It is easy to check that the $\{T_t\}_{t\geqq 0}$ are positive and conservative, so they may be uniquely extended to all of $\mathcal{C}$. The claim is that these operators give rise to a Feller transition semigroup for a spin system on S^V with rates $c_\infty\equiv 0$, and for $n\in\mathbb{Z}^+$,

$$c_n(\xi,\xi(n+1))=\begin{cases}n^2 & \text{if } \xi(n)\neq\xi(n+1)\\ 0 & \text{otherwise.}\end{cases}\tag{17}$$

We will check the Feller property, but leave the semigroup property and the rates to the reader. To see that $\{T_t\}$ is Feller, it suffices to show that for each $\varepsilon>0$, $F\in\mathcal{T}$, $t\geqq 0$ and $\xi\in\Xi$, there is a finite $A\subset V$ such that $|T_tI_F(\xi)-T_tI_F(\xi^A)|<\varepsilon$. By (16), the difference we wish to control is

$$|P_F(A_t\cap C(\xi)=\emptyset)-P_F(A_t\cap C(\xi^A)=\emptyset)|\leqq P_F(A_t\cap A\neq\emptyset)\leqq\sum_{x\in F}P_{\{x\}}(A_t\in A^c).\tag{18}$$

Note that when A contains ∞ we have $P_{\{\infty\}}(A_t\in A^c)=0$, so one can assume that $\infty\notin F$ in (18). Choose $n\in F\cap\mathbb{Z}^+$, and for $k=1,2,\ldots$, let τ_k be exponentially distributed with mean $1/k^2$, the τ_k all governed by a probability measure Q. If $A_M=\{m\in\mathbb{Z}^+: m>M\}\cup\{\infty\}$, $M\in\mathbb{Z}^+$, then $P_{\{n\}}(A_t\in A_M^c)=Q\big(\sum_{n\leq k\leq M-1}\tau_k\leq t, \sum_{n\leq k}\tau_k>t\big)$, which tends to 0 as $M\to\infty$. Therefore, the sum in (18) can be made smaller than ε if M is chosen large enough.

Now the state space for our spin system can be divided into two sets which do not communicate: $\Xi^+=\{\xi\in\Xi:\xi(\infty)=1\}$ and $\Xi^-=\{\xi\in\Xi:\xi(\infty)=-1\}$. When the system starts in Ξ^+, it can be thought of as a Feller spin system on the state space $S^{\mathbb{Z}^+}$ with rates given by (17). But there is a Feller spin system on $S^{\mathbb{Z}^+}$ with the same rates, corresponding to the original system starting in the set Ξ^-. One may easily check using (16) that the first of these systems on $S^{\mathbb{Z}^+}$ has the "all $+1$'s" configuration as a trap, while the second does not, so they are distinct.

The above construction raises the question of a "boundary theory" for spin systems. One way to get spin systems with rates c is to take weak limits of box processes $(\xi_t^{(k)})$, where $(\xi_t^{(k)})$ has boxes $B^{(k)}(n)$ and rates $c_x^{B^{(k)}(0)}$ when $x\in B^{(k)}(0)$, by letting $B^{(k)}(0)\uparrow V$ as $k\to\infty$. The methods of [6] show that $\{(\xi_t^{(k)});\, k\geq 0\}$ always has a weak subsequential limit which solves the martingale problem. Ideally one would hope for an analogous weak compactness theory for the class of all strong Markov processes with given rates, and a description of precisely which processes are weak limits. In particular, is there always (i) a strong Markov process, (ii) a Feller process, with rates c? Another open problem is whether the strong extension property is equivalent to uniqueness of a spin system with prescribed rates. In other words, can it ever happen that the pregenerator G has closure $\bar{G}$ which is not a generator, but such that there is a *unique* extension of $\bar{G}$ which is a generator? These strike us as some of the leading open problems relating to existence and uniqueness of spin systems.

Acknowledgements. Thanks to Professors R. Holley, T. Liggett, F. Spitzer and S. Goldstein for their help.

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Received October 20, 1975