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Limit Theorems for $J-X$ Processes with a General State Space

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1. Preliminaries

Let $(\mathbf{W}, \mathscr{W})$, $(\mathbf{X}, \mathscr{X})$ be two measurable spaces and $\mathbf{Q}$ a transition probability function from $(\mathbf{W}, \mathscr{W})$ to $(\mathbf{W} \times \mathbf{X}, \mathscr{W} \times \mathscr{X})$. It is well known that, for each $\mathbf{w} \in \mathbf{W}$, there exists a probability space $(\Omega, \mathscr{K}, \mathbb{P}_{\mathbf{w}})$ and a $\mathbf{W} \times \mathbf{X}$ valued sequence $(J_n, X_{n+1})_{n \geqq 0}$ of random variables such that

$$\begin{aligned} &\mathbb{P}_{\mathbf{w}}(J_0 = w) = 1 \\ &\mathbb{P}_{\mathbf{w}}(J_{n+1} \in A, X_{n+1} \in B \mid J_0, X_1, J_1, \ldots, X_n, J_n) \\ &= \mathbb{P}_{\mathbf{w}}(J_{n+1} \in A, X_{n+1} \in B \mid X_n, J_n) \\ &= \mathbf{Q}(J_n; A \times B) \qquad \mathbb{P}_{\mathbf{w}}\text{-}a.s. \end{aligned} \tag{1}$$

for any $n \in N$, $A \in \mathscr{W}$ and $B \in \mathscr{X}$ (see e.g. [11], Ch. V. 2).

Definition 1. A Markov chain $(J_n, X_{n+1})_{n \geqq 0}$ satisfying the last equality in (1) will be called a $J-X$ process.

Remark 1. The sequence $(J_n)_{n \geqq 0}$ is also a Markov chain with the transition probability function $\mathbf{P}$ given by

$$\mathbf{P}(w; A) = \mathbf{Q}(w; A \times \mathbf{X}) \tag{2}$$

Remark 2. Let f be a real valued measurable function defined on $\mathbf{X}$. If the chain $(J_n, X_{n+1})_{n \geqq 0}$ is a $J-X$ process, then so is $(J_n, f(X_{n+1}))_{n \geqq 0}$. Since the aim of this paper is to obtain (weak and strong) limit theorems for functions of $(X_n)_{n \geqq 1}$ it turns out that we may (and actually *will*) assume, without any loss of generality, that the space $\mathbf{X}$ is a borelian subset of the real line (maybe $\mathbb{R}$ itself).

The $J-X$ processes were first studied by Janssen [9] in the case of a countable $\mathbf{W}$. He allowed the conditional distribution of X_{n+1} given the past to depend on both J_n and J_{n-1}; but as pointed out in [13] this assumption yields no increase in generality.

Many authors have paid attention to the asymptotic behaviour of the process $(X_n)_{n \geqq 1}$ which is sometimes called a *sequence of random variables defined on a*

Markov chain ([17]) or a *chain-dependent process* ([13]). However all of them required the space **W** to be finite ([17, 18]) or countable ([9, 13]).

In Section 3 of the present paper it is shown that the results obtained in the quoted papers are still valid if **W** is an arbitrary space. On the other hand let us note that the results to be presented are still in force under other conditions ((**A**), (**C**) and (**D**)) than those usually assumed in the literature.

In Section 4 we prove the functional central limit theorem for $(X_n)_{n\geqq 1}$ and derive new results (Theorem 4) from it.

The lemmas proved in Section 2 turn out to be the cornerstone of the entire paper since they show that the process $(X_n)_{n\geqq 1}$ is closely related to a stationary φ-mixing process.

Finally let us mention some noteworthy classes of $J-X$ processes.

1. Let $(Z_t)_{t\geqq 0}$ be a semi-Markov process with an arbitrary state space and $(\tau_n)_{n\geqq 0}$ the sequence of jump moments. Set $X_n=\tau_n-\tau_{n-1}$ for $n\geqq 1$.

It has been proved in [14] that $(Z_{\tau_n}, X_{n+1})_{n\geqq 0}$ is a $J-X$ process with the transition probability function **Q** given by the corresponding semi-Markov kernel.

2. Let $\{(\mathbf{W}, \mathscr{W}), (\mathbf{X}, \mathscr{X}), \mathbf{u}, \mathbf{P}\}$ be a random system with complete connections. Then the associated sequence $(\zeta_n, \xi_n)_{n\geqq 1}$ is a $J-X$ process with **Q** given by

$$\mathbf{Q}(w; A\times B)=\int_B \mathbf{P}(w; dx)\, I_A(\mathbf{u}(w, x)),$$

$I_A(\cdot)$ denoting the indicator of the set A.

(For all the definitions and notations used here the reader is refered to [8], Ch. 2).

For such $J-X$ processes our Theorems 1 and 2 have been already obtained in [3] respectively [7] under some milder conditions, namely assuming the uniform ergodicity of the system. Recently, under the same condition, our last two theorems have been proved in this context in [16].

3. A generalized random system with complete connections is also a $J-X$ process. In this case

$$\mathbf{Q}(w; A\times B)=\int_B \mathbf{P}(w; dx)\, \Pi(w, x; A).$$

Here the notations are those used in [10].

For such processes our approach seems to be the unique appropriate way to studying the asymptotic behaviour of the sequence $(\zeta_n)_{n\geqq 1}$.

2. Auxiliary Results

Let U be the Markov operator associated with the transition probability function **P** given by (2) and let $(\mathbf{M}, \|\cdot\|_{\mathbf{M}})$ a Banach space of real valued functions defined on **W**.

Definition 2. U is said to be regular (with respect to **M**) if

a) U maps **M** into itself boundedly with respect to the norm $\|\cdot\|_{\mathbf{M}}$,

b) there is a linear bounded operator U^∞ which takes **M** into itself and $U^\infty(\mathbf{M})$ is one dimensional,

c) $\|U^n-U^\infty\|_{\mathbf{M}}\to 0$.

Now we introduce some conditions.

Condition **A**. The Markov chain $(J_n)_{n \geqq 0}$ is uniformly φ-reccurent and aperiodic (see [15], p. 26 for the concept of φ-reccurence).

Condition **B**. **W** is a locally compact space and the chain $(J_n)_{n \geqq 0}$ is regular with respect to bounded measurable real valued functions.

In this case regularity holds if and only if *Doeblin's condition* is satisfied, there is only one ergodic set and it is aperiodic (see e.g. [11], Ch. V. 3).

Condition **C**. **W** is a compact metric space, the transition probability function $\mathbf{Q}(\cdot\,; A \times B)$ is continuous for each $A \in \mathbf{W}$ and $B \in \mathbf{X}$ and the chain $(J_n)_{n \geqq 0}$ is regular with respect to continuous real valued functions.

Condition **D**. **W** is a compact metric space, $(J_n, X_{n+1})_{n \geqq 0}$ is a distance diminishing model and $(J_n)_{n \geqq 0}$ is regular with respect to Lipschitz functions.

See [12], p. 31 for the definition of a distance diminishing model.

Without further mention *we will assume from now on that one of these four conditions is fulfiled.*

Let $(\eta_n)_{n \geqq 1}$ be a sequence of real valued random variables defined on $(\Omega, \mathscr{K}, \mathbb{P})$ and let denote by $\mathscr{K}_{[m,n]}$, $m \leqq n$, the σ-algebra generated by $\eta_m, \ldots, \eta_n$; $\mathscr{K}_{[m,\infty)}$ will be the σ-algebra generated by $\eta_r, r \geqq m$.

Definition 3. The sequence $(\eta_n)_{n \geqq 1}$ is said to be φ-mixing with respect to $\mathbb{P}$ if there is a function $\varphi_{\mathbb{P}}$ defined on natural numbers such that $\varphi_{\mathbb{P}}(n) \to 0$ as $n \to \infty$ and

$$|\mathbb{P}(A \cap B) - \mathbb{P}(A)\,\mathbb{P}(B)| \leqq \varphi_{\mathbb{P}}(n)\,\mathbb{P}(A)$$

for every $A \in \mathscr{K}_{[1,k]}$ and $B \in \mathscr{K}_{[k+n,\infty)}$.

Lemma 1. *There is a unique stationary probability measure* π *for* $\mathbf{P}$ *and the process* $(X_n)_{n \geqq 1}$ *is* φ*-mixing with respect to* $\mathbb{P}_\pi$. *Moreover*

$$\varphi_{\mathbb{P}_\pi}(n) = a\,\rho^n$$

where $a > 0$ *and* $\rho \in (0, 1)$.

Proof. It is known that both conditions **A** and **B** imply the existence of a unique stationary probability measure (see [15], p. 31 respectively [11], p. 167). This fact has been proved in [5] under condition **C** and in [12], p. 40, 50–53 under condition **D**.

Let k and n be natural numbers and

$$A = \{X_1 \in C_1, \ldots, X_k \in C_k\}$$

$$B = \{X_{k+n+1} \in C_{k+n+1}, \ldots, X_{k+n+r} \in C_{k+n+r}\} \tag{3}$$

where the C_i's are borelian subsets of the real line. Using the Markov property and the stationarity of π we can write

$$\mathbb{P}_\pi(A \cap B) = \int_{\mathbf{W} \times \mathbf{W}} \mathbb{P}_\pi(A \cap B \mid J_k = u, J_{n+k} = v)\,\mathbb{P}_\pi(J_k \in du, J_{n+k} \in dv)$$

$$= \int_{\mathbf{W} \times \mathbf{W}} \mathbb{P}_\pi(J_k \in du, J_{n+k} \in dv) \int_{C_1 \times \cdots \times C_k} \mathbb{P}_\pi(B \mid J_{n+k} = v, J_k = u, X_1 = x_1, \ldots, X_k = x_k)$$

$$\mathbb{P}_\pi(X_1 \in dx_1, \ldots, X_k \in dx_k \mid J_{n+k} = v, J_k = u)$$

$$= \int_{\mathbf{W} \times \mathbf{W}} \pi(du)\,\mathbf{P}^{(n)}(u; dv)\,\mathbb{P}_\pi(B \mid J_{n+k} = v)\,\mathbb{P}_\pi(A \mid J_k = u).$$

By similar computations we also get

$$\mathbb{P}_\pi(A)\,\mathbb{P}_\pi(B) = \int_{\mathbf{W}} \pi(du)\,\mathbb{P}_\pi(A \mid J_k = u) \int_{\mathbf{W}} \pi(dv)\,\mathbb{P}_\pi(B \mid J_{n+k} = v).$$

Then

$$|\mathbb{P}_\pi(A \cap B) - \mathbb{P}_\pi(A)\,\mathbb{P}_\pi(B)| \leqq \int_{\mathbf{W}} \pi(du)\,\mathbb{P}_\pi(A \mid J_k = u)\, h_n(u) \tag{4}$$

where

$$h_n(u) = \left| \int_{\mathbf{W}} \mathbb{P}_\pi(B \mid J_{n+k} = v)\,[\mathbf{P}^{(n)}(u, dv) - \pi(dv)] \right|.$$

Under condition **A**, Theorem 7.1. in [15] ensures the existence of two constants $a > 0$ and $\rho \in (0, 1)$ such that

$$\|\mathbf{P}^{(n)}(u; \cdot) - \pi(\cdot)\| \leqq a\,\rho^n.$$

On the other hand by Lemma 1.2.1. in [8] we get

$$h_n(u) \leqq \operatorname{essosc} \mathbb{P}_\pi(B \mid J_{n+k} = \cdot)\,\|\mathbb{P}^{(n)}(u; \cdot) - \pi(\cdot)\|.$$

Combining the last two inequalities and (4) we conclude that

$$|\mathbb{P}_\pi(A \cap B) - \mathbb{P}_\pi(A)\,\mathbb{P}_\pi(B)| \leqq a\,\rho^n\,\mathbb{P}_\pi(A) \tag{5}$$

for sets of the form (3).

Under any of the conditions **B**, **C** and **D** regularity is assumed and U^∞ must be given by

$$U^\infty f = \int_{\mathbf{W}} \pi(dw)\, f(w).$$

It is shown in [12], p. 37 that for a regular operator (with respect to **M**) there are $a > 0$ and $\rho \in (0, 1)$ such that

$$\|U^n - U^\infty\|_{\mathbf{M}} \leqq a\,\rho^n.$$

As

$$h_n(u) = |(U^n - U^\infty)\,\mathbb{P}_\pi(B \mid J_{n+k} = \cdot)(u)|$$

it only remains to prove that $\mathbb{P}_\pi(B \mid J_{n+k} = \cdot)$ is bounded under condition **B**, continuous under condition **C** and Lipschitz under condition **D**. The first assertion is obvious and the last two are proved in [5] respectively [8], p. 128.

Hence (5) holds true for sets of the form (3) regardless of which condition we assume. The validity of (5) for arbitrary sets $A \in \mathscr{K}_{[1,k]}$ and $B \in \mathscr{K}_{[k+n+1,\infty)}$ follows by a usual argument of monotone class, q.e.d.

Remark 3. Lemma 1 under condition **B** is to some extent a generalization of the result in [4], except that there the Markov chain $(J_n)_{n \geqq 0}$ has the φ-mixing property for any initial distribution.

Corollary 1. *The tail σ-algebra of the process $(X_n)_{n \geqq 1}$ is $\mathbb{P}_\pi$-trivial.*

Lemma 2. *Let μ be an arbitrary probability measure on $\mathscr{W}$. Then there are two constants $a>0$ and $\rho\in(0,1)$ such that*

$$|\mathbb{P}_\pi(A)-\mathbb{P}_\mu(A)|\leqq a\,\rho^n \tag{6}$$

for any $A\in\mathscr{K}_{[n+k,\infty)}$.

Proof. It is sufficient to prove that (6) holds true for $A=\{X_{n+1}\in C_1,\dots,X_{n+r}\in C_r\}$, where $r\in N$ and the C_i's (borelian subsets of the real line) are arbitrary. For such a set A we get

$$\begin{aligned}|\mathbb{P}_\pi(A)-\mathbb{P}_\mu(A)|&=|\int_{\mathbf{W}}\mathbb{P}_\pi(A\,|\,J_n=u)\,\mathbb{P}_\pi(J_n\in du)\\&\quad-\int_{\mathbf{W}}\mathbb{P}_\mu(A\,|\,J_n=u)\,\mathbb{P}_\mu(J_n\in du)|\\&=|\int_{\mathbf{W}}g(u)\,[\pi(du)-\int_{\mathbf{W}}\mu(dv)\,\mathbf{P}^{(n)}(v;du)]|\\&=|\int_{\mathbf{W}}g(u)\,[\pi(du)-\mu\,\mathbf{P}^{(n)}(du)]|\end{aligned}\tag{7}$$

Here

$$\begin{aligned}g(u)&=\mathbb{P}_\pi(A\,|\,J_n=u)=\mathbb{P}_\mu(A\,|\,J_n=u)\\&=\int_{\mathbf{W}\times C_1}\mathbf{Q}(u;dw_1\times dx_1)\int_{\mathbf{W}\times C_2}\mathbf{Q}(w_1;dw_2\times dx_2)\dots\int_{\mathbf{W}\times C_r}\mathbf{Q}(w_{r-1};dw_r\times dx_r).\end{aligned}$$

A thorough inspection of the arguments in the last part of the proof of the previous lemma shows that (6) can be derived from (7) under any of the conditions **A**$-$**D**, q.e.d.

Corollary 2. *If A is an event in the tail σ-algebra of the process $(X_n)_{n\geqq1}$, then*

$$\mathbb{P}_\pi(A)=\mathbb{P}_\mu(A)$$

for any probability measure μ on $\mathscr{W}$.

Remark 4. The process $(X_n)_{n\geqq1}$ is $\mathbb{P}_\pi$-stationary. Indeed

$$\begin{aligned}\mathbb{P}_\pi(X_{n+1}\in A)&=\int_{\mathbf{W}}\mathbb{P}_\pi(X_{n+1}\in A\,|\,J_n=w)\,\mathbb{P}_\pi(J_n\in dw)\\&=\int_{\mathbf{W}}\pi(dw)\,\mathbf{Q}(w;\mathbf{W}\times A)\end{aligned}$$

and the last term above does not depend on n.

3. Strong Limit Theorems

We restrict ourselves to the proof of the strong law of large numbers and the loglog law; nevertheless other results can be derived in the same way.

Theorem 1. *Assume that $\mathbf{E}_\pi(|X_1|)<\infty$ and let $\mathbf{m}=\mathbf{E}_\pi(X_1)$. Then*

$$(1/n)\sum_{i=1}^{n}X_i\to\mathbf{m}\qquad\mathbb{P}_\mu\text{-a.s.}\tag{8}$$

as $n\to\infty$ for any probability measure μ on $\mathscr{W}$.

Proof. As a result of Remark 4 and Birkhoff's theorem (see e.g. [2], p. 113) (8) takes place $\mathbb{P}_\pi$-*a.s.* But the event in (8) is in the tail σ-algebra of the process $(X_n)_{n \geqq 1}$. The desired conclusion follows from Corollary 2, q.e.d.

Theorem 2. *Assume that* $\mathbf{E}_\pi(|X_1|^{2+\delta}) < \infty$ *for some* $\delta \geqq 0$. *Then the process* $(X_n)_{n \geqq 1}$ *obeys Strassen's version of the loglog law with respect to* $\mathbb{P}_\mu$ *where* μ *is any probability measure on* $\mathscr{W}$.

Proof. By virtue of Lemma 1 and Remark 4 we can easily check that all conditions of Corollary 3 in [6] are verified and therefore so is its conclusion meaning that the loglog law (in Strassen's version) holds true with respect to $\mathbb{P}_\pi$. Use Corollary 2 to conclude the proof, q.e.d.

Remark 5. In [7] it has been proved that some non-stationary sequences (namely the uniform ergodic random systems with complete connections) also obey Strassen's law. Theorem 2 above furnishes other such sequences (classes 1 and 3 mentioned in Section 1).

4. The Functional Central Limit Theorem

For the sake of simplicity we assume that $\mathbf{E}_\pi(X_1)=0$. In the sequel $\mathbb{W}$ is the Wiener measure on $\mathscr{D}[0,1]$ endowed with Skorokhod topology and "$\Rightarrow$" means weak convergence.

Theorem 3. *Assume* $\mathbf{E}_\pi(X_1^2) < \infty$ *and set*

$$\sigma^2 = \mathbf{E}_\pi(X_1^2) + 2 \sum_{k=2}^{\infty} \mathbf{E}_\pi(X_1 X_k),$$

Then

(*i*) $0 \leqq \sigma^2 < \infty$,

(*ii*) *If* $\sigma^2 > 0$ *then*

$$Y_n \circ \mathbb{P}_\mu^{-1} \Rightarrow \mathbb{W} \tag{9}$$

for each probability measure μ *on* $\mathscr{W}$. *Here*

$$Y_n(t) = (1/\sigma\, n^{1/2}) \sum_{0 < k \leqq [nt]} X_k.$$

Proof. Since $(X_n)_{n \geqq 1}$ is $\mathbb{P}_\pi$-stationary and φ-mixing (with $\varphi_{\mathbb{P}_\pi}(n) = a\,\rho^n$) (*i*) is immediate (see e.g. [1], p. 175). Moreover

$$Y_n \circ \mathbb{P}_\pi^{-1} \Rightarrow \mathbb{W}. \tag{10}$$

Choose a sequence $(p_n)_{n \geqq 1}$ of natural numbers such that

$$\lim_{n \to \infty} p_n = \infty$$

$$\lim_{n \to \infty} (p_n / n^{1/2}) = 0 \tag{11}$$

and define

$$Y_n'(t)=(1/\sigma\, n^{1/2}) \sum_{p_n \leqq k \leqq [nt]} X_k.$$

Set

$$\delta_n = \sup_{t\in[0,1]} |Y_n(t) - Y_n'(t)|.$$

By Theorem 1 and (11) we get

$$\delta_n \leqq (1/\sigma\, n^{1/2}) \sum_{k=1}^{p_n} |X_k| \to 0 \qquad \mathbb{P}_\mu\text{-}a.s. \tag{12}$$

Use (10) and (12) to derive

$$Y_n' \circ \mathbb{P}_\pi^{-1} \Rightarrow \mathbb{W}.$$

Select a set A in $\mathscr{D}\,[0,1]$ such that $\mathbb{W}(\mathbf{Fr}\, A)=0$. Since $Y_n'^{-1}(A) \in \mathscr{K}_{[p_n,\infty)}$ we obtain by Lemma 2

$$|\mathbb{P}_\mu(Y_n'^{-1}(A)) - \mathbb{P}_\pi(Y_n'^{-1}(A))| \leqq a\, \rho^{p_n}.$$

From the last two inequalities we have

$$Y_n' \circ \mathbb{P}_\mu^{-1} \Rightarrow \mathbb{W}$$

which in turn combined with (10) and (12) concludes the proof, q.e.d.

Remark 6. By using Theorem 3 we can easily prove the classical central limit theorem as well as Theorem 4 below by the usual procedures described in [1], Chapter 3.

However we need some further notations. Set

$$S_n = \sum_{k=1}^{n} X_k,$$

$$m_n = \min_{0<i\leqq n} (1/\sigma\, n^{1/2})\, S_i,$$

$$M_n = \max_{0<i\leqq n} (1/\sigma\, n^{1/2})\, S_i,$$

$T_n = \min\{i \mid S_n, S_{n-1}, \ldots, S_{i+1}$ have all the same sign and either $S_i=0$ or S_i has the same sign as $S_n\}$ if $S_n \neq 0$
1 otherwise

$$U_n = \operatorname{card}\{i \mid 0<i\leqq n,\ S_i>0\},$$

$$V_n = \operatorname{card}\{i \mid 0<i\leqq T_n,\ S_i>0\},$$

If $\mathbf{X}=\mathbb{R}_+$, then set

$$N(t) = \begin{cases} \max\{k \mid S_k \leqq t\} & \text{if } X_1 \leqq t \\ 0 & \text{otherwise} \end{cases}$$

$$Z_n(t) = \frac{N(n\,t) - n\,t\,\mathbf{m}^{-1}}{n^{1/2}\,\mathbf{m}^{-3/2}}.$$

If $\mathbf{X}=[0,1]$ set

$$F_n^*(t)=(1/n)\sum_{k=1}^{n} I_{[0,t]}\circ X_k, \qquad t\in[0,1],$$

$$F_\pi(t)=\mathbb{P}_\pi(X_1\leqq t).$$

Theorem 4. *For any probability measure μ on $\mathscr{W}$ we have*

(*i*) (*Maximum and minimum distribution*)

$$\lim_{n\to\infty}\mathbb{P}_\mu(a<m_n\leqq M_n<b,\ u<S_n<v)$$

$$=\sum_{k=-\infty}^{+\infty}(1/(2\pi)^{1/2})\int_{u+2k(b-a)}^{v+2k(b-a)}\exp(-t^2/2)\,dt$$

$$-\sum_{k=-\infty}^{+\infty}(1/(2\pi)^{1/2})\int_{2b-v+2k(b-a)}^{2b-u+2k(b-a)}\exp(-t^2/2)\,dt$$

for any $a\leqq 0\leqq b$ *and* $a\leqq u<v\leqq b$.

(*ii*) (*The* arcsin *law*)

$$\lim_{n\to\infty}\mathbb{P}_\mu((1/n)\,T_n<\alpha)=(2/\pi)\arcsin\alpha^{1/2}, \qquad 0<\alpha<1.$$

(*iii*) (*The renewal theorem*)

$$\mathbb{P}_\mu\circ Z_n^{-1}\Rightarrow\mathbb{W}$$

as $n\to\infty$.

(*iv*) (*Kolmogorov-Smirnov's test*).

$n^{1/2}(F_n^*(t)-F_\pi(t))$ *converges weakly to a Gaussian random element in* $\mathscr{D}[0,1]$.

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