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Two Parameter Optimal Stopping and Bi-Markov Processes

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Summary. In this paper, the optimal stopping problem is solved for particular two-parameter processes, here called bi-Markov processes. A subsequent potential theory is developed with respect to a pair of one-parameter semi-groups. We introduce a new notion of harmonicity for two-variable functions and we interpret it in the framework of the theory of bi-Markov processes.

Such a study can be motivated by the following example. Let us suppose that the evolutions of two stochastic systems are modelized by two classical Markov processes X^1 and X^2. We must stop them at – may be different – times T_1 and T_2 such that the average of common satisfaction or the pay-off

$$E(e^{-\alpha_1 T_1 - \alpha_2 T_2} f(X^1_{T_1}, X^2_{T_2}))$$

is maximum; in this formula f is a given positive bounded function, and α_1, α_2 are some positive actualization constants. Time T_1 and T_2 have to be causally chosen; that is to say, knowing the sample paths of X^1 up to T_1 and X^2 up to T_2 only. In other words, for every real t_1, t_2 the event $\{T_1 \leqq t_1, T_2 \leqq t_2\}$ must belong to the σ-field generated by the r.v.'s $(X^1_{s_1}, X^2_{s_2};\ s_1 \leqq t_1,\ s_2 \leqq t_2)$. This problem enters into the framework of the optimal stopping theory for two-parameter processes.

The optimal stopping problem for processes with index set $\mathbb{N}^2$ is now well known. In the sequel we shall refer mainly to the works of Cairoli and Gabriel [16], Krengel and Sucheston [28], Mandelbaum and Vanderbei [34], the author and Szpirglas [38], and Millet [45]. Various existence results and construction methods for optimal solutions can be found in these references. Concerning the processes indexed on $\mathbb{R}^2_+$, A. Millet [45] recently gave a general existence result. Her approach consists of extending the compactification techniques of Baxter and Chacon [3] for two-parameter processes, and the basic tool is the notion of randomized tactics. In this paper we propose a quite different method which generalizes the approach of [38]. We recall that the Snell envelope of a process with parameter set $\mathbb{R}_+^2$ has been defined by Cairoli [14]. We then reduce the general optimal stopping problem of a process Y to the optimal stopping of its Snell envelope J on a given stopping line. This last problem is intrinsically one-dimensional and we solve it by using a straight forward extension of the compactification techniques. Namely, we extend a proof by Meyer [43] on the set of randomized stopping times, and we follow the classical method of Bismut [6] or Edgar, Millet and Sucheston [23].

In the second part of this chapter, we set the two-parameter optimal stopping problem into a general framework inspired by that of the classical theory as treated by Bismut and Skalli [7] and El Karoui [24]. Thus, under appropriate regularity conditions on the Snell envelope we prove that the maximal elements of the set of stopping points T which preserve the martingale property of J, i.e., such that $E(J_T)=E(J_0)$, are solution of the optimal stopping problem. Our approach is appropriate to the case of bi-Markov processes, because we are able to compute the Snell envelope by functional methods. We develop a subsequent potential theory for the purpose.

Roughly speaking a bi-Markov process is the tensor product of two classical one-parameter Markov processes. The well-known bi-Brownian motion enters this class. In the second chapter of this paper, we propose a potential theory related to a two-parameter semi-group or, equivalently, to both the two classical one-parameter semi-groups of the Markov processes which compose the bi-Markov process. We recall various definitions introduced by Cairoli [12, 13] dealing with a one-parameter semi-group constructed as the tensor product of two classical semi-groups. These notions can be interpreted in terms of a bi-Markov process, and we focus on two-parameter supermartingales associated to bi-excessive functions. Trajectorial regularity conditions are given, and two different types of Dynkin formula are obtained. They generalize those studied by Lawler and Vanderbei [31], Vanderbei [51] for two-parameter Markov chains.

Usually one pays attention mainly to the class of bi-harmonic functions. Within the framework of the two-parameter processes theory, we refer to the works of Brossard and Chevalier [11] and Walsh [52] for various properties of this class. If $\mathscr{L}^1$ and $\mathscr{L}^2$ denote the generators of the two underlying Markov processes, a sufficiently smooth function f defined on a product space $E=E^1 \times E^2$ is bi-harmonic on an open subset A if

$$\forall x \in A: \mathscr{L}^1 f(x)=0 \quad \textit{and} \quad \mathscr{L}^2 f(x)=0.$$

More recently Dynkin [22] and Vanderbei [51] have studied another class of harmonic functions. In [51] it is said that a smooth function f is harmonic on a certain domain A if

$$\forall x \in A: \mathscr{L}^1 \mathscr{L}^2 f(x)=0.$$

In connection with the optimal stopping problem, we introduce the notion of weak harmonicity. Accurate conditions will be given farther, but we can say roughly that a sufficiently smooth function f is weakly harmonic on a set A if

$$\forall x \in A: \mathscr{L}^1 f(x)=0 \quad \textit{or} \quad \mathscr{L}^2 f(x)=0.$$

Analogous definitions dealing with the discrete case have been proposed by Mandelbaum and Vanderbei [34]. In this paper we give an interpretation of weak harmonicity in terms of the bi-Markov process and the optimal stopping problem. Moreover, we relate it to the notion of the reduite of a function on a subset. Several results were announced in [35–37].

The third chapter is devoted to the optimal stopping problem for a bi-Markov process defined as the solution of a system of stochastic differential equations. By applying the preceding results, we prove an existence theorem. Moreover, we show that if the optimal pay-off function is sufficiently smooth, then it satisfies a set of partial differential equations with free boundaries as in the classical theory of Benssoussan and Lions [4].

Preliminaries

We refer to Cairoli and Walsh [15], Meyer [14] and Wong and Zakai [53] for preliminary notations, definition and results for the theory of two-parameter stochastic processes. Let us recall the main notions used in this article.

The processes we consider in this paper are indexed on $\mathbb{R}_+^2$. They are extended to its one-point compactification, $\overline{\mathbb{R}}_+^2 = \mathbb{R}_+^2 \cup \{\infty\}$ as being null at infinity. The partial order is defined by

$$\forall s=(s_1, s_2), \quad t=(t_1, t_2): s \leqq t \Leftrightarrow s_1 \leqq t_1 \quad \text{and} \quad s_2 \leqq t_2;$$

with $t \leqq \infty \ \forall t \in \mathbb{R}_+^2$.

Defined on a complete probability space $(\Omega, \mathscr{A}, \mathbb{P})$, a filtration is a family $\mathscr{F} = (\mathscr{F}_t;\, t \in \mathbb{R}_+^2)$ of sub-σ-fields of $\mathscr{A}$, such that [10, 33, 26]: $\mathscr{F}_0$ contains all the $\mathbb{P}$-negligible sets of $\mathscr{A}$ (Axiom F1), family $\mathscr{F}$ increases with respect to the partial order on $\mathbb{R}_+^2$ (Axiom F2), and $\mathscr{F}$ is right-continuous (Axiom F3). The one-parameter filtrations $\mathscr{F}^1 = (\mathscr{F}_{t_1}^1;\, t_1 \in \mathbb{R}_+)$ and $\mathscr{F}^2 = (\mathscr{F}_{t_2}^2;\, t_2 \in \mathbb{R}_+)$ are associated to $\mathscr{F}$ by the following definition:

$$\forall t=(t_1, t_2): \mathscr{F}_{t_1}^1 = \bigvee_u \mathscr{F}_{(t_1, u)} \quad \text{and} \quad \mathscr{F}_{t_2}^2 = \bigvee_u \mathscr{F}_{(u, t_2)}.$$

We also denote $\mathscr{F}_{t_1}^1$ and $\mathscr{F}_{t_2}^2$ by $\mathscr{F}_t^1$ and $\mathscr{F}_t^2$, respectively, and consider $\mathscr{F}^1$ and $\mathscr{F}^2$ as two-parameter filtrations. In this paper, we assume that filtration $\mathscr{F}$ satisfies the following classical [15], [53] conditional independence property (Axiom F4)

> $\forall t=(t_1, t_2)$:
> The σ-fields $\mathscr{F}_{t_1}^1$ and $\mathscr{F}_{t_2}^2$ are conditionally independent given $\mathscr{F}_t$.

The optional (resp. 1-optional, 2-optional) σ-field on $\Omega \times \mathbb{R}_+^2$ associated to $\mathscr{F}$ (resp. $\mathscr{F}^1$, $\mathscr{F}^2$), and the related optional projection (resp. 1-optional projection, 2-optional projection) of a bounded process X, denoted by ${}^oX$ (resp. ${}^{o_1}X$, ${}^{o_2}X$), are defined by Bakry in [1].

A *stopping point* (s.p.) is a random variable (r.v.) T, taking its values in $\overline{\mathbb{R}}_+^2$, such that $\{T \leqq t\} \in \mathscr{F}_t$, $\forall t \in \mathbb{R}_+^2$. The set of all s.p.'s is denoted by $\mathscr{T}$. To any s.p.

T, we associate a σ-field $\mathscr{F}_T$, which is the σ-field of all events A such that $A \cap \{T \leqq t\} \in \mathscr{F}_t$, $\forall t$. All the classical properties of stopping times [20] do not extend to stopping points (see [52]). The graph of a s.p. T, denoted by $[\![T]\!]$, is the optional set defined by:

$$[\![T]\!] = \{(\omega, t): T(\omega) = t, t \in \mathbb{R}_+^2\}.$$

A 1-stopping point is a $\overline{\mathbb{R}}_+^2$-valued r.v. $T = (T_1, T_2)$ such that [1] T_1 is a stopping time with respect to $\mathscr{F}^1$, and T_2 is a $\mathscr{F}^1_{T_1}$-measurable r.v. The set of 1-stopping points is denoted by $\mathscr{T}^1$. The 2-stopping points are defined symmetrically. Recall that [44] $\mathscr{T} = \mathscr{T}^1 \cap \mathscr{T}^2$.

Given a random set H in $\Omega \times \mathbb{R}_+^2$, we denote by $[\![H, \infty[\![$ the random set $[\![H, \infty[\![= \{(\omega, t): \exists s \leqq t$ such that $(\omega, s) \in H\}$. The *début* of H, denoted by L_H, is the lower boundary of the set $[\![H, \infty[\![$, with the convention that $L_H = \infty$ if the section is empty. A *stopping line* (s.l.) is the début of an optional random set (see Merzbach [40] and also [1]). We denote by $\mathscr{L}$ the set of all stopping lines. $\mathscr{T}$ can be taken as a subset of $\mathscr{L}$, by identifying any s.p. T with the s.l. which is the debut of the set $[\![T]\!]$.

The partial order is extended to $\mathscr{T}$ by:

$$\forall T, T' \in \mathscr{T}: T \leqq T' \Leftrightarrow T \leqq T' \quad \text{a.s.},$$

as well as to $\mathscr{L}$ by:

$$\forall L, L' \in \mathscr{L}: L \leqq L' \Leftrightarrow [\![L', \infty[\![\subset [\![L, \infty[\![\quad \text{a.s.}$$

The processes we consider are always supposed to be measurable and real-valued. We do not distinguish two processes which differ only on an evanescent set [44]. For a separable process X, X^* stands for $\sup_t |X_t|$.

Recall that a *supermartingale* (resp. *strong supermartingale*) is a process $J = (J_t; t \in \mathbb{R}_+^2)$ adapted to the filtration $\mathscr{F}$ (resp. optional), integrable (resp. of class (D) i.e., $\{J_T; T \in \mathscr{T}\}$ is uniformly integrable) such that:

$$\forall s, t \in \mathbb{R}_+^2: s \leqq t \Rightarrow E(J_t \mid \mathscr{F}_s) \leqq J_s \quad \text{a.s.},$$

$$(\text{resp.: } \forall S, T \in \mathscr{T}: S \leqq T \Rightarrow E(J_T \mid \mathscr{F}_S) \leqq J_S \text{ a.s.}).$$

To each point $t = (t_1, t_2)$, we can associate the following four quadrants: $Q_t^1 = \{s: t \leqq s\}$, $Q_t^2 = \{s: s_1 < t_1$ and $s_2 \geqq t_2\}$, $Q_t^3 = \{s: s_1 < t_1$ and $s_2 < t_2\}$ and $Q_t^4 = \{s: s_1 \geqq t_1$ and $s_2 < t_2\}$. We say that a process X is $i-j$ limited if for each point t, the process X has limits in the quadrants Q_t^i and Q_t^j $(i, j = 1, \ldots, 4)$. For $i = 2$ and $j = 4$ we also say that X is *laterally limited.* It is said to be right-continuous if it is 1-limited and equal to this limit. Quadrantal limits of bounded martingales have been studied by Millet and Sucheston [46] and Bakry [2], and we refer to [2] for quadrantal limits of optional projections.

The notion of an optional increasing path has been introduced by Walsh [51] as a generalization of the discrete tactics of Krengel and Sucheston [28], and Mandelbaum and Vanderbei [34]. An *optional increasing path* (o.i.p.) is a one-parameter family $(Z_u; u \in \mathbb{R}_+)$ of stopping points, such that the mapping

$u \to Z_u$ is increasing and continuous a.s. Moreover, any o.i.p. can be parametrized "canonically" by taking

$$\forall u \in \mathbb{R}_+ : Z_u = (Z_u^1, Z_u^2) \quad \text{with} \quad u = Z_u^1 + Z_u^2 = |Z_u|.$$

For $m \in \mathbb{N}$, let $\mathbb{D}_m$ denote the set of dyadic numbers of order m $\mathbb{D}_m = \{t = (j 2^{-m}, k 2^{-m}); j, k \in \mathbb{N}\}$. Then, a *tactic of order m* is an increasing sequence of stopping points $(T_n; n \in \mathbb{N})$ such that

$$\forall n: T_n \in \mathbb{D}_m \text{ a.s.} \quad \text{and} \quad T_{n+1} = T_n + (2^{-m}, 0) \quad \text{or} \quad T_n + (0, 2^{-m}),$$

and T_{n+1} is a $\mathscr{F}_{T_n}$-measurable random variable, [28, 34, 52]. By interpolating between each s.p. T_n, we associate to this tactic $(T_n; n \in \mathbb{N})$, an o.i.p. $Z = (Z_u; u \in \mathbb{R}_+)$ whose trajectories are increasing step functions, with corners in $\mathbb{D}_m$ (a corner being a point where Z changes direction). Moreover, by using the definition of a tactic, it can be verified that the corners form a sequence of stopping points. It is proved in [52] that any o.i.p. can be approximated by a sequence of tactics of increasing order. We denote by $\mathscr{Z}$ the set of all o.i.p.'s and by $\mathscr{Z}_m^d$ the set of all tactics of order m. Given an o.i.p. $Z = (Z_u; u \in \mathbb{R}_+)$, $\mathscr{F}^Z$ is the one-parameter filtration defined by $\mathscr{F}^Z = (\mathscr{F}_u^Z = \mathscr{F}_{Z_u}; u \in \mathbb{R}_+)$, and $\mathscr{T}^Z$ is the set of all $\mathscr{F}^Z$-stopping times.

In this paper, we need a new definition of a début. Let H be a random set in $\Omega \times \mathbb{R}_+^2$. For any optional increasing path $Z = (Z_u; u \in \mathbb{R}_+)$, we denote by D_H^Z the random variable defined by

$$D_H^Z = Z_\tau \text{ with } \tau = \inf\{u: Z_u \in H\} \text{ and } D_H^Z = \infty \text{ if the set is empty.}$$

This variable belonging to $Z \cup \{\infty\}$ is called "*the début of H along Z*".

Lemma. *If H is an optional set, then for any optional increasing path Z, D_H^Z is a stopping point.*

Proof. The graph $[\![Z]\!] = \{(\omega, t) \in \Omega \times \mathbb{R}_+^2 : t \in Z(\omega)\}$ is optional and, consequently, so is $[\![Z]\!] \cap H$. Then its début is a stopping line [40]. This stopping line has only one minimal element, which is D_H^Z; this implies that D_H^Z is a stopping point. □

1. Optimal Stopping for Two Parameter Processes

In this chapter we study the optimal stopping problem for processes indexed on the directed set $\mathbb{R}_+^2 \cup \{\infty\} = \overline{\mathbb{R}}_+^2$.

Optimal stopping of processes indexed by directed sets was first studied by G.W. Haggstrom [26]. In the frame of the two-parameter processes theory the optimal stopping problem on $\mathbb{N}^2$ is now well known. A first contribution is due to R. Cairoli and J.P. Gabriel [16]. The basic tools of a tactic and a discrete Snell envelope have been developed by U. Krengel and L. Sucheston [28], and by A. Mandelbaum and R.J. Vanderbei [34]. For processes indexed on the set $\mathbb{N}^2 \cup \{\infty\}$ general solutions have been found and explicitely constructed in Mazziotto-Szpirglas [38]. A similar existence result has recently been proved by A. Millet [45] using a different approach. Moreover her method applies to processes with parameter set $\mathbb{R}_+^2 \cup \{\infty\}$. She gives various conditions which ensure that the optimal stopping problem admits solutions. The easiest condition is when the payoff process is continuous and uniformly bounded on $\mathbb{R}_+^2 \cup \{\infty\}$. Further conditions are also given in Dalang [19].

In this paper we present a different method to solve the optimal stopping problem which follows the approach of [38] for the discrete case. As such optimal stopping points are searched amongst the maximal elements of the subset of stopping points which preserve the martingale property of the Snell envelope. To verify that such a point is a solution, we first reduce the general problem to the following particular one which consists of stopping optimally the restriction of the Snell envelope on a given stopping line. We show that this last problem can be solved by using the classical theory, if we assume a regularity hypothesys on the Snell envelope. Let us now anticipate on the third chapter to remark that fortunately in Markovian situations, the Snell envelope can be computed very precisely by functional methods and the preceding assumptions are easily satisfied. This approach will permit us to solve the optimal stopping problem for a bi-Markov process, under conditions which are not covered by those of [45].

1.1. Optimal Stopping on a Given Stopping Line

On a complete probability space $(\Omega, \mathscr{A}, \mathbb{P})$ endowed with a two-parameter filtration $\mathscr{F} = (\mathscr{F}_t;\, t \in \mathbb{R}_+^2)$ verifying the axioms F1, F2, F3 and F4, let us consider an optional non-negative process of class (D): $Y = (Y_t;\, t \in \overline{\mathbb{R}}_+^2)$. This is called the pay-off process. The optimal stopping problem consists in finding a stopping point T^* such that

$$E(Y_{T^*}) = \sup_{T \in \mathscr{T}} E(Y_T).$$

Such a s.p. will be said to be *optimal*.

In this paragraph we study the particular optimal stopping problem of a pay-off process Y which differs from zero only on a given stopping line L, that is to say

$$\forall\, t \in \mathbb{R}_+^2:\; Y_t = Y_t \mathbb{1}_{\{t \in L\}} \qquad \text{i.e. } Y = Y \mathbb{1}_{[\![L]\!]}{}^1,$$

(recall that $Y_\infty = 0$).

We make the convention that $\infty \in L$ a.s. Let $\mathscr{T}(L)$ denote the set of all stopping points which belongs a.s. to L, that is to say

$$\mathscr{T}(L) = \{T \in \mathscr{T} : [\![T]\!] \subset [\![L]\!]\} = \{T \in \mathscr{T} : T \in L \text{ a.s.}\}.$$

It can be easily verified that $\forall\, T \in \mathscr{T}$, $\{T \in L\} \in \mathscr{F}_T$. Thus, we get

$$\sup_{T \in \mathscr{T}} E(Y_T) = \sup_{T \in \mathscr{T}(L)} E(Y_T).$$

[1] In the sequel, the indicator function of a set A is denoted by $\mathbb{1}_A$

The right-hand side of this equality suggests that the considered optimal stopping problem is one-dimensional. In fact, we solve it by a method which is directly adapted from the classical theory: see Bismut [6] and Edgar-Millet-Sucheston [23], and from Millet [45]. We first introduce a convex set containing $\mathcal{T}(L)$ whose set of extremal elements coincides with $\mathcal{T}(L)$. This set in endowed with a topology called the Baxter-Chacon topology, which makes it compact. In a second step, we prove that the optimization problem admits a solution on this larger space and, by a classical argument, in $\mathcal{T}(L)$ also.

Let $\mathcal{M}^0$ be the set of probabilities on $\Omega \times \overline{\mathbb{R}}_+^2$ whose projections on Ω are equal to $\mathbb{P}$. To any $\mu \in \mathcal{M}^0$ one can associate a unique right-continuous process A, increasing in variation, with $A_0=0$, $A_\infty=1$, and such that

$$\mu(X)=E\int_{\overline{\mathbb{R}}_+^2} X_t\, dA_t$$

for any bounded process X. In the sequel, we often identify μ with A. The space $\mathcal{M}^0$ is endowed with the Baxter-Chacon topology i.e., the coarsest topology such that, for any bounded and continuous process X on $\overline{\mathbb{R}}_+^2$, the application Φ_x defined by

$$\forall A \in \mathcal{M}^0\colon \Phi_x(A)=E\int_{\overline{\mathbb{R}}_+^2} X_t\, dA_t$$

is continuous. We note that $\mathcal{M}^0$ can be embedded in a locally convex topological vector space, namely the dual of the Banach space of continuous bounded processes on $\overline{\mathbb{R}}_+^2 \times \Omega$: see Meyer [43]. Let $\mathcal{M}$ be the subset of $\mathcal{M}^0$ of probabilities μ such that the associated process A is adapted. By analogy with the classical theory, we call the elements of $\mathcal{M}$ randomized stopping points. Indeed $\mathcal{T}$ can be injected into $\mathcal{M}$ by setting:

$$\forall T \in \mathcal{T}\colon A=\mathbb{1}_{[\![T,\infty]\!]} \qquad \text{i.e., } \mu \text{ is the Dirac measure on } T.$$

The sets $\mathcal{M}^0$ and $\mathcal{M}$ are convex and compact. Moreover it has been proved by Ghoussoub [25] that the extremal elements of $\mathcal{M}^0$ are the Dirac measures on any random variable in $\overline{\mathbb{R}}_+^2$. But we do not know if the set of extremal elements of $\mathcal{M}$ is $\mathcal{T}$, as in the one-parameter situation (see Edgar-Millet-Sucheston [23]). This last remark will prevent us from extending the method to the general case. It explains in a certain sense, why the compactification introduced by A. Millet in [45] was quite different.

As concerns the point at stake, we restrict ourselves to the subset $\mathcal{M}(L) \subset \mathcal{M}$ of randomized s.p. A whose support is a.s. contained in $L \cup \{\infty\}$ (i.e., such that $\mu([\![L]\!] \cup (\Omega \times \{\infty\}))=1$). It can easily be checked that the set $\mathcal{M}(L)$ is a convex closed subset of $\mathcal{M}$. Then it is compact for the Baxter-Chacon topology. To prove that $\mathcal{T}(L)$ is exactly the set of all extremal elements of $\mathcal{M}(L)$, property F4 will be of crucial importance.

To begin with we parametrize the stopping line L on an interval of $\mathbb{R}$. Let us consider the plane $\mathbb{R}^2$ endowed with its cartesian coordinate: the index set of our two-parameter processes $\mathbb{R}_+^2$ is the first quadrant. For $\theta\in[0,\pi/2]$, let D_θ be the line defined by

$$D_\theta=\{t=(t_1,t_2)\in\mathbb{R}^2:(t_1+1)\sin\theta=(t_2+1)\cos\theta\}.$$

It is easy to see that for any $\theta\in]0,\pi/2[$, the intersection of D_θ and $L(\omega)$ contains (for each fixed $\omega\in\Omega$) a one only point, noted $L_\theta(\omega)$. Moreover, the r.v. L_θ is a stopping point, and $L_\theta\in\mathscr{T}(L)$. To $\theta=0$ and $\theta=\pi/2$ we associate the point at the infinity ∞. Conversely, any point of L can be represented by one value of the parameter θ, except for point ∞ which has the double representation 0 and $\pi/2$.

On the interval $[0,\pi/2]$ we may consider the classical order relation induced by that of $\mathbb{R}$, and we say that L is described according to the *trigonometrical order* (i.e. θ goes from 0 to $\pi/2$). When the orthogonal order relation on $\mathbb{R}$ is considered, we say that L is described according to the *clockwise order* (i.e. θ goes from $\pi/2$ to 0). In the first case, we will consider the filtration $\tilde{\mathscr{F}}^{\uparrow}$ indexed on $[0,\pi/2]$, defined by

$$\forall\,\theta\in]0,\pi/2[:\ \tilde{\mathscr{F}}_\theta^{\uparrow}=\mathscr{F}_{L_\theta}^2,\ \tilde{\mathscr{F}}_0^{\uparrow}=\mathscr{F}_0\quad\text{and}\quad \tilde{\mathscr{F}}_\infty^{\uparrow}=\mathscr{F}_\infty,$$

and in the second case, the filtration $\tilde{\mathscr{F}}^{\downarrow}$ defined by

$$\forall\,\theta\in]0,\pi/2[:\ \tilde{\mathscr{F}}_\theta^{\downarrow}=\mathscr{F}_{L_\theta}^1,\ \tilde{\mathscr{F}}_0^{\downarrow}=\mathscr{F}_0\quad\text{and}\quad \tilde{\mathscr{F}}_\infty^{\downarrow}=\mathscr{F}_\infty.$$

The restriction to the stopping line L of a given two-parameter process Y can be parametrized on $[0,\pi/2]$ as follows

$$\forall\,\theta\in]0,\pi/2[:\ \tilde{Y}_\theta=Y_{L_\theta},\quad\text{and}\quad \tilde{Y}_0=\tilde{Y}_{\pi/2}=0.$$

If process Y admits limits in the quadrants 1, 2 and 4, then $\tilde{Y}$ admits the limits $\tilde{Y}^{\uparrow-}$ and $\tilde{Y}^{\downarrow-}$ defined by

$$\forall\,\theta\in]0,\pi/2[:\ \tilde{Y}_\theta^{\uparrow-}=\lim_{\substack{\alpha\to\theta\\ \alpha<\theta}}\tilde{Y}_\alpha\quad\text{and}\quad \tilde{Y}_\theta^{\downarrow-}=\lim_{\substack{\alpha\to\theta\\ \alpha>\theta}}\tilde{Y}_\alpha.$$

Such a process will be said to be *bi-limited.*

These one-parameter filtrations $\tilde{\mathscr{F}}^{\uparrow}$ and $\tilde{\mathscr{F}}^{\downarrow}$ satisfy the usual conditions of Dellacherie-Meyer [20], according to the trigonometrical and clockwise sense respectively. Then, the notions of $\tilde{\mathscr{F}}^{\uparrow}$-stopping times and $\tilde{\mathscr{F}}^{\downarrow}$-stopping times are well defined. One can check that a r.v. θ taking its values in $[0,\pi/2]$ is a $\tilde{\mathscr{F}}^{\uparrow}$-stopping time (resp. a $\tilde{\mathscr{F}}^{\downarrow}$-stopping time, both a $\tilde{\mathscr{F}}^{\uparrow}$- and $\tilde{\mathscr{F}}^{\downarrow}$-stopping time) if and only if L_θ is a 2-stopping point (resp. a 1-stopping point, a stopping point).

Let μ be a random probability of $\mathscr{M}(L)$ represented by the increasing process A. To this can be associated two processes $A^{\uparrow}$ and $A^{\downarrow}$ indexed on $[0,\pi/2]$, increasing for the trigonometrical and the clockwise order respectively, $\tilde{\mathscr{F}}^{\uparrow}$- and $\tilde{\mathscr{F}}^{\downarrow}$-adapted respectively, by setting

$$\forall\,\theta\in]0,\pi/2[:\ A_\theta^{\uparrow}=\int_{[L_{\theta_1},\infty]\times[0,L_{\theta_2}]}dA_s\quad\text{et}\quad A_0^{\uparrow}=0,\ A_{\pi/2}^{\uparrow}=1;$$

$$\forall\,\theta\in]\pi/2,0[:\ A_\theta^{\downarrow}=\int_{[0,L_{\theta_1}]\times[L_{\theta_2},\infty]}dA_s\quad\text{et}\quad A_{\pi/2}^{\downarrow}=0,\ A_0^{\downarrow}=1.$$

These processes verify the following relations

$$\forall\,\theta\in[0,\pi/2]\colon A_\theta^\updownarrow + A_\theta^{\downarrow -} = A_\theta^{\updownarrow -} + A_\theta^\downarrow = 1.$$

Conversely, we determine one randomized stopping point of $\mathscr{M}(L)$ by giving a process $A^\uparrow = (A_\theta^\updownarrow;\ \theta\in[0,\pi/2])$ which is $\tilde{\mathscr{F}}^\uparrow$-adapted, right-continuous, increasing from $A_0^\updownarrow = 0$ to $A_{\pi/2}^\updownarrow = 1$, and such that the process $A^\downarrow = 1 - A^{\uparrow -}$ is $\tilde{\mathscr{F}}^\downarrow$-adapted.

We are now able to prove that $\mathscr{M}(L)$ has the same properties as the classical randomized stopping times of Baxter and Chacon [3].

Proposition 1.1.1. i) *Every randomized stopping point A of $\mathscr{M}(L)$ can be represented by a family $(T^x, x\in[0,1])$ of stopping points of $\mathscr{T}(L)$.*

ii) *The set of all extremal elements of the convex $\mathscr{M}(L)$ is exactly $\mathscr{T}(L)$.*

Proof. i) Let A be a randomized s.p. of $\mathscr{M}(L)$, and let $A^\uparrow$ and $A^\downarrow$ be the associated one-parameter processes. For any $x\in[0,1]$ we define a $\tilde{\mathscr{F}}^\uparrow$-stopping time θ^x by setting $\theta^x = \inf\{\theta\colon A_\theta^\updownarrow \geqq x\}$ for the trigonometrical order. But θ^x can be equivalently defined by $\theta^x = \inf\{\theta\colon A_\theta^\downarrow \geqq 1 - x\}$ for the clockwise order. Then θ^x is also a $\tilde{\mathscr{F}}^\downarrow$-stopping time. It follows that the r.v. $T^x = L_\theta^x$ is a s.p. with belongs a.s. to L. It is known that the family $\{\theta^x;\ x\in[0,1]\}$ represents uniquely the process $A^\uparrow$ and $A^\downarrow$, then it represents also A.

ii) Let us consider a randomized stopping point A which is an extremal element of the convex set $\mathscr{M}(L)$. To see that it corresponds to a s.p. $T\in\mathscr{T}(L)$, it is sufficient to prove that A takes only values 0 or 1. For that purpose, let us remark that any randomized s.p. A can be written as a convex combination of two randomized s.p.'s B and C. We consider the processes B and C defined by the following. $\forall\,\theta\in[0,\pi/2]$, let:

$$B_\theta^\updownarrow = 2A_\theta^\updownarrow \wedge 1, \qquad C_\theta^\updownarrow = (2A_\theta^\updownarrow - 1)\vee 0$$

$$B_\theta^\downarrow = (2A_\theta^\downarrow - 1)\vee 0, \qquad C_\theta^\downarrow = 2A_\theta^\downarrow \wedge 1.$$

By construction $B^\uparrow$ and $C^\uparrow$ are increasing $\tilde{\mathscr{F}}^\uparrow$-adapted, and $B^\downarrow$ and $C^\downarrow$ are increasing $\tilde{\mathscr{F}}^\downarrow$-adapted. Moreover,

$$B_\theta^\updownarrow + B_\theta^{\downarrow -} = B_\theta^{\updownarrow -} + B_\theta^\downarrow = C_\theta^\updownarrow + C_\theta^{\downarrow -} = C_\theta^{\updownarrow -} + C_\theta^\downarrow = 1.$$

Then the couples $(B^\uparrow, B^\downarrow)$ and $(C^\uparrow, C^\downarrow)$ represent two randomized stopping points of $\mathscr{M}(L)$: B and C, such that

$$A = 1/2\,B + 1/2\,C.$$

But if A takes values different from 0 or 1, processes B and C are necessarily distinct from A. This would contradict the hypothesis that A is extremal. This completes the proof. □

Now let us come back to the optimal stopping problem on L. For any given positive process Y we associate the linear form Φ_Y defined on $\mathscr{M}(L)$ by the following

$$\forall\,A\in\mathscr{M}(L)\colon \Phi_Y(A) = E\int_{\mathbb{R}_+^2} Y_s\,dA_s = E\int_0^{\pi/2} \tilde{Y}_\theta\,dA_\theta^\updownarrow = E\int_{\pi/2}^{0} \tilde{Y}_\theta\,dA_\theta^\downarrow.$$

If A is a stopping point T of $\mathscr{T}(L)$ i.e., $A = \mathbb{1}_{[\![T,\infty[\![}$, then $\Phi_Y(A) = E(Y_T)$.

As in the classical theory we have a general existence result for the optimal stopping problem on L.

Proposition 1.1.2. *If process Y is such that function Φ_Y is continuous on $\mathcal{M}(L)$, then there exists $T^* \in \mathcal{T}(L)$ such that*

$$\sup_{A \in \mathcal{M}(L)} E(\int Y_s \, dA_s) = E(Y_T^*) = \sup_{T \in \mathcal{T}(L)} E(Y_t).$$

Proof. It is an application of Bourbaki [9] (Prop 1 – Chap II – Parag. 7): $\mathcal{M}(L)$ is a convex compact subset of a locally convex vector space, and the application Φ_Y is linear and continuous. Therefore, Φ_Y reaches its maximum at least on one point amongst the extremal elements of $\mathcal{M}(L)$ (i.e. in $\mathcal{T}(L)$ by Proposition 1.1.1). The proof is completed. □

To conclude this paragraph, we need to give practical conditions on Y ensuring that Φ_Y is continuous on $\mathcal{M}(L)$. Due to the definition of the Baxter-Chacon topology, this is trivially true if Y is bounded continuous on $\overline{\mathbb{R}}_+^2$. Let us give a better condition for Y.

Définition 1.1.1. Process Y is said to be *laterally regular* iff it is right-continuous and has limits in the two lateral quadrants Q^2 and Q^4, and iff for any sequence $(T^n; n \in \mathbb{N})$ of 1-stopping points (resp. 2-stopping points) converging to T, such that

$$\forall n: T^n = (T_1^n, T_2^n): T_1^n \leqq T_1 \quad \text{and} \quad T_2^n \geqq T_2 \text{ (resp. } T_1^n \geqq T \text{ and } T_2^n \leqq T_2),$$

the sequence $(E(Y_{T_n}); n \in \mathbb{N})$ converges to $E(Y_T)$. Process Y is said to be *completely regular* iff it is laterally regular, and iff, in addition, for any increasing sequence $(T^n; n \in \mathbb{N})$ of stopping points converging to T, the sequence $(E(Y_{T_n}); n \in \mathbb{N})$ converges to $E(Y_T)$.

Examples of completely regular processes are furnished by the following result.

Proposition 1.1.3. *The optional projection ${}^oY$ of a bounded continuous process Y is completely regular.*

Proof. Let us verify each item of Definition 1.1.1. First recall that the definitions of the optional projections of a bounded process Y given by [1] imply that $E(Y_T) = E({}^oY_T)$ (resp. $E(Y_T) = E({}^{o_1}Y_T)$, $E(Y_T) = E({}^{o_2}Y_T)$) for any stopping point (resp. 1-stopping point, 2-stopping point) T. The property concerning increasing sequences of stopping points follows immediately. Let $(T^n; n \in \mathbb{N})$ be a sequence of 1-stopping points converging to T, such that $T_1^n \leqq T_1$ and $T_2^n \geqq T_2$, $\forall n$. By definition [1]: ${}^oY = {}^{o_1}({}^{o_2}Y)$, then

$$\lim_n E({}^oY_{T^n}) = \lim_n E({}^{o_1}({}^{o_2}Y)_{T^n}) = \lim_n E({}^{o_2}Y_{T^n}).$$

We now use another result established by Bakry [2] (Theorem 4c) concerning the continuity in the upper half-plan of the 2-optional projection ${}^{o_2}Y$ of a continuous process Y. This yields ${}^{o_2}Y_T = \lim_n {}^{o_2}Y_{T^n}$ a.s., and, by bounded convergence: $\lim_n E({}^{o_2}Y_{T^n}) = E({}^oY_T)$. Finally, we get $\lim_n E({}^oY_{T^n}) = E({}^oY_T)$. The symmetrical case is treated similarly. □

The next proposition is analogous to those of [3] in the classical theory, and the idea of the proof has been prompted by Meyer [43] (Theorem 3).

Proposition 1.1.4. *Let $\mathscr{R}$ be the set of bounded processes Y such that the process ${}^oY$ is lateraly regular, and let L be a fixed stopping line. Then, for any $Y \in \mathscr{R}$, the application Φ_Y: $\mu \in \mathscr{M}(L) \to \mu(Y)$ is continuous on $\mathscr{M}(L)$.*

Proof. Let $\mathscr{U}$ be an ultrafilter on $\mathscr{M}(L)$ converging to a randomized stopping point $\mu \in \mathscr{M}(L)$. Therefore, for any bounded measurable process Y, $\{(v(Y), v \in V); V \in \mathscr{U}\}$ generates an ultrafilter on $[-E(Y^*), E(Y^*)]$, whose limit is denoted by $U(Y)$. To prove that for a given process Y the application Φ_Y is continuous at μ, it is sufficient to verify that $\mu(Y) = U(Y)$, for any ultrafilter $\mathscr{U}$ converging to μ.

From the definition of $U(Y)$ it follows immediatly that

$$|U(Y)| \leqq E(Y^*), \quad U(\mathbb{1}_{[\![L]\!]}) = 1, \quad \text{and} \quad U(Y) = U({}^oY).$$

Moreover, from the definition of the Baxter-Chacon topology, we get

$$U(Y) = \mu(Y) \qquad \forall\, Y \text{ continuous}.$$

Now, let us consider the restrictions $\tilde{Y}$ of these processes Y to the stopping line L with their parametrization on $[0, \pi/2]$. The application $\tilde{Y} \to U(Y)$ defines a linear form $\tilde{U}$ on the set $\tilde{\mathscr{D}}$ of bounded bi-limited processes on $[0, \pi/2]$. As in [43] and by using results from [20] (Sect. VII–4), one can prove that $\tilde{U}$ admits the following representation

$$\tilde{U}(\tilde{Y}) = E\left[\int_0^{\pi/2} \tilde{Y}_\theta\, dB_\theta + \tilde{Y}_\theta^{\uparrow -}\, dC_\theta + \tilde{Y}_\theta^{\downarrow -}\, dD_\theta\right],$$

where B, C and D are right-continuous increasing processes. Moreover, if Y is optional, the processes B, C and D can be taken as adapted to the filtrations $\tilde{\mathscr{F}}^{\downarrow} \cap \tilde{\mathscr{F}}^{\uparrow}$, $\tilde{\mathscr{F}}^{\uparrow}$ and $\tilde{\mathscr{F}}^{\downarrow}$ respectively. If Y is optional and is laterally regular, we are to find a representation involving $\tilde{Y}$ only. With this in view, we propose to treat the integral by dC. It is known [43] that, if for any sequence of $\tilde{\mathscr{F}}^{\uparrow}$-stopping times $(\theta_n; n \in \mathbb{N})$ converging in the trigonometrical order towards a limit θ, $\lim_n (E(\tilde{Y}_{\theta_n}) = E(\tilde{Y}_\theta)$ then, we following equality holds

$$E\left[\int_0^{\pi/2} \tilde{Y}_\theta^{\uparrow -}\, dC_\theta\right] = E\left[\int_0^{\pi/2} \tilde{Y}_\theta\, dC'_\theta\right],$$

where C' is the dual predictible projection of C. This assumption on $\tilde{Y}$ is a fortiori realized if Y is laterally regular. Similar treatments can be done for the integral dD, with respect to the filtration $\tilde{\mathscr{F}}^{\downarrow}$. Finally, we get the following representation for the optional and laterally regular Y:

$$\tilde{U}(\tilde{Y}) = E\left[\int_0^{\pi/2} \tilde{Y}_\theta (dB_\theta + dC'_\theta + dD'_\theta)\right].$$

Coming back to the parameter set $\bar{\mathbb{R}}^2_+$, we deduce there exists an increasing two-parameter process A, which is right-continuous, adapted, with $A_0=0$, and such that

$$U(Y)=E[\int_{\mathbb{R}^2_+} Y_t\,dA_t]$$

for any optional laterally regular process Y. This relation extends trivially to process $Y\in\mathscr{R}$. By Proposition 1.1.3, this holds for bounded continuous process Y in particular. But for such a process we already know that $U(Y)=\mu(Y)$. Then, for all continuous bounded process Y:

$$U(Y)=\mu(Y)=E[\int_{\mathbb{R}^2_+} Y_t\,dA_t],$$

and it follows that A is the increasing process (such that $A_\infty=1$) associated to the randomized stopping point μ. Hence we have proved that $U(Y)=\mu(Y)$ $\forall\, Y\in\mathscr{R}$, and the proof is complete. □

To conclude, let us rely on Proposition 1.1.2 and 1.1.4 in order to give the main existence result of this paragraph.

Theorem 1.1. *Let Y be a bounded optional laterally regular process. Then, the optimal stopping problem on any fixed stopping line L admits a solution, that is to say,*

$$\forall\, L\in\mathscr{L},\ \exists\, T^*\in\mathscr{T}(L) \quad \textit{such that} \quad E(Y_{T^*})=\sup_{T\in\mathscr{T}(L)} E(Y_t).$$

The proof is an easy consequence of what precedes.

1.2. Optimal Stopping Problem on $\bar{\mathbb{R}}^2_+$

This paragraph deals with the optimal stopping problem for a general two-parameter process. An optimality criterium is given, and as in the classical theory [4, 7, 24], the notion of the Snell envelope is the basic tool. Under various hypotheses on the Snell envelope process, we prove the existence of optimal stopping points, namely, solutions are found amongst the maximal elements of the subset of stopping points which preserve the martingale property of the Snell envelope. The assumptions made on this process appear to be analogous to those of the classical theory, but we will unfortunately not be able to connect them to the pay-off process in this paragraph. This will be done for bi-Markov processes in the last chapter.

On a complete probability space $(\Omega, \mathscr{A}, \mathbb{P})$ endowed with a two-parameter filtration $\mathscr{F}=(\mathscr{F}_t;\, t\in\mathbb{R}^2_+)$ verifying the axioms F1, F2, F3 and F4, let us consider an optional non-negative process of class (D): $Y=(Y_t;\, t\in\bar{\mathbb{R}}^2_+)$ the pay-off process. The optimal stopping problem consists of finding a stopping point T^* such that $E(Y_{T^*})=\sup_{T\in\mathscr{T}} E(Y_T)$. Such a s.p. will be said to be *optimal.*

For any s.p. T, let $J(T)$ represent the best expected pay-off, defined by

$$J(T) = \operatorname*{ess\,sup}_{S \in \mathscr{T} : S \geqq T} E(Y_S \mid \mathscr{F}_{\mathscr{T}}).$$

Cairoli [14] has proved that the Mertens theorem [39] extends to two-parameter situations. In particular, there exists one strong supermartingale J, called the Snell envelope of Y, which agregates the family of r.v. $(J(T) : T \in \mathscr{T})$ i.e., $\forall T \in \mathscr{T} : J_T = J(T)$ a.s.

Notice that there could exist several processes which satisfy the preceding formula, and which are not necessarily undistinguishible [14]. In what follows we construct a process I such that $I_T = J_T$, $\forall T \in \mathscr{T}$. This construction is inspired by the classical theory [24, 49]. A similar one was obtained in [31] for the case of two-parameter Markov chains.

Proposition 1.2.1. *Let Y be an optional non-negative uniformly bounded process such that its trajectories are a.s. right lower semi-continuous functions converging to zero at infinity. Let I be the limiting process of the following increasing sequence of optional processes $(I^n; n \in \mathbb{N})$ defined by:*

$$I^0 = Y \quad \textit{and} \quad \forall n \in \mathbb{N} : I^{n+1} = \sup_{r \in \mathbb{D}} \{{}^o(I^n_{r+\cdot})\}$$

where ${}^o(I^n_{r+\cdot})$ denotes the optional projection of the process $I^n_{r+\cdot} = (I^n_{r+t}; t \in \mathbb{R}^2_+)$, and $\mathbb{D}$ is the set of dyadic numbers in $\mathbb{R}^2_+$.

Then I is a strong supermartingale such that

$$\forall T \in \mathscr{T} : I_T = J_T \quad \textit{a.s.}$$

Every optional process having such a property will be called the *Snell envelope of Y.*

Proof. This has been given in full detail in [36] and we only recall the main steps. We define an operator R on the set of all optional bounded processes as follows

$$\forall X \text{ optional bounded: } R(X)_t = \sup_{r \in \mathbb{D}} \{{}^o(X_{r+\cdot})_t\} \qquad \forall t \in \mathbb{R}^2_+.$$

Operator R is positive (i.e. $X \geqq Y \Rightarrow R(X) \geqq R(Y)$), and for any strong supermartingale X, $R(X)$ is a strong supermartingale such that $\forall T \in \mathscr{T} : R(X)_T = X_T$ a.s.

From the fact that $J \geqq Y$, we deduce that

$$\forall T \in \mathscr{T} : J_T \geqq I^n_T, \quad \forall n \in \mathbb{N}, \quad \text{and} \quad J_T \geqq I_T \quad \text{a.s.}$$

On the other hand, it can be proved by a direct computation that:

$$\forall T \in \mathscr{T} : E(Y_S \mid \mathscr{F}_T) \leqq I_T \quad \text{a.s.},$$

for any stopping points S and T such that $S \geqq T$, and $(S-T)$ is dyadic a.s. By using the hypotheses on Y, it is proved in [36] that this relation extends to any pair of stopping points. Then

$$\forall T \in \mathscr{T} : J_T \leqq I_T \quad \text{a.s.}$$

This ends the proof. □

The next result connects the regularity properties of the pay-off process Y with those of its Snell envelope J.

Proposition 1.2.2. *Let the pay-off Y be a non-negative optional process of class (D).*

i) *If for any T and any decreasing sequence $(T_n;\, n\in\mathbb{N})$ of stopping points converging to T,*

$$E(Y_T)\leqq \liminf_n E(Y_{T_n}),$$

then its Snell envelope J verifies

$$E(J_T)=\lim\uparrow E(J_{T_n}).$$

ii) *If Y is a.s. continuous on $\overline{\mathbb{R}}^2_+$ then its Snell envelope J verifies*

$$E(J_T)=\lim\downarrow E(J_{T_n}),$$

for any increasing sequence $(T_n;\, n\in\mathbb{N})$ of stopping points converging to T.

Proof. i) Let $(T_n;\, n\in\mathbb{N})$ be a decreasing sequence of s.p. with limit T: the sequence $(E(J_{T_n});\, n\in\mathbb{N})$ is increasing and majorized by $E(J_T)$. Suppose there exists some constant $\varepsilon>0$ such that

$$\sup_n E(J_{T_n})+2\varepsilon\leqq E(J_T).$$

By the definition of J it is possible to find a s.p. $S\geqq T$ such that

$$E(Y_S)+\varepsilon\geqq E(J_T).$$

Thus by setting $S_n=S\vee T_n$, $\forall_n$, we construct a decreasing sequence of s.p.'s whose limit is S. We get

$$E(Y_{S_n})\leqq E(J_{T_n})\leqq E(J_T)-2\varepsilon\leqq E(Y_S)-\varepsilon.$$

This implies that Y cannot satisfies the prescribed assumption. Therefore,

$$\sup_n E(J_{T_n})+2\varepsilon>E(J_T),\qquad \forall\varepsilon>0.$$

This proves i).

ii) Let $(T_n;\, n\in\mathbb{N})$ be an increasing sequence of s.p. with limit T: the sequence $(E(J_{T_n});\, n\in\mathbb{N})$ is decreasing, let L be the limit. Then $L\geqq E(J_T)$. For each $n\in\mathbb{N}$, one can find a s.p. S_n such that $S_n\geqq T_n$ and $E(J_{T_n})\leqq E(Y_{S_n})+1/n$. Set $U_n=S_n\vee T$: $U_n\in\mathscr{T}$ and $U_n\geqq T$.

By hypothesis, a.s. each trajectory of Y is a uniformly continuous function on $\overline{\mathbb{R}}^2_+$. Let d be a distance on $\overline{\mathbb{R}}^2_+$. By construction

$$d(S_n,\, U_n)\leqq d(T_n,\, T),\qquad \forall n \text{ and } \lim_n d(T_n,\, T)=0\quad \text{a.s.}$$

Therefore, the sequence $(Y_{S_n}-Y_{U_n};\, n\in\mathbb{N})$ converges a.s. to 0 and by uniform integrability this implies $\lim_n E(Y_{S_n}-Y_{U_n})=0$.

We deduce that $\lim_n [E(J_{T_n})-E(Y_{U_n})]=0$, and $\lim_n E(Y_{U_n})=L$.

But $T \leqq U_n$ implies $E(J_T) \geqq E(Y_{U_n})$, $\forall n$. Thus, $E(J_T) \geqq L$, and this provides the proof. □

Note that all these properties do not depend on the chosen representative for the Snell envelope.

In what follows, we consider a different approach to the optimal stopping problem. This consists of reducing the two-parameter problem to the somewhat classical distributed control problem of finding an optional increasing path passing by an optimal stopping point. In addition, this method gives a characterization of the Snell envelope generalizing those of [35, 36].

The main idea of this paragraph is resumed in the following result.

Proposition 1.2.3. *Let Y be a given two-parameter optional, non-negative process of class (D). Then,*

$$\sup_{T \in \mathcal{T}} E(Y_T) = \sup_{Z \in \mathcal{Z}} \{ \sup_{\tau \in \mathcal{T}^Z} E(Y_\tau^Z) \}.$$

Proof. For any o.i.p. Z and any $\mathcal{F}^z$-stopping time τ, Z_τ is a s.p. and, conversely, for any s.p. T, there exists an o.i.p. Z which passes by T (i.e. $T = Z_\tau$ a.s. where τ is a $\mathcal{F}^Z$-stopping time [52]). Then the set $\mathcal{T}$ can be identified with the set $\{(Z, \tau); Z \in \mathcal{Z}$ and $\tau \in \mathcal{T}^Z\}$, and that proves the proposition. □

The equality in Proposition 1.2.3 shows that the general problem can be split up into the two following problems. 1) Finding an optimal optional increasing path. 2) Finding an optimal stopping time on it.

This approach enables the characterization of the behaviour of the Snell envelope J on the set on which Y is strictly less than J. The definition of the debut of a random set along an o.i.p. will be used.

For every $\lambda \in]0, 1[$, set $H^\lambda = \{(\omega, t): Y_t(\omega) \geqq \lambda J_t(\omega)\}$, and denote by D_λ^Z the début of H^λ along the o.i.p. Z. Domain H^λ is optional; therefore D_λ^Z is a stopping point. Moreover, the process $J \mathbb{1}_{H^\lambda}$ is non-negative optional and of class (D). We denote by J^λ its Snell envelope. This process is usually called the reduite of J on the set H^λ. The following result extends classical properties of reduites.

Proposition 1.2.4. *For every stopping point T one has*

$$J_T^\lambda = J_T \quad a.s.$$

Proof. This has been borrowed from [24]. J is a strong supermartingale greater than the process $J \mathbb{1}_{H^\lambda}$, this is necessarily greater than its Snell envelope J^λ. Consequently, for any s.p. S, we get

$$J_S^\lambda \geqq \mathbb{1}_{\{S \in H^\lambda\}} J_S \geqq \mathbb{1}_{\{S \in H^\lambda\}} J_S^\lambda,$$

then

$$J_S^\lambda = J_S \quad \text{a.s. on the set } \{S \in H^\lambda\}.$$

Let I be the strong supermartingale $\lambda J + (1-\lambda) J^\lambda$. Obviously, $J_S \geqq I_S$ $\forall S \in \mathcal{T}$. To prove that $J_S \leqq I_S$ $\forall S \in \mathcal{T}$, it is sufficient to verify that $Y_S \leqq I_S$ $\forall S \in \mathcal{T}$.

On the set $\{S \in H^\lambda\}$, we have $J_S = J_S^\lambda$, then $Y_S \leqq I_S$. On the set $\{S \in H^\lambda\}^c$, we have $Y_S < \lambda J_S$, then $Y_S \leqq I_S$. This achieves the proof. □

From this we deduce the formula which characterizes the behaviour of the Snell envelope J on the domain H^λ. It extends a result given in [35, 36].

Proposition 1.2.5. *For any fixed $\lambda \in]0,1[$, the Snell envelope J of the process Y satisfies*

$$E(J_0) = \sup_{Z \in \mathscr{Z}} E(J_{D_\lambda^Z}),$$

where D_λ^Z is the début of the set $\{Y \geqq \lambda J\}$ along Z.

Proof. The equality is proved for process J^λ, it then holds for J itself by Proposition 1.2.4. For a given s.p. S, let $\mathscr{Z}^S$ denote the set of all o.i.p. passing a.s. by S (i.e. $\forall Z \in \mathscr{Z}^S$, $\exists \sigma \in \mathscr{T}^Z$ such that $S = Z_\sigma$). By definition of J^λ, we have

$$E(J_0^\lambda) = \sup_{S \in \mathscr{T}} E(J_S \mathbb{1}_{\{S \in H^\lambda\}}).$$

Let us prove the following:

$$\forall S \in \mathscr{T}, \quad \forall Z \in \mathscr{Z}^S : \exists T \in \mathscr{T}$$

such that $T \geqq D_\lambda^Z$ and $E(J_S \mathbb{1}_{\{S \in H^\lambda\}}) = E(J_T \mathbb{1}_{\{T \in H^\lambda\}})$.

For this purpose, set for any $S \in \mathscr{T}$ and $Z \in \mathscr{Z}^S$: $T = S$ on $\{S \geqq D_\lambda^Z\}$ and $T = \infty$ on the complementary set. T is a s.p., due to the fact that $\{S \geqq D_\lambda^Z\} \in \mathscr{F}_S$. It is easy to verify $\{S \in H^\lambda\} = \{T \in H^\lambda\} \subset \{S = T\}$.

Then, we obtain

$$E(J_S \mathbb{1}_{\{S \in H^\lambda\}}) = E(J_T \mathbb{1}_{\{T \in H^\lambda\}}).$$

From this formula we deduce the following equalities.

$$E(J_0) = \sup_{S \in \mathscr{T}} E(J_S \mathbb{1}_{\{S \in H^\lambda\}}) = \sup_{T \in \mathscr{T}} E(J_T \mathbb{1}_{\{T \in H^\lambda\}}) = \sup_{Z \in \mathscr{Z}} E(J_{D_\lambda^Z} \mathbb{1}_{\{D^Z \in H^\lambda\}}) = \sup_{Z \in \mathscr{Z}} E(J_{D_\lambda^Z}).$$

This settles the proof. □

Remark 1.2.1. Let L^λ denote the stopping line début of the set H^λ: $\forall Z \in \mathscr{Z}$: $D_\lambda^Z \geqq L^\lambda$. Then we get, a fortiori

$$E(J_0) = \sup_{T \in \mathscr{T}(L^\lambda)} E(J_T)$$

which is identical to the result obtained in [36].

Now let us come back to the original problem on $\overline{\mathbb{R}}_+^2$.

As for the classical theory we have the following optimality criterium.

Proposition 1.2.6. *A stopping point T is optimal if and only if the two following conditions are satisfied*

i) $E(J_T) = E(Y_T)$.

ii) $E(J_0) = E(J_T)$.

The proof is exactly the same as in [24] or [28], and therefore is omitted.

Définition 1.2.1. A stopping point T is called maximal if it is a maximal element of the subset $\{S \in \mathscr{T} : E(J_0) = E(J_S)\}$ i.e., if and only if,

ii) $E(J_0) = E(J_T)$.

ii) $\forall S \in \mathscr{T}$ such that $S \geqq T$ and $\mathbb{P}(\{S = T\}) < 1$: $E(J_T) > E(J_S)$.

The existence of a maximal s.p. follows from appropriate conditions on the Snell envelope in the following result.

Proposition 1.2.7. *If the Snell envelope J is left-continuous in expectation on stopping points, then there exist maximal stopping points.*

Proof. It is a simple application of Zorn's lemma: The set

$$\{T\in\mathcal{T}: E(J_T)=E(J_0)\}$$

is non-empty, and is inductive thanks to the hypothesis on J; therefore it admits maximal elements. □

Assuming further conditions on J we can prove that maximal s.p.'s are optimal.

Theorem 1.2. *If the pay-off process Y is such that its Snell envelope is completely regular and bounded, then there exist maximal stopping points, and any maximal stopping point is optimal for the stopping problem.*

Proof. The existence of maximal s.p.'s follows from Proposition 1.2.2. Let us prove that such an element T is indeed optimal.

By definition, T satisfies the following

$$E(J_0)=E(J_T),$$

and $\forall S\in\mathcal{T}$ such that $S\geqq T$ and $\mathbb{P}\,(S=T)<1$: $E(J_S)<E(J_0)$. To prove that T is actually optimal, we must verify that:

$$E(J_T)=E(Y_T).$$

For that purpose, let us consider the random set

$$H^\lambda=\{(\omega,t)\colon Y_t(\omega)\geqq\lambda J_t(\omega),\ t\geqq T(\omega)\},\qquad \text{for}\ \ \lambda\in]0,1[.$$

This set is optional. Let L^λ be the stopping line début of H^λ. From Remark 1.2.1, we get

$$\forall\lambda\in]0,1[:\ E(J_T)=\sup_{S\in\mathcal{T}(L^\lambda)} E(J_S)=\sup_{S\in\mathcal{T}} E(J_S\,\mathbb{1}_{\{S\in L^\lambda\}}).$$

Recall $\mathcal{T}(L^\lambda)$ denotes the set of s.p. S such that $[\![S]\!]\subset[\![L^\lambda]\!]$. This formula defines an optimal stopping problem on a stopping line which enters the framework of Sect. 1.1. Then, there exists a stopping point S_λ, such that

$$E(J_T)=E(J_{S_\lambda})\quad\text{and}\quad [\![S_\lambda]\!]\subset[\![L^\lambda]\!].$$

By definition, we have $S_\lambda\geqq T$. Because of the maximality of T, this implies $S_\lambda=T$ a.s., and therefore T belongs to the stopping line L^λ a.s. By the construction of H^λ and L^λ, this is possible only if T belongs to H^λ a.s. Then

$$\forall\lambda\in]0,1[:\ Y_T\geqq\lambda J_T\quad\text{a.s.}$$

It follows that $Y_T=J_T$ a.s., completing the proof. □

2. Bi-Markov Processes

Bi-Markov processes are particular two-parameter processes. The well-known bi-Brownian motion, particularly studied by Brossard and Chevalier [11] and Walsh [52], belongs to this class of processes. In this chapter we construct general bi-Markov processes and present various notions of the corresponding bi-potential theory. Then we study various types of supermartingales associated to a bi-Markov process. Finally new notions of reduite and of weak harmonicity for two-variable functions on a open set are proposed and studied in full detail.

2.1. Construction of bi-Markov processes

Roughly speaking, bi-Markov processes are defined as the tensor product of two classical – one parameter – Markov processes. In the sequel, super- or subscript i will take values 1 and 2.

Let $(\Omega^i, \mathcal{M}^i_\infty)$ be a measurable space endowed with a right-continuous filtration $\mathcal{M}^i=(\mathcal{M}^i_u; u\in\mathbb{R}_+)$, and let E^i be a locally compact metric space with Borel σ-field $\mathcal{E}^i$. Let $X^i=(X^i_u; u\in\mathbb{R}_+)$ be an E^i – valued random process on $(\Omega^i, \mathcal{M}^i_\infty)$ which is right-continuous and left-limited, and adapted to the filtration $\mathcal{M}^i$. The set of bounded Borel (resp. continuous, bounded and continuous, bounded and uniformly continuous) functions is denoted by $b(E^i)$ (resp. $C(E^i)$, $C_b(E^i)$, $C_u(E^i)$), and the set of probability measures on $(E^i, \mathcal{E}^i)$ by $M(E^i)$. Let $\mathbb{P}^i=(\mathbb{P}^i_x; x\in E^i)$ be a Borel kernel of probability measures on $(\Omega^i, \mathcal{M}^i_\infty)$. For each $\mu\in M(E^i)\mathbb{P}^i_\mu$ represents the probability measure $\mu\cdot\mathbb{P}=\int_{E^i}\mathbb{P}^i_x\,\mu(dx)$. Let us denote by $\mathcal{M}^{i\mu}=(\mathcal{M}^{i\mu}_u; u\in\mathbb{R}_+)$ the smallest filtration which contains $(\mathcal{M}^i_u; u\in\mathbb{R}_+)$, is right-continuous, and such that all the $\mathbb{P}^i_\mu$-negligible sets are in $\mathcal{M}^{i\mu}_0$. In the sequel, $E^i_x(U)$ (resp. $E^i_\mu(U)$) will represent the expectation $E^i(U)$ with respect to probability $\mathbb{P}^i_x$ (resp. $\mathbb{P}^i_\mu$) of the random variable U. Let us define a family $P^i=(P^i_u; u\in\mathbb{R}_+)$ of operators on $b(E^i)$ by setting

$$\forall u\in\mathbb{R}_+, \quad \forall f\in b(E^i): P^i_u f(x)=E^i_x(f(X^i_u)), \quad \forall x\in E^i.$$

In this paper, we suppose that the Markov property of the collection $(\Omega^i, \mathcal{M}^i_\infty, \mathcal{M}^i, X^i, \mathbb{P}^i)$ is specified by the following hypothesis, drawn from Meyer [42].

Hypothesis H1. $P^i=(P^i_u; u\in\mathbb{R}_+)$ is a semi-group such that for every function $f\in b(E^i)$ and every $u\in\mathbb{R}_+$, the process $P^i_u f(X^i_\cdot)=(P^i_u f(X^i_v); v\in\mathbb{R}_+)$ is the optional projection, with respect to the filtered space $(\Omega^i, \mathcal{M}^{i\mu}, \mathbb{P}^i_\mu)$, of the process $f(X^i_{u+\cdot})=(f(X^i_{u+v}); v\in\mathbb{R}_+)$, for each $\mu\in M(E^i)$.

Denote by $U^i=(U^i_p; p\in\mathbb{R}_+)$ the resolvent family of semi-group P^i, by $\mathcal{L}^i$ its generator, and by $\mathcal{D}(\mathcal{L}^i)$ the domain of this generator. For $p\in\mathbb{R}_+$, we also note $\mathcal{L}^i_p$ the operator $\mathcal{L}^i-pI$ (Identity). Then,

$$\forall f\in\mathcal{D}(\mathcal{L}^i): f=U^i_p g\Leftrightarrow g=-\mathcal{L}^i_p f.$$

Next, we define the bi-Markov process $X=(X_t;\, t\in\mathbb{R}^2_+)$, on the product measurable space $(\Omega=\Omega^1\times\Omega^2,\ \mathscr{A}^0=\mathscr{M}^1_\infty\otimes\mathscr{M}^2_\infty)$ endowed by the two-parameter filtration

$$\mathscr{F}^0=(\mathscr{F}^0_t=\mathscr{M}^1_{t_2}\otimes\mathscr{M}^2_{t_2};\ t=(t_1,t_2)\in\mathbb{R}^2_+)$$

and the family of probabilities $(\mathbb{P}_x=\mathbb{P}^1_{x_1}\otimes\mathbb{P}^2_{x_2};\ x=(x_1,x_2)\in E^1\times E^2)$, by the following.

$$\forall\, t=(t_1,t_2)\colon X_t=(X^1_{t_1}, X^2_{t_2}).$$

Process X takes its values in the product space $E=E^1\times E^2$ endowed with the σ-field $\mathscr{E}=\mathscr{E}^1\otimes\mathscr{E}^2$. Notice that process X enters in the classes of two-parameter Markov processes studied by Nualart and Sanz [48], Korezlioglu, Lefort and Mazziotto [27], Carnal [18] and Dozzi [21]. The set of bounded Borel (resp. continuous, bounded and continuous, bounded and uniformly continuous) functions on E is denoted by $b(E)$ (resp. $C(E)$, $C_b(E)$, $C_u(E)$), and the set of probability measures on $(E,\mathscr{E})$ by $M(E)$. For each $\mu\in M(E)$, $\mathbb{P}_\mu$ represents the probability $\mu\cdot\mathbb{P}=\int_E \mathbb{P}_x\,\mu(dx)$, and $\mathscr{F}^\mu=(\mathscr{F}^\mu_t;\ t\in\mathbb{R}^2_+)$ is the smallest two-parameter filtration which contains filtration $\mathscr{F}^0$, is right-continuous (Axiom F2), and such that all the $\mathbb{P}_\mu$-negligible sets of $(\Omega,\mathscr{A})$ belong to $\mathscr{F}^\mu_0$. In addition, $\mathscr{F}^\mu$ satisfies the conditional independence property of Axiom F4 with respect to $\mathbb{P}_\mu$. Similarly, we define the one-parameter filtration $\mathscr{F}^{\mu_1}=(\mathscr{F}^{\mu_1}_u;\ u\in\mathbb{R}_+)$ (resp. $\mathscr{F}^{\mu_2}=(\mathscr{F}^{\mu_2}_u;\ u\in\mathbb{R}_+)$) to be the smallest filtration which contains filtration $(\mathscr{M}^1_u\otimes\mathscr{M}^2_\infty;\ u\in\mathbb{R}_+)$ (resp. $(\mathscr{M}^1_\infty\otimes\mathscr{M}^2_u;\ u\in\mathbb{R}_+)$), is right-continuous and contains all the $\mathbb{P}_\mu$-negligible sets of $\mathscr{A}$. Finally let $\mathscr{F}=(\mathscr{F}_t;\ t\in\mathbb{R}^2_+)$, $\mathscr{F}^1=(\mathscr{F}^1_{t_1};\ t_1\in\mathbb{R}_+)$ and $\mathscr{F}^2=(\mathscr{F}^2_{t_2};\ t_2\in\mathbb{R}_+)$ be the filtrations defined by

$$\forall\, t=(t_1,t_2)\in\mathbb{R}^2_+\colon \mathscr{F}_t=\bigcap_{\mu\in M(E)}\mathscr{F}^\mu_t,\qquad \mathscr{F}^i_{t_i}=\bigcap_{\mu\in M(E)}\mathscr{F}^{\mu i}_{t_i}\qquad i=1,2.$$

We define a two-parameter semi-group on $b(E)$ $P=(P_t;\ t\in\mathbb{R}^2_+)$ by setting

$$\forall\, t=(t_1,t_2)\colon P_t=P^1_{t_1}\otimes P^2_{t_2}.$$

The associated resolvent is the two-parameter family of operators on $b(E)$ defined by

$$\forall\, p=(p_1,p_2)\colon U_p=U^1_{p_1}\otimes U^2_{p_2}.$$

Operator $P^1_{t_1}$, $U^1_{p_1}$ or $P^2_{t_2}$, $U^2_{p_2}$ will be considered as operating on $b(E)$ as well as on spaces $b(E^1)$ or $b(E^2)$ with no risk of ambiguity. Similarly, generators $\mathscr{L}^1$ and $\mathscr{L}^2$ will be considered on the domain $\mathscr{D}(\mathscr{L}^1,\mathscr{L}^2)$ of functions $f\in C(E)$ such that functions $\mathscr{L}^1 f$ and $\mathscr{L}^2 f$ are well defined and belong to $C_u(E)$. It may be noted that the operators $P^1_{t_1}$, $P^2_{t_2}$, $U^1_{p_1}$ and $U^2_{p_2}$ commute each with other $\forall\, t_1, t_2, p_1, p_2$.

The following result expresses the Markov property of process X. It is a consequence of Hypothesis H1.

Lemma 2.1. *For each $f\in b(E)$ and each $s\in\mathbb{R}^2_+$, the* 1*-optional,* 2*-optional and optional projection of the process $f(X_{s+\cdot})=(f(X_{s+t});\ t\in\mathbb{R}^2_+)$, with respect to probability $\mathbb{P}_\mu$ and filtration $\mathscr{F}^\mu$, are given by the following formulas*

$$^{o_1}(f(X_{s+\cdot}))_t = P^1_{s_1} f(X_{(t_1, t_2+s_2)})$$
$$^{o_2}(f(X_{s+\cdot}))_t = P^2_{s_2} f(X_{(t_1+s_1, t_2)})$$
$$^{o}(f(X_{s+\cdot}))_t = P_s f(X_t),$$

independently of the considered probability $\mu \in M(E)$.

Proof. In case f has a product form (i.e. $f = f^1 \otimes f^2$ with $f^i \in b(E^i)$, $i = 1, 2$) the first two formulas are easy consequences of Hypothesis H1. By a monotone class argument these formulas extend to any $f \in b(E)$. The third formula follows from the definition of an optional projection for two-parameter processes given by Bakry [1]. □

New let us study a strong Markov property for process X.

We denote by $\mathcal{T}$ (resp. $\mathcal{T}^0$, $\mathcal{T}^\mu$ for $\mu \in M(E)$) the set of stopping points on $\overline{\mathbb{R}}^2_+$ with respect to the filtration $\mathcal{F}$ (resp. $\mathcal{F}^0$, $\mathcal{F}^\mu$), and by $\mathcal{T}^i$ (resp. $\mathcal{T}^{0i}$, $\mathcal{T}^{\mu i}$) the set of i-stopping points on $\overline{\mathbb{R}}^2_+$ with respect to the filtration $\mathcal{F}^i$ (resp. $\mathcal{F}^{0i}$, $\mathcal{F}^{\mu i}$), for $i=1$ or 2. It can be easily verified that, for instance, a $\mathcal{F}^{01}$-1-stopping point (i.e. with respect to $\mathcal{F}^{01}$) $T = (T_1, T_2)$ is a r.v. on $(\Omega, \mathcal{A})$ such that:

$$\forall w_2 \in \Omega_2, \text{ fixed: } T_1(\cdot, w_2) \text{ is a } \mathcal{M}^1\text{-stopping time, and}$$
$$T_2(\cdot, w_2) \text{ is } \mathcal{M}^1_{T_1(\cdot, w_2)}\text{-measurable.}$$

A symmetric property holds for $\mathcal{F}^{02}$-2-stopping points, and both are verified for stopping points of $\mathcal{T}^0$.

Proposition 2.1.1. *Process X has the strong Markov property in the following sense.*

$$\forall \mu \in M(E), \quad \forall T \in \mathcal{T}^\mu, \quad \forall S \text{ a } \mathcal{F}^\mu_T\text{-measurable r.v.}, \quad \forall f \in b(E):$$
$$E_\mu(f(X_{T+S}) | \mathcal{F}^\mu_T) = P_S f(X_T) \mathbb{P}_\mu\text{-a.s.}$$

Proof. It is similar to that in the classical theory: see Meyer [41]. First notice that, for given $f \in b(E)$ and $s \in \mathbb{R}^2_+$, the process $M^s = \{M^s_t; t \leqq s)$ defined by.

$$\forall t \leqq s = M^s_t = {}^o(f(X_s))_t = P_{s-t} f(X_t)$$

is a bounded martingale which is right-continuous and left-limited due to the results of Bakry [2] and Millet-Sucheston [46]. Such a two-parameter martingale verifies the optinal sampling theorem with stopping points by Kurtz [29]. Then we achieve the proof as in the classical theory. □

To conclude this paragraph, we study additional regularity properties for process X.

Proposition 2.1.2. *If in addition to Hypothesis $H1$, the processes X^1 and X^2 are quasi-left-continuous (see [8]), the process X satisfies the following.*

Let us give a sequence of random variables $(T^n; n \in \mathbb{N})$ on $\mathbb{R}^2_+ \cup \{\infty\}$ converging to T and let $\mu \in M(E)$. If one of the three following conditions is fullfilled

i) $\forall n \in \mathbb{N}$: $T^n \in \mathcal{T}^{\mu^1}$, $T_2^n \geqq T_2$, *and sequence* $(T_1^n; n \in \mathbb{N})$ *is non-decreasing;*

ii) $\forall n \in \mathbb{N}$: $T^n \in \mathcal{T}^{\mu^2}$, $T_1^n \geqq T_1$, *and sequence* $(T_2^n; n \in \mathbb{N})$ *is non-decreasing;*

iii) $\forall n \in \mathbb{N}$: $T^n \in \mathcal{T}^{\mu}$, *and sequence* $(T^n; n \in \mathbb{N})$ *is non-decreasing on* $\overline{\mathbb{R}}_+^2$;

then the sequence $(X_{T^n}; n \in \mathbb{N})$ *converges to* X_T, $\mathbb{P}_\mu$-a.s.

Proof. Let us prove i). For a.s. all $w_2 \in \Omega_2$, the collection $(T_1^n(\cdot, w_2); n \in \mathbb{N})$ is a non-decreasing sequence of $\mathcal{M}^{1\mu}$-stopping times converging to $T_1(\cdot, w_2)$. Then thanks to the quasi-left-continuity of X^1, the sequence $(X^1_{T_1^n(\cdot, w_2)}; n \in \mathbb{N})$ converges to $X^1_{T_1(\cdot, w_2)}$, $\mathbb{P}^1_{x_1}$-a.s. $\forall x_1 \in E^1$. This implies that $(X^1_{T_1^n}; n \in \mathbb{N})$ converges to $X^1_{T_1}$ $\mathbb{P}_\mu$-a.s., $\forall \mu \in M(E)$. The convergence of the sequence $(X^2_{T_2^n}; n \in \mathbb{N})$ to $X^2_{T_2}$ proceeds directly from the right-continuity of X^2. The proofs of ii) and iii) are similar. □

We conclude by few definitions.

A right-continuous two-parameter process which verifies the conclusion of Proposition 2.1.1, will be called *quasi-left continuous.*

Bi-Markov process X is said to be *normal* if processes X^1 and X^2 are normal [8]. Then the 0–1 law of Blumenthal also holds for X i.e.,

$$\forall A \in \mathcal{F}_0: \mathbb{P}_x(A) = 0 \quad \text{or} \quad 1, \qquad \forall x \in E.$$

A bi-Markov process X is said to be *Fellerian* if the semi-groups P^1 and P^2 are both Feller. It has been proved in [12] that for each $t \in \mathbb{R}_+^2$, the operator P_t maps $C_b(E)$ into $C_b(E)$. The process X is considered as *strongly Feller* if the processes X^1 and X^2 are strongly Feller (i.e. for $i = 1$, 2 and for any $p \in \mathbb{R}_+^2$, operator $U^i_{p_i}$ maps the set of Borel function with compact support on E^i into $C_b(E^i)$).

2.2. *Towards a bi-Potential Theory*

Given two classical semi-groups on two spaces E^1 and E^2 different classes of functions on the product space $E = E^1 \times E^2$ can be defined separately, according to their properties on each space E^1 or E^2. Such a study has been done by Cairoli [12, 13], dealing with the one-parameter semi-group constructed as the tensor product of two classical semi-groups. In this paragraph, we first recall definitions and results of [12] in the frame of two-parameter semi-groups. Later, we study the two-parameter processes associated to the functions mentionned above. Dynkin formulas which generalize the classical one, and these of Lawler and Vanderbei [31] and Vanderbei [51] for Markov chains, are obtained for various classes of supermartingales. Trajectorial regularity properties of these supermartingales are given.

Definition 2.2.1. Let f be a positive function on the product space $E = E^1 \times E^2$, and let $p \in \mathbb{R}_+$. For $i = 1, 2$, f is called *$p-i$-supermedian* (resp. *$p-i$-excessive*) on E, if the function on E^i defined by: $x^i \to f(x^1, x^2)$, $\forall x^i \in E^i$, is p-supermedian (resp. p-excessive) when the other variable is fixed.

Let f be a positive function on E, and let $p=(p_1,p_2)\in\mathbb{R}_+^2$. f is is called *p-bisupermedian* (resp. *p-biexcessive*) if f is both p_1-1-supermedian (resp. p_1-1-excessive) and p_2-2-supermedian (resp. p_2-2-excessive).

We refer to [41] for the definitions of the classical potential theory. It is proved in [12] that any positive function on E which is both p_1-1-excessive and p_2-2-excessive is measurable on E and is lower semi-continuous when processes X^1 and X^2 are strongly Fellerian.

For $p=(p_1,p_2)$ and $t=(t_1,t_2)\in\mathbb{R}_+^2$, denote by $p\cdot t$ the scalar product $p_1t_1+p_2t_2$. It should be noted that if function f is p-bisupermedian then:

$$\forall t\in\mathbb{R}_+^2: e^{-p\cdot t}P_t f\leqq f,$$

and if, moreover, function f is p-biexcessive, then:

$\lim_n e^{-p\cdot t(n)}P_{t(n)}f=f$, for any sequence $(t(n);n\in\mathbb{N})$ decreasing to zero.

A function f on E is said to be *p-biharmonic* iff:

$$\forall t\in\mathbb{R}_+^2: e^{-p\cdot t}P_t f=f.$$

For any function $g\in b(E)$ (not necessarily positive), the function $f=U_p g$ is p-biexcessive (hence positive) iff functions $U_{p_1}^1 g$ and $U_{p_2}^2 g$ are positive. Such a function is called *p-potential* in the sequel. More generally, a function $f\in\mathscr{D}(\mathscr{L}^1,\mathscr{L}^2)$ such that $\mathscr{L}_{p_1}^1 f\leqq 0$ and $\mathscr{L}_{p_2}^2 f\leqq 0$ is p-biexcessive.

The following result proves that any p-biexcessive function can be approximated by p-potentials.

Proposition 2.2.1. *For $p=(p_1,p_2)$ such that $p_1>0$ and $p_2>0$, any p-biexcessive function f is the limit of a non-decreasing sequence of p-potentials $(U_p g^n;n\in\mathbb{N})$.*

Proof. Let f be a bounded function. To any $t=(t_1,t_2)$ not contained by the coordinate axes we associate a function g^t as follows.

$$g^t=(t_1t_2)^{-1}(f+e^{-p\cdot t}P_t f-e^{-p_1t_1}P_{t_1}^1 f-e^{-p_2t_2}P_{t_2}^2 f).$$

The following identity is easily proved.

$$U_p g^t=\int_0^\infty\int_0^\infty e^{-p\cdot s}P_s g^t\,ds=(t_1t_2)^{-1}\int_0^{t_1}\int_0^{t_2} e^{-p\cdot s}P_s f\,ds.$$

Moreover, if the function f is p-biexcessive, then the right-hand side of the previous expression increases to f when (t_1t_2) decreases to zero.. This gives the proof. □

Note that in this proof the functions g^t's associated to a p-biexcessive function f are not necessarily non-negative, as in the classical theory. The subclass of such functions f has been considered in [13], and a Riesz type decomposition obtained.

Let us now study the two-parameter processes associated to these functions and the bi-Markov X. To any function f on E and $p\in\mathbb{R}_+^2$ we associate a process J as follows

$$\forall t\in\mathbb{R}_+^2: J_t=e^{-p\cdot t}f(X_t).$$

It is easy to verify that f is p-supermedian if and only if the associated process J is supermartingale with respect to any probability $\mathbb{P}_x$, $x \in E$.

We shall now study the processes associated to p-biexcessive functions by beginning with the case of p-potentials.

Proposition 2.2.2. *For $p \in \mathbb{R}_+^2$, let J be the process associated to the p-potential $f = U_p g$ with $g \in b(E)$ given. Then, for any probability $\mathbb{P}_\mu$, $\mu \in M(E)$, J is undistinguishable from the optional projection (with respect to $\mathscr{F}^\mu$ and $\mathbb{P}_\mu$) of the process C defined by*

$$\forall t \in \mathbb{R}_+^2: \; C_t = \int_{t_1}^{\infty} \int_{t_2}^{\infty} e^{-p \cdot s} g(X_s)\, ds.$$

Moreover J is a right-continuous bounded supermartingale, with respect to any probability $\mathbb{P}_\mu$, $\mu \in M(E)$.

Proof. Coming back to the definition of optional projection [1], we verify directly that:

$$\forall t \in \mathbb{R}_+^2: \; ({}^oC)_t = \int_{t_1}^{\infty} \int_{t_2}^{\infty} e^{-p \cdot s} \, {}^o(g(X_s))_t \, ds.$$

Then, using Lemma 1.1,

$${}^o(g(X_s))_t = P_{s-t}\, g(X_t) \; \forall s \geqq t,$$

and this leads to the formula: $({}^oC)_t = e^{-p \cdot t} U_p g(X_t) = J_t$. Let $B_t = \int_0^{t_1} \int_0^{t_2} e^{-p \cdot s} g(X_s)\, ds$, $\forall t$. Using the definition of optional projections [1], we get:

$${}^oC = {}^o(B_{\infty\infty} + B - B_{t_1\infty} - B_{\infty t_2}) = {}^oB_{\infty\infty} + B - {}^{o_2}B_{t_1\infty} - {}^{o_1}B_{\infty t_2}.$$

The right-continuity of almost all trajectories of J, for a given $\mu \in M(E)$, follows from results of Bakry [2] because B is continuous. □

The first consequence of this result is a Dynkin-type formula. Let T be any stopping point. Using properties of an optional projection [1], we obtain

$$E_x(e^{-p \cdot T} f(X_T)) = E_x\left(\int_{T_1}^{\infty} \int_{T_2}^{\infty} e^{-p \cdot s} g(X_s)\, ds_1\, ds_2 \right).$$

The second consequence is a decomposition of process J which is analogous, in some sense, to the Doob-Meyer decomposition of supermartingales [20]. Let define the process m by:

$$\forall t \in \mathbb{R}_+^2: \; J_t = m_t + \int_0^{t_1} \int_0^{t_2} e^{-p \cdot s} g(X_s)\, ds_1\, ds_2.$$

It can be verified by straightforward computation that m is a weak martingale (see [44] for this definition).

It can be noticed that enters in these results the fact that $f = U_p g$, but not the fact that f was a p-potential (i.e. $U_p^1 g \geqq 0$ and $U_p^2 g \geqq 0$).

Now let us study the processes associated to p-biexcessive functions. In order to simplify things, we only deal with the case of p not belonging to the coordinate axis in $\mathbb{R}_+^2$.

Proposition 2.2.3. *Let $p=(p_1, p_2)$ be such that $p_1>0$ and $p_2>0$. The process J associated to a bounded p-biexcessive function f is a strong supermartingale, and almost all its trajectories are lower semi-continuous functions for the right topology on $\mathbb{R}_+^2$, with respect to any fixed probability $\mathbb{P}_\mu$, $\mu\in M(E)$.*

Proof. It follows from Proposition 2.2.1 and Proposition 2.2.2 that J is the limit of an increasing sequence of right-continuous supermartingales. Its trajectories are then lower semicontinuous for the right topology on $\mathbb{R}_+^2$. Moreover, the supermartingale property has been verified on stopping points in [52]. □

Remark 2.2.1. In case X^1 and X^2 are both strong Feller processes every p-biexcessive function is lower semi-continuous, [12]. Then Proposition 2.2.3 can be proved directly, as in [37].

For the sake of simplicity we shall only consider points p with equal positive coordinates. Moreover, for any given $p\in\mathbb{R}_+$, we shall make no distinction between the number p and the vector (p, p), and we shall also note $p\cdot t$ for $p(t_1+t_2)$.

By studying the restrictions of a two-parameter supermartingale to optional increasing paths we obtain a second type of Dynkin formula.

Let J be a two-parameter process. For any optional increasing path $Z=(Z_u; u\in\mathbb{R}_+)$, the restriction of J to the o.i.p. Z is the one-parameter process J^Z, defined by $J^Z=(J_u^Z=J_{Z_u}; u\in\mathbb{R}_+)$. This process is $\mathscr{F}^Z$-optional if J is itself $\mathscr{F}$-optional, and is a strong supermartingale with respect to $\mathscr{F}^Z$ if J is a strong supermartingale.

Remark 2.2.2. Let $Z=(Z_u; u\in\mathbb{R}_+)$ be a given optional increasing path. As a consequence of Proposition 2.2.2, the process J^Z corresponding to a p-potential $f=U_p g$ with $g\in b(E)$ is right-continuous. But in addition to Proposition 2.2.3, we can see that the process J^Z corresponding to a p-biexcessive function f is undistinguishable from a right-continuous (one-parameter) process. This proceeds from the classical result (see [20], VI-18) which says that the limit of an increasing sequence of one-parameter right-continuous supermartingales is also a.s. right-continuous, and from Proposition 2.2.1.

In case the strong supermartingale J is associated to a p-biexcessive function f of $\mathscr{D}(\mathscr{L}^1, \mathscr{L}^2)$, we obtain the following Dynkin formula.

Proposition 2.2.4. *Let $Z=(Z_u; u\in\mathbb{R}_+)$ be a given optional increasing path and let f be a p-biexcessive function of $\mathscr{D}(\mathscr{L}^1, \mathscr{L}^2)$ for $p>0$. There exist two one-parameter $\mathscr{F}^Z$-adapted processes, λ^{1Z}, and λ^{2Z}, non-vanishing simultaneously and taking their values in $[0, 1]$, such that for every pair of ordered $\mathscr{F}^Z$-stopping times, $\sigma\leqq\tau$, one has $\forall x\in E$:*

$$E_x(e^{-p\tau}f(X_\tau^Z)-e^{-p\sigma}f(X_\sigma^Z)\,|\,\mathscr{F}_\sigma^Z)$$
$$=E_x\left(\int_\sigma^\tau(\mathscr{L}_p^1 f(X_u^Z)\,\lambda_u^{1Z}+\mathscr{L}_p^2 f(X_u^Z)\,\lambda_u^{2Z})e^{-pu}\,du\,|\,\mathscr{F}_\sigma^Z\right).$$

Proof. If the o.i.p. $Z=(Z_u; u\in\mathbb{R}_+)$ is a tactic of order m, say $(T_n; n\in\mathbb{N})$, the formula can be computed step by step, i.e. between each pair of successive points T_n, T_{n+1} by means of the classical Dynkin formula. We obtain the following:

$$E_x(e^{-p(n+1)2^{-m}}f(X^Z_{(n+1)2^{-m}})-e^{-pn2^{-m}}f(X^Z_{n2^{-m}})|\mathscr{F}^Z_{n2^{-m}})$$
$$=E_x\left(\int_{n2^{-m}}^{(n+1)2^{-m}}(\mathscr{L}^1_p f(X^Z_u)\mathbb{1}_{\{Z^2_{(n+1)2^{-m}}=Z^2_{n2^{-m}}\}}\right.$$
$$\left.+\mathscr{L}^2_p f(X^Z_u)\mathbb{1}_{\{Z^1_{(n+1)2^{-m}}=Z^1_{n2^{-m}}\}})e^{-pu}du|\mathscr{F}^Z_{n2^{-m}}\right).$$

Therefore, we define processes λ^{1Z} and λ^{2Z} by the following formula:

$$\lambda^{1Z}_u=1=1-\lambda^{2Z}_u \quad \text{on} \quad \{Z^2_{(k+1)2^{-n}}=Z^2_{k2^{-n}}\},$$

and

$$\lambda^{2Z}_u=1=1-\lambda^{1Z}_u \quad \text{on} \quad \{Z^1_{(k+1)2^{-n}}=Z^1_{k2^{-n}}\},$$

for $k2^{-n}\leqq u\leqq(k+1)2^{-n}$.

Then the formula is extended to any stopping times, as stated in the proposition.

Now let us consider a general o.i.p. Z. It can be approximated by a sequence of tactics of increasing orders $(Z^n; n\in\mathbb{N})$, such that the paths $((Z^n_u; u\in\mathbb{R}_+); n\in\mathbb{N})$ converge a.s. uniformly on any finite interval to the path $(Z_u; u\in\mathbb{R}_+)$. For each tactic Z^n we can write the preceding Dynkin formula with processes λ^{1Z^n} and λ^{2Z^n}. By continuity, processes $(\mathscr{L}^1_p(X^{Z^n}); n\in\mathbb{N})$ and $(\mathscr{L}^2_p(X^{Z^n}); n\in\mathbb{N})$ converge to processes $\mathscr{L}^1_p(X^Z)$ and $\mathscr{L}^2_p(X^Z)$ respectively. We now need to verify that the sequence of processes $(\lambda^{1Z^n}; n\in\mathbb{N})$ and $(\lambda^{2Z^n}; n\in\mathbb{N})$ converge. For that purpose we modify a method developed in [15] to define stochastic integration on increasing paths. Namely, we remark that processes λ^{1Z^n} and λ^{2Z^n} can be associated to Radon-Nikodym derivates of measures on $\mathbb{R}_+$, with respect to the Lebesgue measure. For arbitrary $u, v\in\mathbb{R}_+$ such that $u\leqq v$ we define the quantity $\Lambda^{1Z^n}([u,v])$ (resp. $\Lambda^{2Z^n}([u,v])$ to be the Lebesgue measure on $\mathbb{R}^2_+$ of the domain determined by Z^n, the vertical lines of abscissas u and v, and the horizontal line of ordinate -1 (resp. determined by Z^n, the horizontal lines of ordinates u and v, and the vertical line of abscissa -1). It is clear that Λ^{1Z^n} and Λ^{2Z^n} are random measures on $\mathbb{R}_+$ absolutely continuous with respect to the Lebesgue measure. Let $\tilde{\lambda}^{1Z^n}$ and $\tilde{\lambda}^{2Z^n}$ be their Radon-Nikodym derivates. It is a matter of verification to see that the processes λ^{1Z^n} and λ^{2Z^n} previously defined coincide with the processes $(\tilde{\lambda}^{1Z^n}_u/1+Z^{n,2}_u; u\in\mathbb{R}_+)$ and $(\tilde{\lambda}^{2Z^n}_u/1+Z^{n,1}_u; u\in\mathbb{R}_+)$. Moreover, the convergence of $(Z^n; n\in\mathbb{N})$ to Z implies that the sequences of measures $(\Lambda^{1Z^n}; n\in\mathbb{N})$ and $(\Lambda^{2Z^n}; n\in\mathbb{N})$ converge weakly to measures Λ^{1Z} and Λ^{2Z} similarly constructed. This allows us to define λ^{1Z} and λ^{2Z}, and the Dynkin formula for the o.i.p. Z follows by arguments of weak convergence. This achieves the proof. □

2.3. *Weak Harmonic Function and Reduite*

The bi-harmonic functions are well known; their connections with one-parameter or two-parameter processes have been widely studied in [11, 52]. Another notion has been introduced in [22, 51]. The definition we propose here is different in its being motivated by the optimal stopping problem; it is analogous to the notion considered in [34] dealing with the optimal stopping of several Markov chains.

Given a subset $A \subset E$ and an o.i.p. Z, the début along Z of the optional random set $\{(\omega, t): X_t(\omega) \in A\}$ is called the *entrance point of* X *in* A *along* Z and denoted by D_A^Z. *The exit point of* X *from* A *along* Z is defined by $S_A^Z = D_{A^c}^Z$, where A^c is the complement of A.

First, we define a harmonic operator similar to those of the classical theory. For $A \subset E$ and $p \in \mathbb{R}_+$, let H_A^p be the operator defined from $b(E)$ in the set of all bounded functions on E, by

$$\forall f \in b(E), \qquad \forall x \in E: H_A^p f(x) = \sup_{Z \in \mathscr{Z}} E_x(e^{-p \cdot D_A^Z} f(X_{D_A^Z})).$$

It may be noticed that $H_A^p f$ has no reason to be measurable. Although H_A^p is non-linear, it verifies several properties of classical harmonic operators.

Proposition 2.3.1. *Operator* H_A^p *satisfies the following;*

i) *If* A *is closed, then* $H_A^p(\mathbb{1}_A f) = H_A^p(f)$

ii) *If* $f \geqq g$, *then* $H_A^p f \geqq H_A^p g$

iii) $\forall x \in A$: $H_A^p f(x) = f(x)$

iv) *If* f *is p-biexcessive; then*

$$\forall x \in E: H_A^p f(x) \geqq \sup_{T \in \mathscr{T}} E_x(e^{-p \cdot T} \mathbb{1}_{\{X_T \in A\}} f(X_T))$$

Proof. If A is closed, then $X_{D_A^Z} \in A$, $\forall Z \in \mathscr{Z}$: this implies i). ii) is obvious. If $x \in A$, then $D_A^Z = 0$. This proves iii). Let Z be an o.i.p., denote by τ the $\mathscr{F}^Z$-stopping time such that $Z_\tau = D_A^Z$, and let σ be any $\mathscr{F}^Z$-stopping time. Then the following inequalities hold:

$$\begin{aligned} E_x(e^{-p \cdot Z_\sigma}(\mathbb{1}_A f)(X_\sigma^Z)) &\leqq E_x(e^{-p \cdot Z_{\sigma \vee \tau}}(\mathbb{1}_A f)(X_{\sigma \vee \tau}^Z)) \\ &\leqq E_x(e^{-p \cdot Z_{\sigma \vee \tau}} f(X_{\sigma \vee \tau}^Z)) \\ &\leqq E_x(e^{-p \cdot Z_\tau} f(X_\tau^Z)). \end{aligned}$$

We deduce from this:

$$\sup_{Z \in \mathscr{Z}} \sup_{\sigma \in \mathscr{T}^Z} E_x(e^{-p \cdot Z_\sigma}(\mathbb{1}_A f)(X_{Z_\sigma})) \leqq H_A^p f(x), \qquad \forall x \in E.$$

Using the fact that for any stopping T there exists $Z \in \mathscr{Z}$ and $\sigma \in \mathscr{T}^Z$ such that $T = Z_\sigma$ a.s. [52], we deduce iv). □

The last assertion of Proposition 2.3.1 suggests that harmonic operators are connected with optimal stopping of functions of bi-Markov processes.

The following definition of weak harmonicity extends those of [35].

Definition 2.3.1. A function f on E will be said to be p-weakly harmonic on an open subset $A \subset E$, if and only if

$$\forall x \in A: f(x) = H^p_{A^c} f(x) \text{ i.e. } f(x) = \sup_{Z \in \mathscr{Z}} E_x(e^{-p \cdot S^Z_A} f(X_{S^Z_A}))$$

where S^Z_A denotes the exit point of X from A along Z.

If a p-biexcessive function f is p-weakly harmonic on A, then f is p-weakly harmonic on any subset B contained in A. Moreover

$$\forall B \subset A: H^p_A H^p_B f = H^p_B H^p_A f = H^p_B f.$$

The set of p-weakly harmonic functions on a given subset contains the set of p-harmonic functions [35].

If a function f is sufficiently smooth, then weak harmonicity on a given subset can be expressed locally. The following proposition is proved by using various results of Sect. 2.2.

Proposition 2.3.2. *Suppose the bi-Markov process X is normal and quasi-left-continuous. Let f be a function in $\mathscr{D}(\mathscr{L}^1, \mathscr{L}^2)$, p a positive number, and let A be an open set, then f is both p-biexcessive and p-weakly harmonic on A if and only if*

$$\mathscr{L}^1_p f \leqq 0 \text{ and } \mathscr{L}^2_p f \leqq 0 \text{ on } E, \text{ and } \max(\mathscr{L}^1_p f, \mathscr{L}^2_p f) = 0 \text{ on } A.$$

Proof. Let us prove the necessary condition; suppose f is p-biexcessive and p-weakly harmonic on an open set A. For $x \in A$ fixed, there exists an open rectangle $B = B^1 \times B^2$ containing x and contained in A. Then f is p-weakly harmonic on B. It is easy to see that the family of exit point $(S^Z_B; Z \in \mathscr{Z})$ forms a stopping line L, which could also be defined by:

$$L = \{t = (t_1, t_2): t_i = \inf\{u: X^i \notin B^i\}, t_j \leqq \inf\{u: X^j \notin B^j\}; i \neq j \in \{0, 1\}\}.$$

According to Proposition 2.1.2 and Theorem 1.1, the optimal stopping problem on L associated to f and X admits a solution T. Let Z be an o.i.p. passing by $T: Z_\tau = T = S^Z_B$. Using Proposition 2.2.4 and Definition 2.3.1 we get

$$f(x) = E_x(e^{-p \cdot S^Z_B} f(X_{S^Z_B})) = f(x) + E_x\left(\int_0^\tau (\mathscr{L}^1_p f(X_u) \lambda^{1Z}_u + \mathscr{L}^2_p f(X_u) \lambda^{2Z}_u) e^{-pu} du\right)$$

and necessarily:

$$\int_0^\tau (\mathscr{L}^1_p f(X^Z_u) \lambda^{1Z}_u + \mathscr{L}^2_p f(X^Z_u) \lambda^{2Z}_u) du = 0 \quad \text{a.s.}$$

X^1 and X^2 being Normal and B^1, B^2 being open, we have $\forall x \in B: \mathbb{P}_x(\{0 \in L\}) = 0$. This implies that $\tau > 0$ a.s. Now let us work with a fixed $\omega \in \Omega$ chosen in the set of $\mathbb{P}_x$-probability 1 where the above expression is strictly negative. Suppose $\mathscr{L}^1_p f(x) < 0$ and $\mathscr{L}^2_p f(x) < 0$. Then the right-continuity of process X^Z and the continuity of the functions $\mathscr{L}^1_p f$, $\mathscr{L}^2_p f$ imply there exists a time σ such that $0 < \sigma \leqq \tau$ and

$$\mathscr{L}^1_p f(X^Z_u) < 0 \quad \text{and} \quad \mathscr{L}^2_p f(X^Z_u) < 0 \qquad \forall u \leqq \sigma.$$

In accordance with what precedes this is possible only if

$$\lambda^{1Z}_u = \lambda^{2Z}_u = 0 \qquad \text{for } u \leqq \sigma.$$

This contradicts the result of Proposition 2.2.4 which says that λ^{1Z} and λ^{2Z} cannot be simultaneously null. Thus, we have proved that either $\mathscr{L}_p^1 f(x)$ or $\mathscr{L}_p^2 f(x)$ must be equal to zero.

Conversely, let us suppose f satisfies the system of the proposition. Obviously f is p-biexcessive, and it remains to be proved that f is p-weakly harmonic on A. For $x \in A$ and $\varepsilon > 0$ given, we can construct an o.i.p. Z such that:

$$f(x) - E_x(e^{-p \cdot S_A^Z} f(X_{S_A^Z})) \leqq \varepsilon.$$

For the purpose, consider the following open sets:

$$B = \{y : |\mathscr{L}_p^1 f(y)| < \varepsilon\} \quad \text{and} \quad C = \{y : |\mathscr{L}_p^2 f(y)| < \varepsilon\}.$$

Obviously $B \cup C \supset A$; suppose that $x = (x^1, x^2) \in B$. Let us construct Z as follows. Let $T^1 = (T_1^1, T_2^1)$ be defined by

$$T_1^1 = \inf\{u : (X_u^1, x^2) \in B^c\} \quad \text{and} \quad T_2^1 = 0,$$

let $T^2 = (T_1^2, T_2^2)$ be defined by

$$T_1^2 = T_1^1 \quad \text{and} \quad T_2^2 = \inf\{u : (X_{T_1^1}^1, X_u^2) \in C^c\}.$$

It can easily be verified that T^1 and T^2 are s.p.'s. By iterating the foregoing procedure, we construct an increasing sequence $(T^n; n \in \mathbb{N})$ of s.p.'s, which induces an o.i.p. Z, as in Paragraph 2.2.

Everything has been done to ensure the following inequality

$$\left| E_x \left(\int_0^\infty (\mathscr{L}_p^1 f(X_u^Z) \lambda_u^{1Z} + \mathscr{L}_p^2 f(X_u^Z) \lambda_u^{2Z}) e^{-pu} du \right) \right| \leqq \varepsilon / p.$$

Thus, we deduce that

$$\sup_{Z \in \mathscr{Z}} E_x(e^{-p \cdot S_A^Z} f(X_{S_A^Z})) \leqq f(x) \leqq \sup_{Z \in \mathscr{Z}} E_x(e^{-p \cdot S_A^Z} f(X_{S_A^Z})) + \varepsilon / p,$$

which leads to the desired conclusion. □

According to the classical potential theory we set the following definition for the reduite of a given function.

Definition 2.3.2. Let f be a given positive function on E. If the set of p-biexcessive functions majorizing f is non-void and has one minimal element, this p-biexcessive function is called the p-reduite of f, and is denoted by Rf.

By interpreting this notion of reduite in the framework of the two-parameter optimal stopping, we obtain an existence result and a construction of the p-reduite.

Proposition 2.3.3. *Let f be a non-negative Borel bounded function on E such that the process $f(X) = (f(X_t); t \in \mathbb{R}_+^2)$ is lower semi-continuous for the right topology a.s. for any probability $\mathbb{P}_x$, $x \in E$. Then, the function q defined as the limit of the increasing sequence $(q^n; n \in \mathbb{N})$, defined by the following recurrence formulas*

$$q^0 = f \quad \text{and for} \quad n \geqq 0: q^{n+1} = \sup_{r \in \mathbb{D}} e^{-p \cdot r} P_r q^n,$$

is the p-reduite of f. Moreover, q verifies

$$\forall x \in E: q(x) = \sup_{T \in \mathcal{T}^x} E_x(e^{-p \cdot T} f(X_T)).$$

Proof. For $x \in E$ fixed, let us consider the optimal stopping problem on the probability space $(\Omega, \mathcal{F}^x, \mathbb{P}_x)$ associated to the following pay-off process Y:

$$\forall t \in \mathbb{R}^2_+: Y_t = e^{-p \cdot t} f(X_t).$$

Let J^x denote its Snell envelope constructed as the limit of the sequence of processes $(I^n; n \in \mathbb{N})$ defined by Proposition 1.2.1. We verify immediately that

$$\forall n \in \mathbb{N}: I^n_t = e^{-p \cdot t} q^n(X_t), \qquad t \in \mathbb{R}^2_+.$$

By construction $(q^n;\ n \in \mathbb{N})$ is an increasing sequence of uniformly bounded Borel functions: the limit q is also in $b(E)$, and

$$\forall x \in E: J^x_t = e^{-p \cdot t} q(X_t), \qquad t \in \mathbb{R}^2_+.$$

This proves that the Snell envelope J^x can be chosen independently from x, and in particular

$$\forall x \in E: q(x) = \sup_{T \in \mathcal{T}^x} E_x(e^{-p \cdot t} f(X_T)).$$

Now let us prove that q is the least p-biexcessive majorant of f.

By the supermartingale property of J, we get, $\forall x \in E$:

$$\forall t \in \mathbb{R}^2_+: E_x(J_0) \geqq E_x(J_t) \text{ i.e.,} \qquad q(x) \geqq e^{-p \cdot t} P_t q(x).$$

Moreover we proved in Proposition 1.2.2 that the right lower semi-continuity of the pay-off process Y implies the right-continuity on $\mathbb{R}^2_+$ of the function $t \to E_x(J_t)$. Then, for any sequence $(t(n); n \in \mathbb{N})$, decreasing to zero we have

$$\forall x \in E: q(x) = E_x(J_0) = \lim E_x(J_{t(n)}) = \lim e^{-p \cdot t(n)} P_{t(n)} q(x).$$

Hence, q is p-biexcessive. It remains to be verified that q is the least p-biexcessive majorant of f. Suppose q' is a p-biexcessive function greater than f. Then the process J' defined by $J'_t = e^{-p \cdot t} q'(X_t)$, $\forall t \in \mathbb{R}^2_+$, is a strong supermartingale which majorizes Y. This implies that J' majorizes the Snell envelope J, and

$$q'(x) = E_x(J'_0) \geqq E_x(J_0) = q(x).$$

This gives the proof. □

This result can be used to define the notion of p-reduite of a p-biexcessive function on a given open subset, and links with the weak-harmonicity. The following proposition extends a result given in [17], the idea of the proof is similar.

Proposition 2.3.4. *Let q be a p-biexcessive bounded function, and let A be an open subset in E. There exists a function q_A which is the least p-biexcessive majorant of function q on the subset A. Moreover*

$$\forall x \in E: q_A(x) = H^p_A q(x).$$

Proof. Function q being p-biexcessive bounded, the process $Y=(e^{-p\cdot t}q(X_t);\ t\in\mathbb{R}_+^2)$ is lower semi-continuous for the right-topology by Proposition 2.2.3. A being open, this also holds for the process

$$Y^A=(e^{-p\cdot t}q(X_t)\,\mathbb{1}_{\{X_t\in A\}};\ t\in\mathbb{R}_+^2).$$

Then, existence of q_A proceeds from Proposition 2.3.3. We get $q_A\leqq H_A^p q$ by Proposition 2.3.1 iv). Let us prove the converse. By definition, q_A is p-biexcessive and majorizes q on A. Let $Z=(Z_u;\ u\in\mathbb{R}_+)$ be any given optional increasing path. From Remark 2.2.2 we know that processes $q(X_{Z.})$ and $q_A(X_{Z.})$ are one-parameter a.s. right-continuous supermartingales and

$$\forall u\in\mathbb{R}_+:\ q_A(X_{Z_u})\geqq q(X_{Z_u})\quad \text{on}\quad \{Z_u\in A\}.$$

By definition, the time $D_A^Z=\inf\{u: Z_u\in A\}$ is a.s. adherent, with respect to the right-topology on $\mathbb{R}_+$, to the set $\{u: Z_u\in A\}$. Therefore

$$q_A(X_{D_A^Z})\geqq q(X_{D_A^Z})\quad \text{a.s.}$$

This implies $\forall Z\in\mathscr{Z}$ and $\forall x\in E$:

$$E_x(e^{-p\cdot D_A^Z}q_A(X_{D_A^Z}))\geqq E_x(e^{-p\cdot D_A^Z}q(X_{D_A^Z})).$$

A fortiori,

$$q_A(x)\geqq E_x(e^{-p\cdot D_A^Z}q(X_{D_A^Z}))\qquad \forall Z\in\mathscr{Z}.$$

From this, we deduce, $\forall x\in E$:

$$q_A(x)\geqq H_A^p(x).$$

This achieves the proof. □

The evolution of the p-reduite Rf of a function f, on the subset in E where it majorizes strictly f, is described by the following result.

Proposition 2.3.5. *Let f be such as in Proposition 2.3.3. Then, for any $\lambda\in]0,1[$, the p-reduite Rf of f is p-weakly harmonic on the set $\{x\in E: f(x)<\lambda Rf(x)\}$.*

The proof is a straightforward application of Proposition 1.2.5.

Without further assumption on the regularity of functions f and Rf, it seems difficult to characterize Rf on the set $\{f<Rf\}$. But the converse is true.

Proposition 2.3.6. *Let f be such as in Proposition 2.3.3, and suppose there exists a function q which verifies:*

i) $q\geqq f$.

ii) *q is p-biexcessive.*

iii) *q is p-weakly harmonic on $\{q>f\}$, then q is the p-reduite of f.*

Proof. Using iii) and Proposition 2.3.3, we get

$$\begin{aligned} q(x)=H_{\{q=f\}}^p q(x) &=\sup_{T\in\mathscr{T}} E_x(\mathbb{1}_{\{q(X_T)=f(X_T)\}}\,e^{-p\cdot T}q(X_T))\\ &=\sup_{T\in\mathscr{T}} E_x(\mathbb{1}_{\{q(X_T)=f(X_T)\}}\,e^{-p\cdot T}f(X_T))\leqq \sup_{T\in\mathscr{T}} E_x(e^{-p\cdot T}f(X_T)).\end{aligned}$$

Thus, $q\leqq Rf$. Taking into account i) and ii), it follows that $q=Rf$. □

3. Optimal Stopping of Bi-Markov Processes

In this chapter we give an existence result for the optimal stopping problem of a bi-Markov process X defined as solution of stochastic differential equations.

Moreover we prove that, if the optimal cost function is sufficiently smooth, it then satisfies a set of partial differential equations with a free boundary similar to those of the classical theory (see Bensoussan-Lions [4]).

For $i=1$ and 2, let $B^i=(B^i_u; u\in\mathbb{R}_+)$ be a Brownian motion and $N^i=(N^i(dz,du))$ the stochastic martingale measure associated to a Poisson process (see Jacod [30]), defined on the probability space $(\Omega^i, \mathcal{M}^i, \mathbb{P}^i)$ endowed with the filtration $(\mathcal{M}^i_u; u\in\mathbb{R}_+)$. Consider the stochastic differential equation on $E^i=\mathbb{R}^{d_i}$:

$$\text{(I)} \qquad dX^i_u=b^i(X^i_u)\,du+a^i(X^i_u)\cdot dB^i_u+\int_{E^i\setminus\{0\}} c^i(v, X^i_{u-})\cdot N^i(dv, du)$$

where a^i, b^i, c^i are matrices of appropriate dimensions, such that for any $y\in E^i$, there exists a unique strong solution $X^{iy}=(X^{iy}_u; u\in\mathbb{R}_+)$ with initial value $X^{iy}_0=y$. For $i=1$ or 2, the processes $(X^{iy}; y\in E^i)$ form a Markov flow to which one can associate a canonical Markov process X^i. Thus, by setting

$$\forall t=(t_1, t_2)\colon X_t=(X^1_{t_1}, X^2_{t_2}),$$

we get a bi-Markov process as treated in Sect. 2. In the sequel, it will be more convenient to work with the families of processes $(X^{1x^1}; x^1\in E^1)$ and $(X^{2x^2}; x^2\in E^2)$ directly. For that purpose we define the following family of processes

$$\forall x=(x^1, x^2)\in E=E^1\times E^2\colon X^x_t=(X^{1x^1}_{t_1}, X^{2x^2}_{t_2}),$$

on the probability space $(\Omega=\Omega^1\times\Omega^2,\ \mathcal{A}=\mathcal{M}^1\otimes\mathcal{M}^2,\ \mathbb{P}=\mathbb{P}^1\otimes\mathbb{P}^2)$ endowed with the smallest two-parameter filtration $\mathcal{F}=(\mathcal{F}_t; t\in\mathbb{R}^2_+)$ satisfying Axioms F1, F2, F3, and containing the filtration $(\mathcal{M}^1_{t_1}\otimes\mathcal{M}^2_{t_2}; (t_1, t_2)\in\mathbb{R}^2_+)$. In addition, $\mathcal{F}$ satisfies Axiom F4. Let $\mathcal{T}$ be the set of stopping points with respect to $\mathcal{F}$.

The following result is a straightforward application of Gronwall's inequality. If the coefficients a^i, b^i, c^i are bounded and Lipshitzian then, for any constant $A>0$ and for any $\mathcal{M}^i$-stopping time T such that $T\leqq A$ a.s.,

$$E(|X^{ix}_T-X^{iy}_T|)\leqq|x-y|\,e^{K(A)} \qquad \forall x, y\in E^i$$

where K is a function of A, strictly increasing positive, and depending only on a^i, b^i, c^i. (See Stroock-Varadhan [50], Corollary 5.1.2 for the Brownian case).

This result extends to X as follows.

Lemma 3.1. *For $i=1$ and 2, let a^i, b^i, c^i be bounded and lipshitzian. Then there exists a positive strictly increasing function K such that,*

$$\forall A>0, \quad \forall T\in\mathcal{T} \quad \text{with } |T|=T_1+T_2\leqq A$$
$$E(|X^x_T-X^y_T|)\leqq|x-y|\,e^{K(A)}, \qquad \forall x, y\in E.$$

Proof. Let us assume that the norm on E is defined by: $\forall x=(x^1, x^2)\in E: |x| = |x^1|+|x^2|$, where $|x^i|$ represents the norm of x^i on E^i. It is sufficient to verify that, for $i=1, 2$:

$$E_{\mathbb{P}}(|X^{ix^i}_{T_i} - X^{iy^i}_{T_i}|) \leqq |x^i - y^i| \, e^{K(A)}, \qquad \forall x^i, y^i \in E^i.$$

Let us derive this formula for $i=1$.

It can easily be verified that if T is a stopping point, then $T_1(\cdot, \omega_2)$ is a $\mathscr{M}^1$-stopping time on $(\Omega^1, \mathscr{M}^1, \mathbb{P}^1)$ for $\mathbb{P}^2$-almost all ω_2. Moreover $T_1 \leqq A$ a.s. Then we get

$$E_{\mathbb{P}^1}(|X^{1x^1}_{T_1(\cdot,\omega_2)} - X^{1y^1}_{T_1(\cdot,\omega_2)}|) \leqq |x^1 - y^1| \, e^{K(A)}.$$

By integrating by $\mathbb{P}^2$, we obtain the inequality. □

For a positive real number and a function $f\in b(E)$, we define a family $Y=(Y^x; x\in E)$ of pay-off processes by

$$\forall x\in E, \qquad \forall t\in\mathbb{R}^2_+: Y^x_t = e^{-p\cdot t} f(X^x_t).$$

f is called the pay-off function, and p the actualization factor. Let us denote by J^x the Snell envelope of the process Y^x, for $x\in E$.

Suppose that f is continuous. Then, by Proposition 2.3.3 we find that there is a function $q\in b(E)$ which is the p-reduite of f, such that

$$\forall x\in E: J^x_T = e^{-p\cdot T} q(X^x_T), \qquad \forall T\in\mathscr{T}.$$

This formula holds for general bi-Markov processes, but under additional hypotheses on processes X^1 and X^2 the Snell reduite q has a better regularity property.

Proposition 3.1.1. *If the pay-off function f is bounded and uniformly continuous on E, and if the bi-Markov family X satisfies the condition of Lemma* 3.1, *then the p-reduite q of f is also uniformly continuous on E.*

Proof. It is analogous to that of [47] for the classical case. Let T be a s.p., A a positive constant, and let x, y be two distinct points in E. Let us study the random variable U defined by

$$U = |e^{-p\cdot T} f(X^x_T) - e^{-p\cdot T} f(X^y_T)|.$$

It can be noticed that, for a given s.p. T and a constant A, there exists a s.p. T_A such that $T_A \leqq T$ with $T_A = T$ on $\{|T|\leqq A\}$ and $|T_A| = A$ on $\{|T|>A\}$. One can obtain such a s.p. by choosing an o.i.p. Z passing through T, and by defining $T_A = Z_{|T|\wedge A}$. Then, we get the following inequalities.

$$\begin{aligned} U &\leqq |e^{-p\cdot T} f(X^x_T) - e^{-p\cdot T_A} f(X^x_{T_A})| \\ &\quad + |e^{-p\cdot T_A} f(X^x_{T_A}) - e^{-p\cdot T_A} f(X^y_{T_A})| + |e^{-p\cdot T_A} f(X^y_{T_A}) - e^{-p\cdot T} f(X^y_T)| \\ &\leqq 4 e^{-pA} \|f\| + |e^{-p\cdot T_A}(f(X^x_{T_A}) - f(X^y_{T_A}))|. \end{aligned}$$

The function f being uniformly continuous on E, we have

$$\forall \varepsilon>0, \qquad \exists\, \delta \text{ such that } |z-z'|<\delta \Rightarrow |f(z)-f(z')|<\varepsilon.$$

Using the Chebyschev inequality, we thus obtain:

$$E(|f(X^x_{T_A})-f(X^y_{T_A})|)\leqq \varepsilon+\frac{2}{\delta}\,\|f\|\,E(X^x_{T_A}-X^y_{T_A}|).$$

According to Lemma 3.1, we then deduce a majorant of $E(U)$, independent of the s.p. T:

$$E(U)\leqq 4e^{-pA}\,\|f\|+\varepsilon+\frac{2}{\delta}\,\|f\|\,|x-y|\,e^{K(A)}.$$

Therefore,

$$\begin{aligned}|q(x)-q(y)|&=|\sup_{T\in\mathcal{T}} E(e^{-p\cdot T}f(X^x_T))-\sup_{T\in\mathcal{T}} E(e^{-p\cdot T}f(X^y_T))|\\&\leqq \sup_{T\in\mathcal{T}} E(e^{-p\cdot T}\,|f(X^x_T)-f(X^y_T)|)\\&\leqq 4e^{-p\cdot A}\,\|f\|+\varepsilon+\frac{2}{\delta}\,\|f\|\,|x-y|\,e^{K(A)}.\end{aligned}$$

By choosing A, then ε and finally x and y, we can make this majorant as small as we want. This implies the uniform continuity for q. □

As an application of Chapter I, we obtain an existence result for a bi-Markov process defined as solutions of two independent stochastic differential equations.

Theorem 3.1. *Let $X=(X^x;\ x\in E)$ be the flow associated to the system (I) of two independent stochastic differential equations with bounded and lipschitzian coefficients. Let f be a bounded uniformly continuous function, and let p be a positive real number. Then, the optimal stopping problem associated to the pay-off process Y defined by*

$$\forall\, t\in\mathbb{R}^2_+:\ Y_t=e^{-p\cdot t}f(X_t)$$

admits solutions. Moreover the maximal stopping points are optimal.

Proof. In order to apply Theorem 1.1, we must only verify that the Snell envelope is completely regular. By Proposition 3.1.1, we know that, for any $x\in E$, the Snell envelope J^x is given by

$$J^x_t=e^{p\cdot t}\,q(X^x_t)$$

where q is a uniformly continuous function. Let us prove that if $(T^n;\ n\in\mathbb{N})$ is a sequence of 1-stopping points converging to T such that $T^n_1\leqq T_1$ and $T^n_2\geqq T_2$, then $\lim_n E(J^x_{T^n})=E(J^x_T)$. Recall that the solutions X^1 and X^2 of equations of system (I) are quasi-left continuous and right-continuous [30]. Then, by Proposition 2.1.2, we get

$$\lim_n X^x_{T^n}=X^x_T\quad \text{a.s. on } (\Omega,\mathcal{A},\mathbb{P}),\qquad \forall\, x\in E.$$

Function q being continuous bounded, we get by dominated convergence:

$$\lim_n E(e^{-p\cdot T^n}\,q(X^x_{T^n}))=E(e^{-p\cdot T}q(X^x_T)).$$

The other situations are treated similarly. This achieves the proof. □

It may be underlined that we proved an existence result for a pay-off process which does not enter into the existence conditions studied by A. Millet [45]. Moreover Theorem 3.1 gives optimal solutions as the maximal elements of the set of stopping points which preserve the martingale property of the Snell envelope. This characterization is similar to that given by Mazziotto-Szpirglas [38] in the discrete parameter set case. Unfortunately, for the moment, we do not know how to generalize the effective construction of [38] to obtain a solution. But let us notice that ε-optimal tactics could be found, such as in the proof of Proposition 2.3.2, and in Bismut [5].

By using the results of Chapter II, we can characterize the Snell reduite q associated to the preceding optimal stopping problem. From Proposition 2.3.5, we know that q is p-weakly harmonic on any subset $\{f<\lambda q\}$ for $\lambda \in]0,1[$. If we assumed further regularity conditions on q, we could then characterize it as the solution of a system of variational inequalities similar to those studied by Bensoussan-Lions [4].

Theorem 3.2. *If the pay-off function f is such that its p-reduite q belongs to the domain $\mathscr{D}(\mathscr{L}^1, \mathscr{L}^2)$, then it verifies the following system of partial differential inequations with free boundary*

(S1) $q \geqq f$.

(S2) $\mathscr{L}^1 q \leqq pq$ *and* $\mathscr{L}^2 q \leqq pq$ *on* E.

(S3) $\max(\mathscr{L}^1 q - pq, \mathscr{L}^2 q - pq) = 0$ *on* $\{q > f\}$.

Conversely if system (S1, S2, S3) *admits a solution in* $\mathscr{D}(\mathscr{L}^1, \mathscr{L}^2)$, *then it is the Snell reduite* q.

Proof. By Proposition 2.3.5, we have (S1), (S2), and (S3) on $\{f<\lambda q\}$ for any $\lambda \in]0,1[$. But if $q \in \mathscr{D}(\mathscr{L}^1, \mathscr{L}^2)$, functions $\mathscr{L}^1 q$ and $\mathscr{L}^2 q$ are continuous and the inequality extends to $\{q>f\}$. The reciprocal assertion follows from Proposition 2.3.6 directly. □

System (S1, S2, S3) is partly similar to the one considered in the classical theory of optimal stopping [2]. The main difference seems to be the existence of a non-linear operator in relation (S3). Such an operator appears commonly in classical distributed control problem, more precisely in the Hamilton-Jacobi-Bellman equation.

The following associated Dirichlet problem (i.e. with a smooth fixed boundary) is well known and solved by Brezis-Evans [10], and by Lions-Menaldi [22].

(S′1) $q = f$ on the boundary of a smooth domain A.

(S′2) $\mathrm{Max}(\mathscr{L}^1 q - pq, \mathscr{L}^2 q - pq) = 0$ in A.

System (S1, S2, S3) is a free boundary problem which can be interpreted as follows

(S″1) $f - q \leqq 0$, $\mathscr{L}^1 q - pq \leqq 0$, $\mathscr{L}^2 q - pq \leqq 0$.

(S″2) $\mathrm{Max}(f - q, \mathscr{L}^1 q - pq, \mathscr{L}^2 q - pq) = 0$ in E.

Such a system has been recently studied and solved by P.L. Lions [33] in appropriate Sobolev spaces.

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