Z. Wahrscheinlichkeitstheorie verw. Gebiete
69, 19–22 (1985)

Zeitschrift für
Wahrscheinlichkeitstheorie
und verwandte Gebiete

A Lower Bound for the Critical Probability of the Square Lattice Site Percolation

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Summary. We prove by elementary combinatorial considerations that the critical probability of the square lattice site percolation is larger than 0.503478.

1. Introduction

Throughout this note $\mathscr{G}$ and $\mathscr{G}^*$ will denote the square lattice and its matching pair (the graph obtained from the first one by adding edges on every diagonal of elementary squares) respectively. p_c and p_c^* will denote the critical probability of the Bernoulli site percolation on these graphs. In [3] and [4] Russo proved that

$$p_c + p_c^* = 1 \tag{1}$$

(see also [2]).

Due to the fact that $\mathscr{G}$ is a subgraph of $\mathscr{G}^*$, from (1.1) trivially follows that $p_c \geqq \frac{1}{2}$. Higuchi [1] improved this, showing that

$$p_c > \tfrac{1}{2}. \tag{1.2}$$

(Simulation results suggest $p_c \approx 0.58$ [5].)

In the present note we shall improve Higuchi's lower bound. The gain in the numerical value is rather small, but we consider that our method of proof is simpler and more elementary than that of Higuchi's.

Let x_0 be the root between 0 and 1 of the polynomial

$$3x^8 - 8x^7 + 6x^6 + x^4 - 1. \tag{1.3}$$

(The fact that (1.3) has a single root in the interval (0, 1) is shown in the Appendix.)

Our result is the following

* Work supported by the Central Research Found of the Hungarian Academy of Sciences (Grant No. 476/82)

Proposition 1. $p_c \geqq x_0^4 > 0.503478$.

In the next sections we shall prove this statement by coupling simultaneously percolation processes on $\mathscr{G}$ and $\mathscr{G}^*$ to a percolation process on a third, auxiliary lattice.

2. Description of the Graphs

Let us consider the three lattices shown in Fig. 1.

$\mathscr{G}$ is a square lattice. Its vertices are placed on the geometric points of coordinates

$$(2i, 2j), \qquad i+j=2k. \tag{2.1}$$

Two vertices of $\mathscr{G}$ are joined by an edge if their distance equals $2\sqrt{2}$. The set of vertices of $\mathscr{G}$ is denoted by $\mathscr{V}$.

$\mathscr{L}$ is the auxiliary lattice. The set of its vertices is partitioned in two subclasses: $\mathscr{U}=\mathscr{U}_b \cup \mathscr{U}_w$. Black vertices of $\mathscr{L}$ are placed on the same geometric points and are connected by the same edges as those of $\mathscr{G}$'s. The white ones are placed in the points

$$(i, j), \qquad i+j=2k+1 \tag{2.2}$$

and two of them are connected by an edge if their distance is $\sqrt{2}$ or if $i=i'=2k+1$ $(j=j'=2k+1)$ and $|j-j'|=2$ $(|i-i'|=2)$. (The subgraph induced by the white sites is the covering graph of a square lattice.) Sites of different colours are connected by an edge if their distance equals $\sqrt{5}$.

$\mathscr{G}^*$ is the matching pair of a square lattice. Its vertex set is also partitioned: $\mathscr{V}^*=\mathscr{V}_b^* \cup \mathscr{V}_w^*$. The black vertices and their connections are the same as those on $\mathscr{L}$ (or $\mathscr{G}$). White vertices are placed in the points

$$(2i, 2j) \qquad i+j=2k+1 \tag{2.3}$$

and two of them are connected if their distance equals $2\sqrt{2}$. Two vertices of different colours are connected on this graph if their distance equals 2. (The subgraphs induced by the black vertices on $\mathscr{L}$ and $\mathscr{G}^*$ are copies of $\mathscr{G}$.)

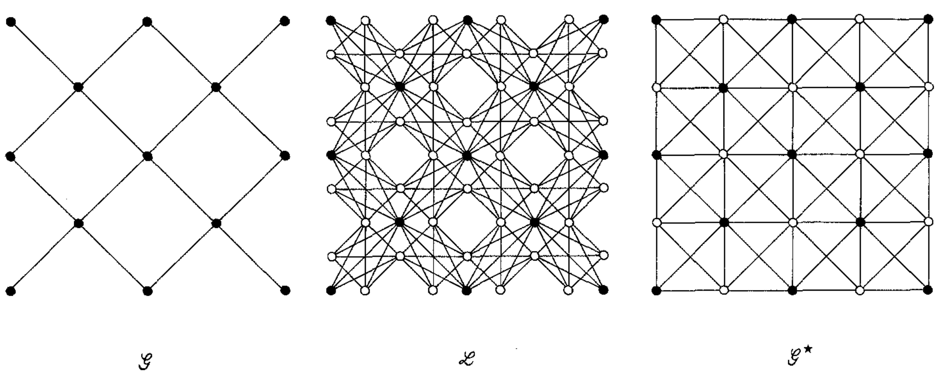

Fig. 1

If v is a vertex of any of the three graphs, $\pi(v)$ will denote the geometric point on which it is placed.

We shall define now some mappings from the vertex set of $\mathscr{L}$ on the vertex set of the other two graphs.

$$f\colon \mathscr{U}\to\mathscr{V};\quad f(u)=\begin{cases}\text{that } v\in\mathscr{V} \text{ for which } \pi(u)=\pi(v) & \text{if } u\in\mathscr{U}_b\\ \text{that } v\in\mathscr{V} \text{ for which } \operatorname{dist}(\pi(u),\pi(v))=1 & \text{if } u\in\mathscr{U}_w\end{cases} \tag{2.4}$$

$$g\colon \mathscr{U}\to\mathscr{V}^*;\quad g(u)=\begin{cases}\text{that } v\in\mathscr{V}_b^* \text{ for which } \pi(u)=\pi(v) & \text{if } u\in\mathscr{U}_b\\ \text{that } v\in\mathscr{V}_w^* \text{ for which } \operatorname{dist}(\pi(u),\pi(v))=1 & \text{if } u\in\mathscr{U}_w.\end{cases} \tag{2.5}$$

The images of the graph $\mathscr{L}$ under these mappings are exactly $\mathscr{G}$ and $\mathscr{G}^*$ respectively.

The configuration spaces of the percolation problems on these graphs are

$$\Omega_{\mathscr{G}}=\{-1,+1\}^{\mathscr{V}};\quad \Omega_{\mathscr{L}}=\{-1,+1\}^{\mathscr{U}};\quad \Omega_{\mathscr{G}^*}=\{-1,+1\}^{\mathscr{V}^*} \tag{2.6}$$

with the natural σ-algebras.

3. Proof

Consider the Bernoulli percolation problem on $\mathscr{L}$ with the measure

$$P_{\mathscr{L}}=\prod_{u\in\mathscr{U}}\nu_u;\quad \nu_u(\omega_{\mathscr{L}}(u)=+1)=\begin{cases}p & \text{if } u\in\mathscr{U}_b\\ q=1-(1-p)^{1/4} & \text{if } u\in\mathscr{U}_w\end{cases} \tag{3.1}$$

and couple with it the following percolation problems:

1. On $\mathscr{G}$

$$\text{for } v\in\mathscr{V}\quad \omega_{\mathscr{G}}(v)=\max\{\omega_{\mathscr{L}}(f^{-1}(\{v\})\cap\mathscr{U}_b),\ \max_{u\in f^{-1}(\{v\})\cap\mathscr{U}_w}(\min_{u'\in f^{-1}(\{v\})\cap\mathscr{U}_w\setminus\{u\}}\omega_{\mathscr{L}}(u'))\}. \tag{3.2}$$

Shortly: we consider $v\in\mathscr{V}$ occupied iff the copy of v in $\mathscr{L}(\in\mathscr{U}_b)$ is occupied or at least three of the four white sites surrounding it are occupied.

2. On $\mathscr{G}^*$

$$\text{for } v\in\mathscr{V}^*\quad \omega_{\mathscr{G}^*}(v)=\max_{u\in f^{-1}(\{v\})}\omega_{\mathscr{L}}(u). \tag{3.3}$$

Or: we declare occupied a black site of $\mathscr{G}^*$ iff its copy in $\mathscr{L}$ is occupied, a white site of $\mathscr{G}^*$ iff at least one of the four white sites of $\mathscr{L}$ surrounding the geometric point where it is placed is occupied.

It is easily seen that the percolation problems defined in this way on $\mathscr{G}$ and $\mathscr{G}^*$ are Bernoulli ones with probability measures

$$P_{\mathscr{G}}=\prod_{v\in}\mu_v;\qquad \mu_v(\omega_{\mathscr{G}}(v)=+1)=p+(1-p)\,q^3(4-3q) \tag{3.4}$$

$$P_{\mathscr{G}^*}=\prod_{v\in\mathscr{V}^*}\mu_v^*;\qquad \mu_v^*(\omega_{\mathscr{G}^*}(v)=+1)=p. \tag{3.5}$$

By simple inspection one can convince himself of the truth of the following

Lemma. *Let $\omega_{\mathscr{L}}\in\Omega_{\mathscr{L}}$ and $\omega_{\mathscr{G}}\in\Omega_{\mathscr{G}}$, $\omega_{\mathscr{G}^*}\in\Omega_{\mathscr{G}^*}$ derived from $\omega_{\mathscr{L}}$ in the manner defined above. Then:*

i) if $v\in\mathscr{V}$ and $v'\in\mathscr{V}$ are $+$-connected in $\omega_{\mathscr{G}}$ then $f^{-1}(\{v\})\subset\mathscr{U}$ and $f^{-1}(\{v'\})\subset\mathscr{U}$ are also $+$-connected in $\omega_{\mathscr{L}}$.

ii) if $u\in\mathscr{U}$ and $u'\in\mathscr{U}$ are $+$-connected in $\omega_{\mathscr{L}}$ then $g(u)\in\mathscr{V}^*$ and $g(u')\in\mathscr{V}^*$ are also $+$-connected in $\omega_{\mathscr{G}^*}$.

In both cases the formal proof goes through induction on the length of the shortest $+$-path between v and v' respectively u and u'.

Consequently if percolation occurs on $\mathscr{G}$ it must occur also on $\mathscr{G}^*$. Or, equivalently

$$p+(1-p)(1-(1-p)^{1/4})^3(4-3(1-(1-p)^{1/4}))>p_c\Rightarrow p>p_c^*. \tag{3.6}$$

By (1.1) this is equivalent to:

$$3p_c^2-8p_c^{7/4}+6p_c^{3/2}+p_c-1\geqq 0 \tag{3.7}$$

from which the result follows.

Appendix

We show that the polynomial

$$f(x)=3x^8-8x^7+6x^6+x^4-1 \tag{A.1}$$

has a single root in the interval (0, 1). This follows trivially from the facts that

$$f(0)=-1;\quad f(1)=+1 \tag{A.2}$$

and the derivative of f

$$\begin{aligned} f'(x)&=24x^7-56x^6+36x^5+4x^3\\ &=4x^5[(2x-3)^2+(x-1)^2+1]+4x^3(x^2-1)^2 \end{aligned} \tag{A.3}$$

is strictly positive in (0, 1).

Acknowledgement. The author is indebted to his colleagues at the Mathematical Institute of the HAS for valuable discussions on percolation problems.

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Received June 6, 1984