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Time-Revealed Convergence Properties of Normalized Maxima in Stationary Gaussian Processes*

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Let $\{X(s), s \geqq 0\}$ denote a real-valued, separable, stationary Gaussian Process with mean zero and variance one. Let $\{r(s), s \geqq 0\}$ be the covariance function, $M_T = \{\sup X(s)\colon 0 \leqq s \leqq T\}$ and $c_T = (2 \ln T)^{\frac{1}{2}}$. Suppose

(1) $1 - r(s) \sim C|s|^\alpha$ as $s \to 0$ for some constant $C > 0$ and $0 < \alpha \leqq 2$ and

(2) $r(s) \ln s \to 0$ as $s \to \infty$. Then

$\liminf_{T \to \infty} c_T(M_T - c_T)/\ln\ln T = \frac{1}{\alpha} - \frac{1}{2}$ a.s. and

(3) $\limsup_{T \to \infty} c_T(M_T - c_T)/\ln\ln T = \frac{1}{\alpha} + \frac{1}{2}$ a.s.

Suppose (1) holds and

(4) $r(s)(\ln s)^{1+\varepsilon} \to 0$ as $s \to \infty$ for some $\varepsilon > 0$. Then

(5) $\lim_{T \to \infty} E(\exp(t U_T)) = E(\exp(t X))$

for $t > 0$ sufficiently small where X is a random variable with distribution function $\exp(-\exp(-x))$; $-\infty < x < \infty$ and U_T is defined as

$$U_T = c_T(M_T - a_T) \quad \text{and} \quad a_T = c_T + \left(\frac{1}{\alpha} - \frac{1}{2}\right) \ln(V_\alpha \ln T)/c_T$$

V_α being a positive constant. The condition $r(s)(\ln s)^{1-\varepsilon} \to 0$ as $s \to \infty$ is not sufficient for these results. Result (3) improves that of Pickands (Trans. Amer. Math. Soc. **145**, 75–86 (1969)) and Qualls and Watanabe (Ann. Math. Stat. **42**, 2029–2035 (1971)).

1. Introduction

Let $\{X(s), s \geqq 0\}$ denote a real-valued, separable stationary Gaussian process with $EX(s) = 0$ and $EX^2(s) = 1$. Let $\{r(s), s \geqq 0\}$ be the covariance function $r(s) = E(X(s)X(0))$ and let $M_T = \{\sup X(s)\colon 0 \leqq s \leqq T\}$. We consider processes for which

$$1 - r(s) \sim C|s|^\alpha \quad \text{as } s \to 0, \tag{1.1}$$

for some constant $C > 0$ and $0 < \alpha \leqq 2$. Condition (1.1) ensures the continuity of sample paths for $X(s)$ and hence M_T is finite almost surely.

Convergence in distribution of suitably normalized M_T was considered by Cramér [3], Belyaev [1], Qualls [11] and Pickands [10]. Pickands proved that $r(s) \ln s \to 0$ as $s \to \infty$, along with (1.1), is sufficient for $c_T(M_T - a_T)$ to converge in distribution to a random variable X where X has distribution function

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$\exp(-\exp(-x))$, $-\infty<x<\infty$. Here c_T and a_T are constants defined as follows:

$$c_T=\max\{(2\ln T)^{\frac{1}{2}};0\};$$
$$a_T=c_T+\left(\frac{1}{\alpha}-\frac{1}{2}\right)\ln(V_\alpha \ln T)/c_T \quad \text{if } T\geqq e \qquad (1.2)$$
$$=0 \quad \text{if } T<e,$$

V_α being a positive constant given explicitly in [10].

M_T is said to be respectively "stable" or "relatively stable" if $M_T-c_T\to 0$ or $M_T/c_T\to 1$ as $T\to\infty$. The modes of convergence considered are convergence in probability and almost sure convergence. Progressively weaker conditions on the covariance function were used by several authors for results in this area. (Ref. Šur [13], Cramér [3], Pickands [8], Nisio [6], Marcus [4] and Watanabe [15].) Analogous results for the discrete parameter case are referred to in [5]. For the continuous parameter case one has $M_T-c_T\to 0$ a.s. as $T\to\infty$ if (1.1) holds and $r(s)\ln s\to 0$ as $s\to\infty$. Also $M_T/c_T\to 1$ a.s. as $T\to\infty$ under a very weak local condition and $r(s)\to 0$ as $s\to\infty$.

Our interest lies in the rate at which $M_T-c_T\to 0$ as $T\to\infty$. In this direction we prove

Theorem 1. *If* (1.1) *holds and*

$$r(s)\ln s\to 0 \quad \text{as } s\to\infty, \qquad (1.3)$$

then

$$\liminf_{T\to\infty} c_T(M_T-c_T)/\ln\ln T=\frac{1}{\alpha}-\frac{1}{2} \text{ a.s.}, \qquad (1.4)$$

$$\limsup_{T\to\infty} c_T(M_T-c_T)/\ln\ln T=\frac{1}{\alpha}+\frac{1}{2} \text{ a.s.} \qquad (1.5)$$

Theorem 1 extends results of Pickands [10] and Qualls and Watanabe [12]. The extension lies in the weakening of the mixing condition previously used, viz., $r(s)s^\gamma\to 0$ as $s\to\infty$ for some $0<\gamma<1$. It is made possible by the following two lemmas giving information about the tails of the distribution of M_T.

Lemma 1. *Let* (1.1) *hold and suppose*

$$r(s)(\ln s)^{1+\varepsilon}\to 0 \quad \textit{as } s\to\infty \textit{ for some } \varepsilon>0. \qquad (1.6)$$

Then

$$\exp(tA^2)P(M_T\leqq a_T-A/c_T)\to 0 \quad \textit{as } A\to\infty \qquad (1.7)$$

uniformly in T for all sufficiently small values of t.

Lemma 2. *If* (1.1) *and* (1.3) *hold, then*

$$P\left(\exists\tau_0\ni X(T)\leqq c_T+\left(\frac{1}{\alpha}+\frac{1}{2}\right)\frac{\ln\ln T}{c_T}\ \forall T\geqq\tau_0\right)=0. \qquad (1.8)$$

Lemma 2 is a particular case of Theorem A of Pathak and Qualls [7]. Our result was obtained independently using different techniques.

In support of the sharpness of the Condition (1.3) we give the following theorem.

Theorem 2. *Let the covariance function* $r(s)$ *be non-increasing, satisfying* (1.1) *and*

$$r(s) \to 0 \quad \textit{but} \quad (r(s)\ln s)/\ln\ln s \to \infty \qquad \textit{as } s \to \infty. \tag{1.9}$$

Then

$$\liminf_{T\to\infty} c_T(M_T - c_T)/\ln\ln T = -\infty \text{ a.s.} \tag{1.10}$$

We also establish the convergence of the moment generating function of suitably normalized M_T. Since this depends heavily on Lemma 1 we require the stronger mixing Condition (1.6) instead of (1.3). Corollaries 1 and 2 then estimate asymptotically the variance and the moments of M_T.

Theorem 3. *Under the Conditions* (1.1) *and* (1.6) *we have*

$$\lim_{T\to\infty} E(\exp(t\,Y_T)) = E(\exp(t\,X)) \tag{1.11}$$

for all sufficiently small t where X is a random variable with distribution function $\exp(-\exp(-x))$, $-\infty < x < \infty$ *and* $Y_T = c_T(M_T - a_T)$.

Corollary 1. *For any random variable Z let* $\sigma^2(Z)$ *denote* $EZ^2 - (EZ)^2$. *Then*

$$\lim_{T\to\infty} \sigma^2(M_T) = (\pi^2 - 6)/12.$$

Corollary 2. *For all* $k \geqq 1$

$$a_T^{-k} E(M_T^k) = 1 + O\left(\frac{1}{\ln T}\right) \qquad \textit{as } T \to \infty.$$

The results here are extentions to the continuous parameter case of those obtained in [5] for discrete parameter processes. The proofs are similar to those in [5] but require further refinement because of the local condition on the covariance function.

Section 2 contains the proofs of Theorems 1, 2, 3 and Lemma 1. The proof of Lemma 2 is omitted in view of Theorem A of [7] and since it uses similar techniques to those in Lemma 1.

2. Proofs

Proof of Lemma 1. For ease of notation below we set

$$\lambda_T = a_T - A/c_T; \qquad E(A, T) = \exp(t A^2) P(M_T \leqq \lambda_T) \tag{2.1}$$

and

$$\varphi(u) = (2\pi)^{-\frac{1}{2}} \exp(-u^2); \qquad \Phi(u) = \int_{-\infty}^{u} \varphi(x)\,dx.$$

Our aim is to prove $E(A, T) \to 0$ as $A \to \infty$ uniformly in T for all sufficiently small values of t assuming that (1.1) and (1.6) hold. We observe from (1.1) that there exists a positive θ such that

$$1 - r(s) \geqq \frac{C}{2} |s|^\alpha \;\; \forall 0 \leqq |s| \leqq \theta. \tag{2.2}$$

If $L(T) = [T^\gamma]$ for some $0 < \gamma < 1$, $[\cdot]$ denoting the integral part, and

$$\delta_x = \sup\{|r(s)|:\; s \geqq x\}$$

then (1.6) implies

$$\delta_{L(T)}(\ln T)^{1+\varepsilon}\to 0 \quad \text{as } T\to\infty \tag{2.3}$$

for some $\varepsilon>0$.

A major portion of the proof consists in showing that given $\eta>0$, there exists A^* and T_0 (both depending on η alone) so that for all $A\geqq A^*$ and $T\geqq T_0$

$$E(A,T)<\eta. \tag{2.4}$$

Assuming (2.4) together with (2.14) and (2.17), we complete the proof as follows. Let $0\leqq T\leqq T_0$ and $A>2a_{T_0}c_{T_0}$ (a_T and c_T defined in (1.2)). Set

$$d_T=\left(a_T-\frac{3}{2}\frac{A}{c_T}\right)(1-\delta_{L(T)})^{-\frac{1}{2}}. \tag{2.5}$$

Observe that $A>2a_Tc_T\,\forall\,0\leqq T\leqq T_0$, so (2.14) and (2.17) imply that

$$\begin{aligned}E(A,T)&\leqq \exp(tA^2)\,\Phi\big(-A/(4c_T\delta^{\frac{1}{2}}_{L(T)})\big)+\exp(tA^2)\,\Phi^m(d_T)\\&=\exp(tA^2)\big(1-\Phi(A/(4c_T\delta^{\frac{1}{2}}_{L(T)}))\big)+\exp(tA^2)\{1-\Phi(-d_T)\}^m\\&\leqq\frac{4c_T\delta^{\frac{1}{2}}_{L(T)}}{A(2\pi)^{\frac{1}{2}}}\exp\left\{tA^2-\frac{A^2}{32c_T^2\delta_{L(T)}}\right\}\\&\quad+\big(2(1-\delta_{L(T)})^{\frac{1}{2}}a_T^{-1}\big)^m(2\pi)^{-\frac{1}{2}}\exp\{tA^2-m\lambda_T^2/(2(1-\delta_{L(T)}))\}.\end{aligned}$$

Note that $\sup\{c_T^2\delta_{L(T)}: T\geqq 0\}<\infty$ so $tA^2-A^2/(32c_T^2\delta_{L(T)})\to-\infty$ as $A\to\infty$ for all sufficiently small t. Also $A>2a_Tc_T$ implies

$$tA^2-m\lambda_T^2/(2(1-\delta_{L(T)})\leqq tA^2-mA^2/(8c_T^2)\to-\infty \quad \text{as } A\to\infty.$$

Thus for all $0\leqq T\leqq T_0$, we can find $A^{**}(>2a_{T_0}c_{T_0})$ such that $A>A^{**}$ implies

$$E(A,T)<\eta.$$

This together with (2.4) clearly establishes the result. We turn our attention now to the verification of (2.4). As intermediate stages we will have (2.14) and (2.17).

We use two types of blocking and partitioning. The first results in a function which bounds $E(A,T)$ for $0\leqq A\leqq(\ln\ln T)^{\frac{1}{4}}$. The second, similar to that done in the discrete case (5; (2.19)) produces suitable dominating functions on different sections of the remaining values of A (cf. (2.17)).

First divide the interval $[0,T]$ into $[T]$ intervals of unit length each. Cut off a small portion $\omega, 0<\omega<1$, from the right hand side of each interval. Let I_1 denote the union of these smaller intervals, viz., $\bigcup_{i=1}^{[T]}[(i-1),(i-\omega))$. Define the set G_1 as

$$G_1=\{jc_T^{-2/\alpha}\,|\,j=0,1,\dots[Tc_T^{2/\alpha}]\}.$$

Clearly

$$E(A,T)\leqq P\big(\max_{s\in G_1\cap I_1}X(s)\leqq\lambda_T\big). \tag{2.6}$$

Now consider a process $\{Y(s), 0\leqq s\leqq[T]\}$ made up of $[T]$ independent pieces of unit duration, each having the same structure as X on $0\leqq s\leqq 1$. We will use this Y process and the following version of a result of Berman (cf. 10, Lemma 3.7) to bound the right hand side of (2.6).

Lemma (Berman 1964). *Let $\{\chi_n, n\geqq 1\}$ and $\{\zeta_n, n\geqq 1\}$ be Gaussian sequences satisfying $E\chi_n=E\zeta_n=0$; $E\chi_n^2=E\zeta_n^2=1$; $E\chi_i\chi_j=r_{ij}$ and $E\zeta_i\zeta_j=s_{ij}$. Then for every real number a*

$$\begin{aligned}&|P\{\max_{1\leqq i\leqq n}\chi_i\leqq a\}-P\{\max_{1\leqq i\leqq n}\zeta_i\leqq a\}|\\&\leqq\sum_{i=1}^{n}\sum_{j=1}^{n}|r_{ij}-s_{ij}|(1-w_{ij}^2)^{-\frac{1}{2}}\exp(-a^2/(1+w_{ij}))\end{aligned}\tag{2.7}$$

where $w_{ij}=\max(r_{ij}, s_{ij})$.

(2.7) implies that the right hand side of (2.6) is at most

$$\begin{aligned}&\exp(tA^2)P(\max_{s\in G_1\cap I_1}Y(s)\leqq\lambda_T)\\&+\exp(tA^2)[Tc_T^{2/\alpha}]\sum_{j=[\omega c_T^{2/\alpha}]}^{[Tc_T^{2/\alpha}]}(1-r^2(jc_T^{-2/\alpha}))^{-\frac{1}{2}}|r(jc_T^{-2/\alpha})|\\&\cdot\exp\{-\lambda_T^2/(1+r(jc_T^{2/\alpha}))\}.\end{aligned}\tag{2.8}$$

We first look at the second term in (2.8). The sum involved is a nondecreasing function in $|r|$. We split this sum into two parts, $[\omega c_T^{2/\alpha}]\leqq j\leqq[L(T)c_T^{2/\alpha}]$ and $[L(T)c_T^{2/\alpha}]<j\leqq[Tc_T^{2/\alpha}]$. δ_ω is an approximate upper bound for values of $|r|$ over the first part and $\delta_{L(T)}$ is such a bound for the second part. Hence the second term in (2.8) is no bigger than

$$\begin{aligned}&\exp(tA^2)(1-\delta_\omega^2)^{-\frac{1}{2}}(Tc_T^{2/\alpha}]\,[L(T)c_T^{2/\alpha}]\exp(-\lambda_T^2/(1+\delta_\omega))\\&+\exp(tA^2)\delta_{L(T)}(1-\delta_{L(T)}^2)^{-\frac{1}{2}}[Tc_T^{2/\alpha}]^2\exp(-\lambda_T^2(1+\delta_{L(T)})^{-1}).\end{aligned}\tag{2.9}$$

But by definition of a_T,

$$\begin{aligned}\exp(-\lambda_T^2(1+\delta_\omega)^{-1})&=\exp(-a_T^2(1+\delta_\omega)^{-1}+2Aa_T/(c_T(1+\delta_\omega))-A^2/(c_T^2(1+\delta_\omega)))\\&<\{(Tc_T)^{(2/\alpha)-1}\}^{\frac{-2}{1+\delta_\omega}}\exp(2a_TA/c_T).\end{aligned}$$

Thus an upper bound for the first term in (2.9) is

$$\text{const}\cdot T^{\left(1+\gamma-\frac{2}{1+\delta_\omega}\right)}c_T^{4/\alpha}\exp\left(tA^2+\frac{2a_T}{c_T}A\right)\tag{2.10}$$

and an upper bound for the second term in (2.9) is

$$\text{const}\cdot\delta_{L(T)}c_T^2\exp\{tA^2+\delta_{L(T)}c_T^2(1+\delta_{L(T)})^{-1}+2a_Tc_T^{-1}A\}.\tag{2.11}$$

Consider the values of A for which $0\leqq A\leqq A_1(T)=(\ln\ln T)^{\frac{1}{4}}$. By choosing $0<\gamma<1$ so that $1+\gamma-2(1+\delta_\omega)^{-1}<0$ we see that the maximum of (2.10) over the values of A under consideration tends to zero as $T\to\infty$. Also such a maximum of (2.11) tends to zero as $T\to\infty$ in view of (2.3). Thus for $0\leqq A\leqq A_1(T)$, the second term of (2.8) is bounded above by a function, say $h(T)$, where $h(T)\to 0$ as $T\to\infty$.

Now consider the first term of (2.8). By definition of γ, it is equal to

$$\exp(tA^2)P^{[T]}(\mu_1\leqq\lambda_T)=\exp\{tA^2+[T]\ln P(\mu_1\leqq\lambda_T)\}$$

where $\mu_1=\max\{X(s): s\in G_1\cap[0,1-\omega)\}$.

If $0 \leqq A \leqq A_1(T)$, $\lambda_T \to \infty$ as $T \to \infty$; hence $P(\mu_1 \leqq \lambda_T) \to 1$ and

$$-\ln P(\mu_1 \leqq \lambda_T) = -\ln\{1 - P(\mu_1 > \lambda_T)\} \qquad (2.12)$$
$$\sim P(\mu_1 > \lambda_T) \sim H_\alpha (1-\omega)\, \lambda_T^{(2/\alpha)-1}\, \varphi(\lambda_T).$$

The last statement follows by Pickands (9, Lemma 2.9) with H_α some positive constant. Substituting in (2.12) we see that for T sufficiently large and $0 \leqq A \leqq A_1(T)$, the first term of (2.8) is asymptotically equal to

$$\exp\{t A^2 - \text{const} \cdot [T]\, c_T^{(2/\alpha)-1} \exp(-\lambda_{T/2}^2)\} \leqq \exp\{t A^2 - \Gamma_1 e^{A/2}\},$$

Γ_1 being some positive constant. (Notice that for $0 \leqq A \leqq A_1(T)$, $\lambda_T/c_T \to 1$, $a_T/c_T \to 1$ and $[T]\, c_T^{(2/\alpha)-1} e^{-a_T^2/2} \to$ const as $T \to \infty$.) Thus there exists T_1 such that for all $T \geqq T_1$ and $0 \leqq A \leqq A_1(T)$,

$$E(A, T) \leqq h(T) + \exp(t A^2 - \Gamma_1 e^{A/2}). \qquad (2.13)$$

For the remaining values of A we will now consider a different blocking and partitioning. Divide the interval $[0, T]$ into $[T/L(T)]$ consecutive blocks of length $L(T)$ each. Exclude every other interval. We then have $m = [\frac{1}{2}([T/L(T)]+1)]$ intervals left. Divide each such interval into $[L(T)/\theta]$ intervals of length θ. Exclude every other so that there are $m_1 = [\frac{1}{2}([L(T)/\theta]+1)]$ intervals inside every interval of length $L(T)$ chosen above. Let the union of these intervals be denoted by I_2, i.e.,

$$I_2 = \bigcup_{i=0}^{m} \bigcup_{j=0}^{m} [2iL(T) + 2j\theta, 2iL(T) + (2j+1)\theta).$$

Define

$$G_2 = \{j\theta_0(A, T) \mid j = 0, 1, \ldots, [T\theta_0(A, T)]\}$$

where $\theta_0(A, T) = \min(\theta, e^{A/2} c_T^{-2/\alpha})$. Then

$$P(M_T \leqq \lambda_T) \leqq P(\max_{s \in G_2 \cap I_2} X(s) \leqq \lambda_T).$$

Consider the variable

$$Z_{ijk} = (1-\delta_{\theta_0})^{\frac{1}{2}}\, Y_{ijk} + (\delta_{\theta_0} - \delta_\theta)^{\frac{1}{2}}\, W_{ij} + (\delta_\theta - \delta_{L(T)})^{\frac{1}{2}}\, U_i + \delta_{L(T)}^{\frac{1}{2}}\, V$$

where Y_{ijk}'s, W_{ij}'s, U_i's and V are all mutually independent normal variables with mean zero variance one, $1 \leqq i \leqq m$; $1 \leqq j \leqq m_1$; $1 \leqq k \leqq m_2 = \max\{1, [\theta c_T^{2/\alpha} e^{-A/2}]\}$ and $\delta_{\theta_0} = \delta_{\theta_0(A,T)}$. The covariance matrix of $\{X(s)\colon s \in G_2 \cap I_2\}$ is bounded above by that of the Z_{ijk}'s. (See the discussion leading to (2.1), (2.2) and (2.3) in [5].) Using Slepian's Lemma [14, Lemma 1], we have

$$P(\max_{s \in G_2 \cap I_2} X(s) \leqq \lambda_T) \leqq P(Z_{ijk} \leqq \lambda_T \ \forall\, i, j, k)$$
$$\leqq \Phi(-A/(4 c_T \delta_{L(T)}^{\frac{1}{2}})) \qquad (2.14)$$
$$+ \{P[(1-\delta_{\theta_0})^{\frac{1}{2}}\, Y_{jk} + (\delta_{\theta_0} - \delta_\theta)^{\frac{1}{2}}\, W_j + \delta_\theta^{\frac{1}{2}}\, U \leqq d_T \ \forall i, j]\}^m$$

where Y_{jk}'s, W_i's and U are again independent standard normal variables and d_T is defined in (2.5). By the same arguments as in [5] it is sufficient to show that $\exp(tA^2) \times \{$second term in 2.14$\}$ tends to zero uniformly in T as $A \to \infty$ for t sufficiently small. We will bound this by $f(A, T)$ for $A_{\frac{1}{2}} \leqq A \leqq 4(1-\rho)\, a_T c_T/3$ and by $g(A, T)$ for $A > 4(1-\rho)\, a_T c_T/3$ (cf. (2.17)). ρ is to be in $(0, 1)$ and will be chosen later, $A_1 = A_1(T) = (\ln\ln T)^{\frac{1}{4}}$. Using (2.7) we get the following upper bound for the second term of (2.14):

$$\{\delta_\theta(1-\delta_\theta^2)(m_1 m_2)^2 \exp(-d_T^2(1-\delta_{L(T)})(1+\delta_\theta)^{-1}) \\ + \delta_{\theta_0}(1-\delta_{\theta_0}^2) m_1 m_2^2 \exp(-d_T^2(1-\delta_{L(T)})/(1+\delta_{\theta_0})) + \Phi^{m_1 m_2}(d_T)\}^m. \tag{2.15}$$

From now up to (2.17), we assume $A_1/2 \leqq A \leqq 4(1-\rho)\, a_T c_T/3$. Since $a_T - 3A/(4c_T) \geqq \rho\, a_T$, the first term of (2.15) is bounded above by

$$\text{const} \cdot m_2^2 (T c_T^{(2/\alpha)-1})^{2\gamma - 2\rho^2/(1+\delta_\theta)}.$$

This last expression, call it $S_1(T)$, is $o(T^{-(1-\gamma)})$ as $T \to \infty$ since we choose ρ such that $2\rho^2/(1+\delta_\theta) - 2\gamma > 1-\gamma$.

We show that the second term in (2.15) is also $o(T^{-(1-\gamma)})$ as $T \to \infty$. First notice that if $A > 2\ln(\theta c_T^{2/\alpha})$ then the term in question is just $S_1(T)/m_1$. Secondly for $A < 2\ln(\theta c_T^{2/\alpha})$ it is no bigger than

$$\text{const} \cdot m_1 m_2^2 (e^{A/2} c_T^{-2/\alpha})^{-\alpha/2} \exp(-d_T^2(1-\delta_{L(T)})/(1+\delta_{\theta_0})) \tag{2.16}$$

in view of (2.2). Notice that $m_1 m_2^2 \sim L(T)\, c_T^{4/\alpha} e^A$ and

$$\exp(-d_T^2(1-\delta_{L(T)})/(1+\delta_{\theta_0})) \\ \sim (T c_T^{(2/\alpha)-1})^{-1} \exp\{3a_T - A/(4c_T) - (1-\delta_{\theta_0})\, a_T^2/2(1+\delta_{\theta_0})\}$$

as $T \to \infty$, for the values of A under consideration. We also see that

$$1 - \delta_{\theta_0} = 1 - r(e^{A/2} c_T^{-2/\alpha}) \geqq \{C \exp(\alpha A_1/4)\}/(2c_T^2).$$

Thus (2.16) is at most

$$\text{const} \cdot T^{-(1-\gamma)} c_T^{(2/\alpha)+2} \exp\{-\text{const} \cdot \exp(\alpha A_1/4)\}.$$

Call this bound $S_2(T)$. Recall $A_1 = (\ln\ln T)^{\frac{1}{4}}$ and note that $S_2(T)$ is $o(T^{-(1-\gamma)})$ as $T \to \infty$.

Lastly, by similar arguments as in (5) (see (2.9) and (2.10)), for all $A \leqq 4(1-\rho) \cdot a_T c_T/3$, and $\rho^2 > \gamma$

$$\Phi^{m_1 m_2}(d_T) \geqq \Phi^{m_1 m_2}(\rho\, a_T) \to 1 \quad \text{as } T \to \infty.$$

Thus for T sufficiently large, (2.15) is no bigger than

$$\Phi^{m m_1 m_2}(d_T)\{1 + 2S_1(T) + 2S_2(T)\}^m \leqq 2\{\Phi(d_T)\}^{\frac{mL(T)}{2}} c_T^{2/\alpha} e^{-A/2}.$$

This is our first bound for the second term in (2.14). Next, note that we arrived at this second term in (2.14) by selecting $m_1 m_2$ variables in each block of length $L(T)$. For the second bound we select only one variable in each block of length $L(T)$. Thus there exists T_2 such that

$$\exp(tA^2) \times \{\text{second term of (2.14)}\} \leqq \begin{cases} f(A, T) & \forall (T, A) \in \mathscr{R} \\ g(A, T) & \forall A \text{ and } T \end{cases} \tag{2.17}$$

where

$$f(A, T) = 2\exp(tA^2)\{\Phi(d_T)\}^{\frac{mL(T)}{2}} c_T^{2/\alpha} e^{-A/2}, \tag{2.18}$$

$$g(A, T) = \exp(tA^2)\, \Phi^m(d_T) \tag{2.19}$$

and
$$(T, A)\in\mathscr{R} \quad \text{if } T\geqq T_2 \quad \text{and} \quad \frac{A_1}{2}\leqq A\leqq\frac{4(1-\rho)\,a_T c_T}{3}.$$

We study the function $f(A, T)$ first.

$$\begin{aligned}\frac{df(A,T)}{dA}=f(A,T)\{&2tA-\tfrac{1}{4}e^{-A/2}mL(T)\,c_T^{2/\alpha}\ln\Phi(d_T)\\ &-\tfrac{1}{2}e^{-A/2}mL(T)\,c_T^{2/\alpha}\Phi^{-1}(d_T)\,\varphi(d_T)\big(3/(4c_T(1-\delta_{L(T)})^{\frac{1}{2}})\big)\}.\end{aligned}\tag{2.20}$$

Since $a_T-3A/(4c_T)\to\infty$ as $T\to\infty$ for all $A\leqq 4(1-\rho)\,a_T c_T/3$, $-\ln\Phi(d_T)\sim\varphi(d_T)/d_T$ and for T large enough the term multiplying $f(A, T)$ in (2.20) is at most

$$2tA-\tfrac{1}{2}e^{-A/2}mL(T)\,c_T^{2/\alpha}\varphi(d_T)\,d_T^{-1}\{-\tfrac{1}{2}+3d_T\Phi^{-1}(d_T)/(4c_T(1-\delta_{L(T)})^{\frac{1}{2}})\}. \tag{2.21}$$

Now as $T\to\infty$ $d_T\Phi^{-1}(d_T)/\big(c_T(1-\delta_{L(T)})^{\frac{1}{2}}\big)\to 1$ and

$$\begin{aligned}\varphi(d_T)&\sim(Tc_T^{(2/\alpha)-1})^{-1}\exp\left\{\frac{3}{4}\frac{a_T}{c_T}A\left(1-\frac{3}{8}\frac{A}{c_T a_T}\right)-\frac{1}{2}\delta_{L(T)}a_T^2\right\}\\ &\geqq(Tc_T^{(2/\alpha)-1})^{-1}\exp\left\{\frac{3(1+\rho)}{8}\frac{a_T}{c_T}A\right\}\text{const}\end{aligned}$$

for the values of A under consideration. Choosing $\rho>\frac{1}{2}$ and substituting we see that (2.21) is at most

$$2TA-\text{const}\cdot\exp\{A/16\} \tag{2.22}$$

where the constant is positive. We assume that $0<\rho<1$, $0<\gamma<1$ are chosen so that $\rho^2>\gamma$, $\rho>\frac{1}{2}$ and $(2\rho^2/(1+\delta_\theta))-2\gamma>1-\gamma$. We will also require later that $1-\gamma-(\rho^2/(1+\delta_{L(T)}))>0$. Such a choice is possible if e.g. $1>\rho^2>(1+\delta_\theta)/2$ and $0<\gamma<\min\left(\frac{2\rho^2}{1+\delta_\theta}-1;\,1-\frac{\rho^2}{1-\delta_{L(T)}}\right)$. By choosing t sufficiently small we see that (2.22) is at most $-\varepsilon_0$ $\forall A\geqq 0$ and some small $\varepsilon_0>0$. Thus for all $A\leqq 4(1-\rho)\cdot a_T c_T/3$ and T large

$$\left\{\frac{d}{dA}f(A,T)\right\}\Big/f(A,T)\leqq-\varepsilon_0.$$

Integrating both sides between $A_1/2$ and A we have

$$f(A,T)\leqq f(A_1/2,T)\exp\left(-\varepsilon_0\left(A-\frac{A_1}{2}\right)\right) \tag{2.23}$$

for all $A_1/2\leqq A\leqq 4(1-\rho)\,a_T c_T/3$ and T large. Now

$$\begin{aligned}f(A_1/2,T)&=2\exp(tA_1^2)\,\{\Phi(d_T)\}^{\frac{mL(T)}{2}c_T^{2/\alpha}e^{-A_1/2}}\\ &=2\exp\left\{tA_1^2+\frac{mL(T)}{2}c_T^{2/\alpha}e^{-A_1/2}\ln\Phi(d_T)\right\}\\ &\leqq 2\exp\{tA_1^2-\text{const}\cdot mL(T)\,c_T^{2/\alpha}e^{-A_1/2}\varphi(d_T)\,d_T^{-1}\}.\end{aligned}$$

Expanding $\varphi(d_T)$ for $A=A_1$ we see that $f(A_1/2, T)\to 0$ as $T\to\infty$. Thus there exists T_3 such that $\forall T\geqq T_3$,

$$\max\{f(A,T)\,|\,A_1\leqq A\leqq 4(1-\rho)\,a_T c_T/3\}\leqq\exp\big(-(\varepsilon_0 A_1)/2\big). \tag{2.24}$$

By similar arguments as in [5], (see discussion after (2.16)) we have for sufficiently small values of t and $A > 4a_T c_T$, $\frac{d}{dA} g(A, T) < 0$. Also

$$\mathcal{G}(T) \to 0 \quad \text{as } T \to \infty \tag{2.25}$$

where

$$\mathcal{G}(T) = \max\{g(A, T) \mid 4(1-\rho)\, a_T c_T/3 \leqq A \leqq 4 a_T c_T\}.$$

Combining (2.13), (2.14), (2.24) and (2.25) it follows that there exists T_4 such that $\forall T \geqq T_4$

$$E(A, T) \leqq \begin{cases} h(T) + \exp(tA^2 - \Gamma_1 e^{A/2}) & \text{if } 0 \leqq A \leqq A_1 \\ \exp(tA^2)\, \Phi\big(-A/(4c_T \delta^{\frac{1}{2}}_{L(T)})\big) + \exp\big(-(\varepsilon_0 A_1)/2\big) & \\ \qquad \text{if } A_1 \leqq A \leqq 4(1-\rho)\, a_T c_T/3 & \\ \exp(tA^2)\, \Phi\big(-A/(4c_T \delta^{\frac{1}{2}}_{L(T)})\big) + \mathcal{G}(T) & \\ \qquad \text{if } A > 4(1-\rho)\, c_T a_T/3. & \end{cases} \tag{2.26}$$

Given $\eta > 0$, choose A^* so large that for all $A \geqq A^*$

$$\exp(tA^2 - \Gamma_1 e^{A/2}) < \eta/2 \quad \text{and} \quad \exp(tA^2)\, \Phi\big(-A/(4c_T \delta^{\frac{1}{2}}_{L(T)})\big) < \eta/2.$$

The functions $\mathcal{G}(T)$, $h(T)$ and $\exp\big(-(\varepsilon_0 A_1)/2\big)$ depend only on T. Thus we can choose T_0 so large that for all $T \geqq T_0$ each one of these functions is no bigger than $\eta/2$. Substituting in (2.26) we get (2.4) and the lemma is proved.

Proof of Theorem 1. Let $U_T = 2(M_T - b_T)\, c_T/\ln\ln T$ where $b_T = c_T + \frac{1}{\alpha} \frac{\ln\ln T}{c_T}$ (c_T is defined in (1.2)). We will establish that

$$\limsup_{T\to\infty} U_T = 1 \quad \text{a.s.} \tag{2.27}$$

and

$$\liminf_{T\to\infty} U_T = -1 \quad \text{a.s.} \tag{2.28}$$

We assume (1.1) and (1.3) hold.

Our first consideration is the lim sup. Lim sup $U_T \geqq 1$ a.s. is a consequence of Lemma 2 and lim sup $U_T \leqq 1$ a.s. follows from Theorem 3.1 of [10].

Next we show that $\liminf_{T\to\infty} U_T \geqq -1$ a.s. By Lemma 3.6 of [10] it is sufficient to show that for every $\varepsilon > 0$, there exists $\Delta > 1$ so that

$$\lim_{T\to\infty} (\ln T)^{\Delta}\, P\left(M_T \leqq c_T + \left(\frac{1}{2} - \frac{1}{\alpha} - \varepsilon\right) \frac{\ln\ln T}{c_T}\right) = 0$$

or that

$$\lim_{T\to\infty} \exp(\Delta \ln\ln T)\, P\big(M_T \leqq a_T \leqq a_T - (\varepsilon \ln\ln T)/c_T\big) = 0. \tag{2.29}$$

Lemma 1 will obviously imply (2.29) under the Conditions (1.1) and (1.6). We show here that for the weaker conclusion (2.29), we can replace (1.6) by (1.3). If one inspects the proof of Lemma 1, all the arguments leading to (2.14), (2.17) and (2.24) use only the fact that $r(s) \ln s \to 0$ as $s \to \infty$. Hence there exists S_0 fixed so that for all $T \geqq S_0$ and for all A such that $(\ln\ln T)^{\frac{1}{4}} \leqq A \leqq 4(1-\rho)\, a_T c_T/3$,

$$\begin{aligned} \exp(tA^2)\, P(M_T \leqq a_T - A/c_T) &< \exp(tA^2)\, \Phi\big(-A/(4c_T \delta^2_{L(T)})\big) \\ &\quad + \exp\big(-\varepsilon_0 (\ln\ln T)^{\frac{1}{4}}/2\big). \end{aligned} \tag{2.30}$$

Furthermore, given $\varepsilon>0$, there exists S_1 such that for all $T\geqq S_1$, $(\ln\ln T)^{\frac{1}{2}}\leqq \varepsilon(\ln\ln T)\leqq 4(1-\rho)\,a_T c_T/3$. Substituting $A=\varepsilon\ln\ln T$ in (2.30) we get

$$\exp\left(t\varepsilon^2(\ln\ln T)^2\right)P\left(M_T\leqq a_T-\frac{\varepsilon\ln\ln T}{c_T}\right)\leqq v(T)$$

for all $T\geqq\max(S_0,S_1)$ where $v(T)\to 0$ as $T\to\infty$. This implies (2.29).

Lastly, $\liminf_{T\to\infty} U_T\leqq -1$ a.s. if $\lim_{T\to\infty} P\left(U_T>-(1-\varepsilon)\right)=0$. Given positive constants η and ε, let τ_1^* be so large that for all $T\geqq\tau_1^*$

$$1-\exp\left(-\exp\left(-\frac{\varepsilon}{2}\ln\ln\tau_1^*\right)\right)<\eta/2.$$

The limit distribution of the normalized M_T (cf. Theorem 2.1 of [10]) enables us to find τ_2^* (depending on ε and η) such that for all $T\geqq\tau_2^*$

$$P\left(M_T>a_T+(\varepsilon\ln\ln\tau_1^*)/(2c_T)\right)\leqq\left\{1-\exp\left(-\exp\left(-\frac{\varepsilon}{2}\ln\ln\tau_1^*\right)\right)\right\}+\eta/2.$$

Thus for $T\geqq\max(\tau_1^*,\tau_2^*)$

$$\begin{aligned}P\left(U_T>-(1-\varepsilon)\right)&=P\left(M_T>a_T+(\varepsilon\ln\ln T)/c_T\right)\\&\leqq P\left(M_T>a_T+(\varepsilon\ln\ln\tau_1)/(2c_T)\right)\leqq\eta.\end{aligned}$$

This completes the proof of Theorem 1.

Proof of Theorem 2. Our aim is to show that $\liminf_{T\to\infty} U_T=-\infty$ a.s. if (1.1) and (1.9) hold. Given K, let A_T be the event $(M_T-c_T)c_T\geqq -K\ln\ln T$. We will show that for any K, and every fixed T_0,

$$P(A_T,\forall T\geqq T_0)\leqq\lim_{T\to\infty}P(A_T)=0.$$

We approximate M_T by the maximum over a dense enough subset of $[0,T]$. Let $\tau=[Tc_T^{5/\alpha}]$, $l_T=K\ln\ln T/c_T$ and $m_\tau=\max\{X(ic_T^{-5/\alpha})\colon 1\leqq i\leqq\tau\}$. Then

$$P(A_T)=P(M_T\geqq c_T-l_T)\leqq P(m_\tau\geqq c_T-2l_T)+D_\tau$$

where

$$D_\tau=P\{M_T\geqq c_T-l_T;\ m_\tau-2l_T\}.$$

For a fixed T, let Z_i, $i=1,2\ldots\tau$ be independent standard normal variables and let $M_\tau^*=\max\{Z_i\colon 1\leqq i\leqq\tau\}$. Let U be a standard normal variable independent of all Z_i's. m_τ is the maximum of τ joint normal variables with correlations at least $r(T)$, and so by Slepian's Lemma, as in the proof of Theorem 2 of [5], we have

$$\begin{aligned}P(m_\tau\geqq c_T-2l_T)&\leqq P\{(1-r_{(T)})^{\frac{1}{2}}M_\tau^*+r^{\frac{1}{2}}(T)U\geqq c_T-2l_T\}\\&\leqq\left\{1-\Phi\left(c_T r^{\frac{1}{2}}(T)/4\right)\right\}+P\{(M_\tau^*-c_\tau)c_\tau\geqq\Gamma_1 r(T)\ln T-\Gamma_2\ln\ln T\}\end{aligned}$$

for some positive constants Γ_1 and Γ_2. Both the terms in right hand side above tend to zero and Theorem 2 will be proved if we show that $D_\tau\to 0$ as $T\to\infty$. By stationarity of X,

$$\begin{aligned}D_\tau\leqq\tau P\{&X(0)<c_T-2l_T;\ X(c_T^{-5/\alpha})<c_T-2l_T;\\&\max(X(s)\colon 0\leqq s\leqq c_T^{-2/\alpha})>c_T-l_T\}.\end{aligned}\tag{2.31}$$

Following Berman [4], Lemma 3.5, the event described in (2.31) implies that for some $n \geqq 1$, some j, $1 \leqq j \leqq 2^n$ and every Δ, $0<\Delta<1$,

$$X(j2^{-n}c_T^{-5/\alpha}) - X((j-1)2^{-n}c_T^{-5/\alpha}) > l_T \Delta^{n-1}(1-\Delta).$$

For, if the alternative inequality held for every n and j and for some Δ, $0<\Delta<1$, we would have

$$\sup_{j,n} X(j2^{-n}c_T^{-5/\alpha}) \leqq c_T - l_T.$$

The continuity of X then would imply a contradiction, viz.,

$$\max\{X(s):\ 0 \leqq s \leqq c_T^{-5/\alpha}\} \leqq c_T - l_T.$$

Thus the right hand side of (2.31) is at most

$$\tau \sum_{n=1}^{\infty} 2^n \left\{1 - \Phi\left(\frac{l_T \Delta^{n-1}(1-\Delta)}{(2(1-r(2^{-n}c_T^{-5/\alpha})))^{\frac{1}{2}}}\right)\right\}. \tag{2.32}$$

We can choose T so large that for all n

$$(1-r(2^{-n}c_T^{-5/\alpha}))^{\frac{1}{2}} \leqq (2\,C)^{\frac{1}{2}}(2^{-n}c_T^{-5/\alpha})^{\alpha/2}.$$

Then (2.32) is at most

$$\frac{2\tau C^{\frac{1}{2}}\Delta}{(1-\Delta)\, l_T c_T^{5/\alpha}} \sum_{n=1}^{\infty} (2^{(1-\alpha/2)}\Delta^{-1})^n \exp\{-\mathrm{const}\cdot(\ln\ln T)^2(\Delta^2 2^\alpha)^n c_T^3\},$$

the constant above being positive.

By the Cauchy-Schwarz inequality, (2.32) is at most

$$\tau\left\{\sum_{n=1}^{\infty}(2^{(1-(\alpha/2))}\Delta^{-1})^n \exp\{-(\Delta^2 2^\alpha)^n\}\right\}^{\frac{1}{2}} \left\{\sum_{n=1}^{\infty}\exp(-(\Delta^2 2^\alpha)^n c_T^3)\right\}^{\frac{1}{2}}$$

for large T. Each of the series above converges if we choose Δ so that $\Delta^2 2^\alpha > 1$. The last expression then tends to zero as $T \to \infty$ since $\tau = \exp\{\frac{1}{2}c_T^2 + 5\ln c_T/\alpha\}$. Hence the result.

Proof of Theorem 3. We show that

$$\lim_{T\to\infty} E(\exp(t\,Y_T)) = E(\exp(t\,X))$$

for all t sufficiently small if (1.1) and (1.6) hold. $Y = c_T(M_T - a_T)$ and a_T is defined in (1.2).

Following the arguments of Theorem 3 [5], Lemma 1 and (2.33) will be sufficient for the desired conclusion.

$$\exp(t\,A)\,P(M_T > a_T + A/c_T) \to 0 \quad \text{as } A \to \infty \tag{2.33}$$

for all $t>0$ sufficiently small, the convergence being uniform in T. By stationary,

$$\exp(t\,A)\,P(M_T > a_T + A/c_T) \leqq ([T]+1)\exp(t\,A)\,P(M_1 > a_T + A/c_T), \tag{2.34}$$

$M_1 = \max\{X(s): 0 \leqq s \leqq 1\}$. According to Lemma 2.9 of (9) there exists a positive constant and T_0 fixed such that for all $T > T_0^*$,

$$P(M_1 > a_T + A/c_T) \leqq \text{const} \cdot (a_T + A/c_T)^{(2/\alpha)-1} \exp(-(a_T + A/c_T)^2/2)$$
$$\leqq \text{const} \cdot (Tc_T^{(2/\alpha)-1})^{-1} (a_T + A/c_T)^{(2/\alpha)-1} \exp(-A/2).$$

Thus given $\eta > 0$ there exists a^* such that

$$(\text{Left Hand Side of } (2.34)) < \eta \tag{2.35}$$

for $t < \frac{1}{2}$, $A > a^*$ and $T \geqq T_0^*$. For $T \leqq T_0^*$ notice that

$$P(M_T > a_T + A/c_T) \leqq P(M_{T_0^*} > A/c_{T_0^*}). \tag{2.36}$$

Again using Lemma (2.9) of [9], we can choose A large enough that the right hand side of (2.36) is no bigger than η, and the result follows.

Corollaries 1 and 2 can be proved in the same manner as Corollaries 1 and 2 of [5].

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