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Some Invariance Principles for Rank Statistics for Testing Independence

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1. Introduction

In the context of testing the hypothesis of stochastic independence, along with a martingale property of a class of rank order statistics, a functional central limit theorem, an almost sure (a.s.) invariance principle and a law of iterated logarithm for such statistics are established. Almost sure convergence of these statistics to appropriate centering constants is also proved under weaker regularity conditions. These results are then incorporated in the study of the asymptotic theory of some sequential rank tests for independence.

Let $\{\mathbf{Z}_i=(X_i, Y_i), i\geqq 1\}$ be a sequence of independent and identically distributed random vectors (iidrv) with each $\mathbf{Z}_i$ having a continuous (bivariate) distribution function (df) $H(x,y)$, $-\infty<x, y<\infty$. We denote the two marginal df's by $F(x)=H(x,\infty)$ and $G(y)=H(\infty,y)$. The corresponding empirical df's based on a sample of size $n(\geqq 1)$ are denoted by

$$H_n(x,y)=n^{-1}\sum_{i=1}^{n} u(x-X_i)\,u(y-Y_i),\qquad F_n(x)=H_n(x,\infty)=n^{-1}\sum_{i=1}^{n} u(x-X_i)$$

and

$$G_n(y)=H_n(\infty,y)=n^{-1}\sum_{i=1}^{n} u(y-Y_i),$$

where $u(t)$ is equal to 1 when $t\geqq 0$ and is 0, otherwise. Let $R_{n,i}=\sum_{j=1}^{n} u(X_i-X_j)$ $\left(\text{and } S_{n,i}=\sum_{j=1}^{n} u(Y_i-Y_j)\right)$ be the rank of X_i (and Y_i) among $X_1,\ldots,X_n$ (and $Y_1,\ldots,Y_n$), for $i=1,\ldots,n$. By virtue of the assumed continuity of F and G, ties among the X_i (or the Y_i) can be neglected in probability, so that the ranks are the natural integers $1,\ldots,n$, permuted in certain order. For testing the hypothesis of stochastic independence, viz.,

$$H_0\colon\ H(x,y)=F(x)\,G(y)\qquad \text{for all } -\infty<x, y<\infty, \tag{1.1}$$

a general class of nonparametric tests are based on the following type of rank order statistics:

$$T_n=\sum_{i=1}^{n} J_n(R_{n,i})\,L_n(S_{n,i}),\qquad n\geqq 1, \tag{1.2}$$

* Part of this work was done while the first author was visiting the Indian Statistical Institute, Calcutta.

where $J_n(i)=EJ(U_{ni})$, $L_n(i)=EL(U_{ni})$, $i=1,\dots,n$, $U_{n1}\leqq\cdots\leqq U_{nn}$ are the ordered random variables of a sample of size n from the rectangular (0, 1) df, and $J(u)$, $L(u)$, $0<u<1$, are non-decreasing score functions such that for some $r(\geqq 2)$, to be specified later on,

$$\int_0^1 |J(u)|^r du<\infty \quad \text{and} \quad \int_0^1 |L(u)|^r du<\infty. \tag{1.3}$$

Without any loss of generality, we may standardize the score functions by letting

$$\int_0^1 J(u)\,du=\int_0^1 L(u)\,du=0 \quad \text{and} \quad \int_0^1 J^2(u)\,du=\int_0^1 L^2(u)\,du=1. \tag{1.4}$$

Well-known particular cases of (1.1) are the Spearman rank covariance and the normal scores statistics which correspond to $J(u)=L(u)=\sqrt{12}(u-1/2)$ and $\Phi^{-1}(u)$, respectively, where $\Phi(x)$ is the standard normal df.

Under (1.1), when (1.3) holds for $r=2$, $n^{-1/2}T_n$ has asymptotically the standard normal distribution [cf. Hájek and Šidák (1967, p. 168)]. Under no extra regularity conditions, in our Theorem 1, we strengthen this result to a Donsker type invariance principle (or a functional central limit theorem) for the process $\{n^{-1/2}T_k, 1\leqq k\leqq n\}$. Strassen (1967) has considered an elegant a.s. invariance principle for sums of independent random variables and martingales. In our Theorem 2, a similar result is proved for the tail sequence $\{T_k, k\geqq n\}$ when (1.3) holds for some $r>2$ and an appropriate growth condition is imposed on the first derivative of the score functions $J(u)$ and $L(u)$. A basis for the proofs of the two theorem is a fundamental martingale property of $\{T_k, k\geqq 1\}$, which is proved in Lemma 3.1. Let us now define

$$\mu=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} J(F(x))\,L(G(y))\,dH(x,y). \tag{1.5}$$

Note that by (1.4), when H_0 in (1.1) holds, $\mu=0$. When (1.1) does not hold, μ appears as a centering constant in the asymptotic normality of the standardized form of T_n [viz., Chapter 8 of Puri and Sen (1971)]. The a.s. convergence of $n^{-1}T_n$ to μ is established in Theorem 3 under milder regularity conditions.

The last section of the paper deals with some sequential procedures for testing the null hypothesis in (1.1). First, along the lines of Darling and Robbins (1968), a class of sequential rank tests for (1.1) is proposed for which the power is equal to 1 and the type I error can be made arbitrarily close to 0. In this context, Theorems 2 and 3 are of great help. Secondly, along the lines of Sen (1973) and Sen and Ghosh (1974), a class of sequential rank tests for (1.1) is proposed which asymptotically attains a prescribed strength and behaves like the sequential probability ratio test. In this context, Theorems 1 and 2 play a fundamental role. Finally, Theorems 1 and 2 are also of considerable value in studying the asymptotic behavior of T_n when the sample size n is itself a random variable.

2. Statement of the Results

Consider the space $C[0,1]$ of real valued continuous functions on $I=[0,1]$ and associate with it the uniform topology with the metric

$$\rho(x,y)=\sup_{t\in I}|x(t)-y(t)|, \qquad x,y\in C[0,1]. \tag{2.1}$$

Let $T_0=0$ and for every $n\geqq 1$, define a process $W_n=\{W_n(t), t\in I\}$ by letting

$$W_n(k/n)=n^{-1/2}T_k, \quad k=0,1,\ldots,n; \tag{2.2}$$

$$W_n(t)=W_n(k/n)+(nt-k)\left[W_n((k+1)/n)-W_n(k/n)\right], \quad t\in[k/n,(k+1)/n], \tag{2.3}$$

for $k=0,1,\ldots,n-1$. Note that by (1.2) and (1.4), $T_1=J_1(1)L_1(1)=0$, and hence, $W_1(t)=0$ for $t\in I$. Also, let $W=\{W(t), t\in I\}$ be a standard Brownian motion on I. Then, we have the following functional central limit theorem.

Theorem 1. *Under* (1.1) *and* (1.4), *as* $n\to\infty$, $W_n \xrightarrow{\mathscr{D}} W$ *in the uniform topology on* $C[0,1]$.

The weak convergence of W_n to W in Theorem 1 demands nothing more than the square integrability and non-degeneracy of $J(u)$ and $L(u)$, and is a natural extension of a theorem in Hájek and Šidák (1967, p. 168). Under comparatively stringent regularity conditions and in the same spirit as in Theorem 4.4 of Strassen (1967), we have the following a.s. invariance principle for $\{T_k, k\geqq n\}$ as $n\to\infty$.

For every $n\geqq 1$, we define

$$V_n=\sum_{i=1}^{n} E[(T_i-T_{i-1})^2 \mid T_1,\ldots,T_{i-1}], \quad \tilde{T}_{V_n}=T_n, \tag{2.4}$$

and complete the definition of $\tilde{T}_t$ for every $t\in[V_n, V_{n+1}]$ by linear interpolation. Note that $T_1=0$ implies that $V_1=0$. Assume further that $J(u)$ and $L(u)$ admit continuous first derivatives $J^{(1)}(u)$ and $L^{(1)}(u)$, such that on writing $J(u)=J^{(0)}(u)$ and $L(u)=L^{(0)}(u)$, we have for $r=0,1$,

$$|J^{(r)}(u)|\leqq K[u(1-u)]^{-1/2+\delta-r}, \quad |L^{(r)}(u)|\leqq K[u(1-u)]^{-1/2+\delta-r}, \tag{2.5}$$

where K and $\delta\,(0<\delta\leqq 1/2)$ are positive constants. Then, we have the following.

Theorem 2. *Under* (1.1) *and* (2.5), *there exists a standard Brownian motion* $W=\{W(t)\colon 0\leqq t<\infty\}$ *on* $[0,\infty)$ *such that*

$$\tilde{T}_t=W(t)+o(t^{1/2}) \text{ a.s.,} \quad \text{as } t\to\infty. \tag{2.6}$$

By virtue of (2.4) [viz., $\tilde{T}_{V_n}=T_n$ for $n\geqq 1$], (2.6) and a further result that $n^{-1}V_n\to 1$ a.s., as $n\to\infty$ [see Lemma 3.2], the celebrated law of iterated logarithm for the standard Brownian motion process leads us to a similar result for the rank statistics $\{T_n\}$ which is stated below.

Corollary. *Under* (1.1) *and* (2.5),

$$P\{\limsup_n\,[2n\log\log n]^{-1/2}\,T_n=1\}=1, \tag{2.7}$$

$$P\{\liminf_n\,[2n\log\log n]^{-1/2}\,T_n=-1\}=1. \tag{2.8}$$

Finally, we consider the a.s. convergence of $n^{-1}T_n$ when (1.1) may not hold.

Theorem 3. *If both* $J(u)$ *and* $L(u)$ *are continuous and square integrable, then*

$$n^{-1}T_n\to\mu \text{ a.s.,} \quad \text{as} \quad n\to\infty, \tag{2.9}$$

where μ *is defined by* (1.5).

The proofs of the theorems are deferred to the following section. The sequential procedures are considered in the last section.

3. Proofs of the Theorems

Let $\mathscr{F}_n^{(1)}$ be the σ-field generated by $(R_{n,1}, \dots, R_{n,n})$, $\mathscr{F}_n^{(2)}$ be the σ-field generated by $(S_{n,1}, \dots, S_{n,n})$, and $\mathscr{F}_n$ be the σ-field generated by $(R_{n,1}, \dots, R_{n,n}; S_{n,1}, \dots, S_{n,n})$ when (1.1) holds i.e., when $(R_{n,1}, \dots, R_{n,n})$ and $(S_{n,1}, \dots, S_{n,n})$ are stochastically independent, for $n \geqq 1$. Then $\mathscr{F}_n$ is non-decreasing in n.

Lemma 3.1. *Under* (1.1), $\{T_n, \mathscr{F}_n; n \geqq 1\}$ *is a martingale.*

Proof. By (1.2), for every $n \geqq 1$,

$$E(T_{n+1}|\mathscr{F}_n) = \sum_{i=1}^{n} E\{J_{n+1}(R_{n+1,i}) L_{n+1}(S_{n+1,i})|\mathscr{F}_n\} + E\{J_{n+1}(R_{n+1,n+1}) L_{n+1}(S_{n+1,n+1})|\mathscr{F}_n\}. \tag{3.1}$$

Now, under (1.1),

$$\begin{aligned} &E(J_{n+1}(R_{n+1,n+1}) L_{n+1}(S_{n+1,n+1})|\mathscr{F}_n\} \\ &= E\{J_{n+1}(R_{n+1,n+1})|\mathscr{F}_n^{(1)}\} E\{L_{n+1}(S_{n+1,n+1})|\mathscr{F}_n^{(2)}\} \\ &= \left[(n+1)^{-1} \sum_{i=1}^{n+1} J_{n+1}(i)\right] \left[(n+1)^{-1} \sum_{i=1}^{n+1} L_{n+1}(i)\right] \\ &= \left[\int_0^1 J(u)\, du\right] \left[\int_0^1 L(u)\, du\right] = 0, \end{aligned} \tag{3.2}$$

where the last line follows from the definition of J_n, L_n and from (1.4). Also, given $\mathscr{F}_n$, $R_{n+1,i}$ can assume the two values $R_{n,i}$ and $(R_{n,i}+1)$ with respective conditional probabilities $1-(n+1)^{-1} R_{n,i}$ and $(n+1)^{-1} R_{n,i}$; $S_{n+1,i}$ can assume the two values $S_{n,i}$ and $(S_{n,i}+1)$ with respective conditional probabilities $1-(n+1)^{-1} S_{n,i}$ and $(n+1)^{-1} S_{n,i}$, and $R_{n+1,i}$, $S_{n+1,i}$ are stochastically independent, for $i=1, \dots, n$. Hence,

$$\begin{aligned} &\sum_{i=1}^{n} E\{J_{n+1}(R_{n+1,i}) L_{n+1}(S_{n+1,i})|\mathscr{F}_n\} \\ &= \sum_{i=1}^{n} \{[(1-(n+1)^{-1} R_{n,i}) J_{n+1}(R_{n,i}) + (n+1)^{-1} R_{n,i} J_{n+1}(R_{n,i}+1)] \\ &\quad \cdot [(1-(n+1)^{-1} S_{n,i}) L_{n+1}(S_{n,i}) + (n+1)^{-1} S_{n,i} L_{n+1}(S_{n,i}+1)]\} \\ &= \sum_{i=1}^{n} J_n(R_{n,i}) L_n(S_{n,i}), \end{aligned} \tag{3.3}$$

as by the recurrance relation among the expected order statistics [cf. David (1970, p. 36)], for every $i(=1, \dots, n-1)$; $n \geqq 2$,

$$(1-i/n) J_n(i) + (i/n) J_n(i+1) = J_{n-1}(i) \quad \text{when} \quad J_n(i) = EJ(U_{ni}),\ 1 \leqq i \leqq n, \tag{3.4}$$

and a similar relation holds for the $L_n(i)$. Hence the lemma follows from (3.1), (3.2) and (3.3).

Let us now consider the proof of Theorem 1. By virtue of Theorem 8.1 of Billingsley (1968, p. 54), it suffices to show that (*i*) the finite dimensional distributions (f.d.d.) of $\{W_n\}$ converge weakly to those of the standard Brownian motion W, and (*ii*) $\{W_n\}$ is tight. To prove (*i*), let us define

$$T_n^0 = \sum_{i=1}^{n} J_n(R_{n,i})\, L(G(Y_i)), \qquad n \geqq 1. \tag{3.5}$$

Then, by Theorem 3.1 of Hájek (1961), under (1.1) and (1.4),

$$n^{-1} E(T_n - T_n^0)^2 \to 0 \quad \text{as} \quad n \to \infty. \tag{3.6}$$

Let us also introduce

$$T_n^* = \sum_{i=1}^{n} J(F(X_i))\, L(G(Y_i)), \qquad n \geqq 1. \tag{3.7}$$

Since, under (1.1) and given $Y_1, \ldots, Y_n$, all possible $n!$ permutations of $(R_{n,1}, \ldots, R_{n,n})$ [over $(1, \ldots, n)$] are conditionally equally likely, and

$$n^{-1} \sum_{i=1}^{n} L^2(G(Y_i)) \to \int_0^1 L^2(u)\, du = 1 \text{ a.s.}, \quad \text{as} \quad n \to \infty$$

[by the Kintchine strong law of large numbers], first on working with the conditional distribution of (T_n^0, T_n^*) [given $(Y_1, \ldots, Y_n)$] with Theorem 3.1 of Hájek (1961), and then taking expectation over $(Y_1, \ldots, Y_n)$, it follows that under (1.1),

$$n^{-1} E(T_n^0 - T_n^*)^2 \to 0 \quad \text{as} \quad n \to \infty. \tag{3.8}$$

Consequently, by (3.6) and (3.8), under (1.1), $n^{-1} E(T_n - T_n^*)^2 \to 0$ as $n \to \infty$. As a result, for every fixed $m (\geqq 1)$ and arbitrary $t_1, \ldots, t_m (\varepsilon I)$, the two vectors $n^{-1/2}(T_{[nt_1]}, \ldots, T_{[nt_m]})$ and $n^{-1/2}(T^*_{[nt_1]}, \ldots, T^*_{[nt_m]})$ have the same limiting distribution, if they have any at all. Since, T_k^* involves a sum over iidrv's with mean 0 and variance 1, for every $k \geqq 1$, $n^{-1/2}(T^*_{[nt_1]}, \ldots, T^*_{[nt_m]})$ converges in law to a multinormal distribution (as $n \to \infty$) whose covariance matrix has the form $\min(t_i, t_j)$ for $i, j = 1, \ldots, m$. Hence, the convergence of the f.d.d. follows readily.

To prove (ii), we note that by (2.2) and Lemma 3.1, for every $2 \leqq q \leqq k \leqq n$,

$$\begin{aligned} E[W_n(k/n) - W_n(q/n)]^2 &= EW_n^2(k/n) - EW_n^2(q/n) \\ &= [k/(k-1)]\,(k/n) A_k^2 B_k^2 - [q/(q-1)]\,(q/n) A_q^2 B_q^2, \end{aligned} \tag{3.9}$$

where

$$A_k^2 = k^{-1} \sum_{i=1}^{k} J_k^2(i) \quad \text{and} \quad B_k^2 = k^{-1} \sum_{i=1}^{k} L_k^2(i), \qquad k \geqq 1. \tag{3.10}$$

Note that by the definition of the $J_k(i)$,

$$\begin{aligned} A_k^2 &= k^{-1} \sum_{i=1}^{k} [EJ(U_{ki})]^2 \leqq k^{-1} \sum_{i=1}^{k} E[J^2(U_{ki})] \\ &= k^{-1} \sum_{i=1}^{k} k \binom{k-1}{i-1} \int_0^1 J^2(u)\, u^{i-1} (1-u)^{k-i}\, du \\ &= \int_0^1 J^2(u)\, du = 1, \end{aligned} \tag{3.11}$$

and by the results of Hoeffding (1953), as $n\to\infty$, $A_n^2\to 1$. Similarly,

$$B_k^2\leqq 1 \quad \text{for every } k\geqq 1 \quad \text{and} \quad \text{as } n\to\infty,\ B_n^2\to 1. \tag{3.12}$$

Consequently, on noting that $EW_n^2(1/n)=EW_n^2(0/n)=0$ for every $n\geqq 1$, we obtain on letting $[n\delta]\leqq k-q\leqq[n\delta]+1$, $\delta>0$, that (3.9) converges to δ as $n\to\infty$. The proof of (ii) then follows from Lemma 4, (25) and (26) of Brown (1971) along with our Lemma 3.1 and (3.9). Since the steps are identical, the details are omitted.

Consider now the proof of Theorem 2 where we mainly use Theorem 4.4 of Strassen (1967). Here, we require to show that V_n, defined by (2.3), goes to ∞ a.s. as $n\to\infty$. We prove the following stronger result to be used later on proving (2.7)–(2.8).

Lemma 3.2. *Under* (1.1) *and* (2.5), $n^{-1}V_n\to 1$ a.s., *as* $n\to\infty$.

Proof. Let $Q_n=T_{n+1}-T_n$, $n\geqq 1$, so that $V_n=\sum_{i=1}^{n-1}E(Q_i^2\mid\mathscr{F}_i)$, $Q_0=T_1=0$, and for $n\geqq 1$,

$$\begin{aligned}Q_n&=J_{n+1}(R_{n+1,n+1})\,L_{n+1}(S_{n+1,n+1})\\&\quad+\sum_{i=1}^{n}[J_{n+1}(R_{n+1,i})\,L_{n+1}(S_{n+1,i})-J_n(R_{n,i})\,L_n(S_{n,i})]=Q_{n1}+Q_{n2},\quad\text{say}.\end{aligned}\tag{3.13}$$

Now, by the same arguments as in the proof of Lemma 3.1,

$$E(Q_{n1}^2\mid\mathscr{F}_n)=\left[(n+1)^{-1}\sum_{i=1}^{n+1}J_{n+1}^2(i)\right]\left[(n+1)^{-1}\sum_{i=1}^{n+1}L_{n+1}^2(i)\right]=A_{n+1}^2B_{n+1}^2. \tag{3.14}$$

Hence, by (3.11)–(3.12) and (3.14),

$$n^{-1}\sum_{i=1}^{n-1}E(Q_{i1}^2\mid\mathscr{F}_i)\to 1 \quad \text{as } n\to\infty. \tag{3.15}$$

Also, on making use of (3.3), (3.15) and the fact that

$$\begin{aligned}[\{E(Q_{n1}^2\mid\mathscr{F}_n)\}^{1/2}-\{E(Q_{n2}^2\mid\mathscr{F}_n)\}^{1/2}]^2&\leqq E[(Q_{n1}+Q_{n2})^2\mid\mathscr{F}_n]\\&\leqq[\{E(Q_{n1}^2\mid\mathscr{F}_n)\}^{1/2}+\{E(Q_{n2}^2\mid\mathscr{F}_n)\}^{1/2}]^2,\end{aligned}\tag{3.16}$$

the proof of the lemma follows provided it is shown that

$$E(Q_{n2}^2\mid\mathscr{F}_n)\to 0 \quad \text{as } n\to\infty, \tag{3.17}$$

and for this, it suffices to show that as $n\to\infty$,

$$\sum_{i=1}^{n}E\{[J_{n+1}(R_{n+1,i})\,L_{n+1}(S_{n+1,i})-J_n(R_{n,i})\,L_n(S_{n,i})]^2\mid\mathscr{F}_n\}\to 0 \quad \text{a.s.}, \tag{3.18}$$

$$\begin{aligned}\sum_{i\neq j=1}^{n}&E\{[J_{n+1}(R_{n+1,i})\,L_{n+1}(S_{n+1,i})-J_n(R_{n,i})\,L_n(S_{n,i})][J_{n+1}(R_{n+1,j})\,L_{n+1}(S_{n+1,j})\\&-J_n(R_{n,j})\,L_n(S_{n,j})]\mid\mathscr{F}_n\}\to 0 \quad \text{a.s.}\end{aligned}\tag{3.19}$$

Now, on proceeding as in Lemma 3.1, we obtain that

$$\begin{aligned} V[J_{n+1}(R_{n+1,1})|\mathscr{F}_n] &= (n+1)^{-1}(n+1-R_{n,i})[J_{n+1}(R_{n,i})-J_n(R_{n,i})]^2 \\ &\quad + (n+1)^{-1} R_{n,i}[J_{n+1}(R_{n,i}+1)-J_n(R_{n,i})]^2 \\ &= (n+1)^{-2} R_{n,i}(n+1-R_{n,i})[J_{n+1}(R_{n,i}+1)-J_{n+1}(R_{n,i})]^2, \end{aligned}$$

by (3.4). Similarly,

$$V[L_{n+1}(S_{n+1,i})|\mathscr{F}_n] = (n+1)^{-2} S_{n,i}(n+1-S_{n,i})[L_{n+1}(S_{n+1,i})-L_{n+1}(S_{n,i})]^2.$$

Hence, making use of the fact that for independent $U_1, U_2, E(U_1U_2)^2-(EU_1EU_2)^2 = (EU_1)^2 V(U_2)+(EU_2)^2 V(U_1)+V(U_1)V(U_2)$, we have the left hand side of (3.18) equal to

$$\begin{aligned} &\sum_{i=1}^{n} L^2_{n+1}(S_{n+1,i})(n+1)^{-2} R_{n,i}(n+1-R_{n,i})[J_{n+1}(R_{n,i}+1)-J_{n+1}(R_{n,i})]^2 \\ &\quad + \sum_{i=1}^{n} J^2_{n+1}(R_{n+1,i})(n+1)^{-2} S_{n,i}(n+1-S_{n,i})[L_{n+1}(S_{n,i}+1)-L_{n+1}(S_{n,i})]^2 \\ &\quad + \sum_{i=1}^{n}(n+1)^{-4} R_{n,i}(n+1-R_{n,i}) S_{n,i}(n+1-S_{n,i})[J_{n+1}(R_{n,i}+1)-J_{n+1}(R_{n,i})] \\ &\quad \cdot [L_{n+1}(S_{n,i}+1)-L_{n+1}(S_{n,i})] = Q^*_{n1}+Q^*_{n2}+Q^*_{n3}, \quad \text{say.} \end{aligned} \tag{3.20}$$

Note that under (1.1),

$$E(Q^*_{n1}) = B^2_{n+1}\left[\sum_{j=1}^{n}(n+1)^{-2} j(n+1-j)\{J_{n+1}(j+1)-J_{n+1}(j)\}^2\right] = O(n^{-2\gamma}), \tag{3.21}$$

where $\gamma>0$, and the last step follows from Sen and Ghosh (1972, p. 342, lines 4 and 5). In the same manner, it follows by the same technique as in Section 2 (namely, (2.35) through (2.40)) of Sen and Ghosh (1972) that under (2.5), for every positive integer k, $E|Q^*_{n1}|^k = O(n^{-2k\gamma})$, so that if we select k such that $k\gamma = 1+\gamma_0$, $\gamma_0>0$, we have

$$\begin{aligned} P\{\sup_{n\geq n_0}|Q^*_{n1}|>Cn^{-\gamma}\} &\leq \sum_{n\geq n_0} P\{|Q^*_{n1}|>Cn^{-\gamma}\} \\ &\leq \sum_{n\geq n_0} C^{-k} n^{k\gamma} E|Q^*_{n1}|^k = O(n_0^{-\gamma_0}) \to 0 \quad \text{as } n_0\to\infty. \end{aligned} \tag{3.22}$$

Consequently, $|Q^*_{n1}| = O(n^{-\gamma})$ a.s., as $n\to\infty$. Similarly, $|Q^*_{n2}| = O(n^{-\gamma})$ a.s., as $n\to\infty$. Finally, the treatment of Q^*_{n3} follows on the same line, but with added simplicity, because each term of Q^*_{n3} is bounded by $Cn^{-1-2\gamma}$, for some positive $C(<\infty)$. Thus, (3.18) holds. To prove (3.19), we use of the fact that if U_1, U_2, U_3, U_4 are such that U_1, U_2 are independent and U_3, U_4 are also so, then

$$\begin{aligned} &E(U_1U_2U_3U_4) - EU_1EU_2EU_3EU_4 \\ &\quad = EU_1EU_3\,\mathrm{Cov}(U_2,U_4) + EU_2EU_4\,\mathrm{Cov}(U_1,U_3) + \mathrm{Cov}(U_1,U_3)\,\mathrm{Cov}(U_2,U_4). \end{aligned}$$

Thus, writing $U_i = J_{n+1}(R_{n+1,i}) - J_n(R_{n,i})$ and $V_i = L_{n+1}(S_{n+1,i}) - L_n(S_{n,i})$, $i=1,\ldots,n$, and noting that $E(U_i|\mathscr{F}_n)=0$ and $E(V_i|\mathscr{F}_n)=0$ for $i=1,\ldots,n$, we obtain from the above identity that the left hand side of (3.19) is equal to

$$\sum_{i \neq j=1}^{n} \operatorname{Cov}(U_i, U_j|\mathscr{F}_n) \operatorname{Cov}(V_i, V_j|\mathscr{F}_n)$$

whose absolute value is bounded by

$$\begin{aligned}
&\sum_{i \neq j=1}^{n} [\operatorname{Var}(U_i|\mathscr{F}_n)\operatorname{Var}(U_j|\mathscr{F}_n)\operatorname{Var}(V_i|\mathscr{F}_n)\operatorname{Var}(V_j|\mathscr{F}_n)]^{1/2}\\
&= \sum_{i \neq j=1}^{n} \{(n+1)^{-8} R_{n,i}(n+1-R_{n,i}) R_{n,j}(n+1-R_{n,j}) S_{n,i}(n+1-S_{n,i}) S_{n,j}(n+1-S_{n,j})\\
&\qquad \cdot [J_{n+1}(R_{n,i}+1) - J_{n+1}(R_{n,i})]^2 [J_{n+1}(R_{n,j}+1) - J_{n+1}(R_{n,j})]^2 [L_{n+1}(S_{n,i}+1)\\
&\qquad - L_{n+1}(S_{n,i})]^2 [L_{n+1}(S_{n,j}+1) - L_{n+1}(S_{n,j})]^2\}^{1/2} \qquad (3.23)\\
&\leqq \{\max_{1\leqq i\leqq n} (n+1)^{-2}\, i(n+1-i)[L_{n+1}(i+1)-L_{n+1}(i)]^2\}\\
&\quad \cdot \left\{ \sum_{i \neq j=1}^{n} [(n+1)^{-4}\, i(n+1-i)\, j(n+1-j)\right.\\
&\quad \left. \cdot [J_{n+1}(i+1) - J_{n+1}(i)]^2 [J_{n+1}(j+1) - J_{n+1}(j)]^2]^{1/2} \right\}.
\end{aligned}$$

Now, as in (3.21), the first factor on the right hand side of (3.23) is $O(n^{-2\gamma})$, for some $\gamma > 0$, while the second factor is less than

$$\left\{ \sum_{i=1}^{n} (n+1)^{-1} [i(n+1-i)]^{1/2} [J_{n+1}(i+1) - J_{n+1}(i)] \right\}^2 = O(1),$$

which follows along the lines of Section 2 of Sen and Ghosh (1972). Hence, (3.19) follows and the proof of the lemma is complete.

Lemma 3.3. *Under* (1.1) *and* (2.5), *for every* $\delta > 0$, *there exist two numbers* ε *and* η, *such that* $0 < \varepsilon < \eta < \delta/2 \leqq 1/4$, *and*

$$\sum_{n \geqq 1} n^{-1+\varepsilon} E[Q_n^2 I(Q_n^2 \geqq 4n^{1-\varepsilon})|\mathscr{F}_n] < \infty \quad \text{a.s.},$$

where $I(A)$ *stands for the indicator function of a set* A.

Proof. By virtue of (3.13), (3.14), (3.16), the inequality $Q_n^2 \leqq 2[Q_{n1}^2 + Q_{n2}^2]$ and the fact that $E(Q_{n2}^2|\mathscr{F}_n)$ exists for every $n \geqq 1$, it suffices to show that as $n_0 \to \infty$,

$$\sum_{n \geqq n_0} n^{-1+\varepsilon} E[Q_{nj}^2 I(Q_{nj}^2 \geqq n^{1-\varepsilon})|\mathscr{F}_n] \to 0 \text{ a.s.}, \quad \text{for } j=1,2. \qquad (3.24)$$

Note that as in (3.14), $Q_{01}=0$ and for $n \geqq 2$,

$$E(Q_{n-1}^2|\mathscr{F}_{n-1}) = n^{-2} \sum_{i=1}^{n} \sum_{j=1}^{n} \chi_n(i,j) J_n^2(i) L_n^2(j), \qquad (3.25)$$

where $\chi_n(i,j)$ is 1 or 0 according as $J_n^2(i)\,L_n^2(j)$ is $\geqq n^{1-\varepsilon}$ or not. It is well known [viz., Puri and Sen (1971, pp. 408–413)] that under (2.5), for every $\delta>0$, there exists a $\gamma>0$, such that $\max_{1\leqq j\leqq n}|J_n(i)-J(i/(n+1))|=O(n^{-1/2-\gamma})$, which, along with the C_2-inequality, leads us to

$$\begin{aligned} J_n^2(i) &= \left[J\big(i/(n+1)\big)+\{J_n(i)-J\big(i/(n+1)\big)\}\right]^2 \\ &\leqq 2J^2\big(i/(n+1)\big)+O(n^{-1-2\gamma}), \qquad \text{for } i=1,\ldots,n. \end{aligned} \tag{3.26}$$

Now, for $n^{1/2}\leqq i\leqq n-n^{1/2}$, by (2.5), $J^2\big(i/(n+1)\big)\leqq K\,[i(n+1-i)/(n+1)^2]^{-1+2\delta}\leqq C\,n^{1/2-\delta}$, $0<C<\infty$, so that by (3.26), $J_n^2(i)\leqq 2\,C\,n^{1/2-\delta}+O(n^{-1-2\gamma})$, and a similar bound holds for the $L_n^2(i)$. Further, $1-2\delta<1-\delta/2<1-\varepsilon<1$, so that there exists an n_0, such that for $n\geqq n_0$, $\chi_n(i,j)=0$ for every $n^{1/2}\leqq i,j\leqq n-n^{1/2}$, and hence, (3.25) is bounded by

$$\begin{aligned} & n^{-2}\sum_{i=1}^{n}\Big\{\sum_{j\leqq n^{1/2}}+\sum_{j\geqq n-n^{1/2}}[J_n^2(i)\,L_n^2(j)+J_n^2(j)\,L_n^2(i)]\Big\} \\ &= A_n^2\Big\{n^{-1}\Big[\sum_{j\leqq n^{1/2}}L_n^2(j)+\sum_{j\geqq n-n^{1/2}}L_n^2(j)\Big]\Big\} \\ &\quad + B_n^2\Big\{n^{-1}\Big[\sum_{j\leqq n^{1/2}}J_n^2(j)+\sum_{j\geqq n-n^{1/2}}J_n^2(j)\Big]\Big\}. \end{aligned} \tag{3.27}$$

But, by (2.5) and (3.26), $n^{-1}\sum_{j\leqq n^{1/2}}J_n^2(j)\leqq 2n^{-1}\sum_{j\leqq n^{1/2}}J^2\big(j/(n+1)\big)+O(n^{-3/2-2\gamma})=O(n^{-\delta})+O(n^{-3/2-2\gamma})=O(n^{-\delta})$, and the same bound applies to the other three terms on the right hand side of (3.27). Thus, by (3.11), (3.12) and (3.27), (3.25) is bounded, for $n\geqq n_0$, by $C^*\,n^{-\delta}$, where $C^*<\infty$. Therefore, by noting that

$$0<\varepsilon<\delta/2<\delta,$$

and hence, $\sum_{n\geqq 1}n^{-1+\varepsilon-\delta}<\infty$, we conclude that (3.24) holds for $j=1$. For $j=2$, we note that (3.24) is bounded by $\sum_{n\geqq n_0}n^{-1+\varepsilon}\,E(Q_{n2}^2\,|\,\mathscr{F}_n)$. Now, in course of the proof of Lemma 3.2, we have observed that as $n\to\infty$, $E(Q_{n2}^2\,|\,\mathscr{F}_n)=O(n^{-\gamma})$ a.s., where $\gamma(>0)$ depends on δ in (2.5). Thus, for n_0 adequately large,

$$\sum_{n\geqq n_0}n^{-1+\varepsilon}\,E(Q_{n2}^2\,|\,\mathscr{F}_n)\leqq C\sum_{n\geqq n_0}n^{-1+\varepsilon-\gamma}\ \text{a.s.};\qquad C<\infty, \tag{3.28}$$

and as $0<\varepsilon<\gamma$, the right hand side of (3.28) converges to 0 as $n_0\to\infty$. Q.E.D.

Returning now to the proof of Theorem 2, we note that by Lemma 3.1, $\{T_n, \mathscr{F}_n, n\geqq 1\}$ is a martingale, and by Lemma 3.2, $\left\{\sum_{i=1}^{n}E[(T_i-T_{i-1})^2\,|\,\mathscr{F}_{i-1}]\right\}\Big/n\to 1$ a.s., as $n\to\infty$, and hence, (2.6) follows directly from our Lemma 3.3 and Theorem 4.4 of Strassen (1967). To prove (2.7)–(2.8), we note that by (2.4) and (2.6), as $n\to\infty$,

$$\sup_{k\geqq n}\,[T_k-W(V_k)]/[2\,V_k\log\log V_k]^{1/2}\to 0 \qquad \text{with probability 1.} \tag{3.29}$$

Also, by definition and (3.14)–(3.17), we obtain that as $n\to\infty$,

$$V_{n+1}-V_n=E(Q_n^2\,|\,\mathscr{F}_n)\to A^2B^2=1 \text{ a.s., i.e., } (V_{n+1}-V_n)/V_{n+1}=O(n^{-1}) \text{ a.s.}$$

Consequently, by Lemma 4.2 of Strassen (1967), as $n\to\infty$,

$$\sup_{V_n\le t\le V_{n+1}} |W(t)-W(V_n)|/[2V_n\log\log V_n]^{1/2}\to 0 \text{ a.s.}$$

Hence, by the law of iterated logarithm for the standard Brownian motion process, as $n\to\infty$,

$$\sup_{k\ge n}[2V_k\log\log V_k]^{-1/2}T_k=1 \text{ a.s.,}\qquad \inf_{k\ge n}[2V_k\log\log V_k]^{-1/2}T_k=-1 \text{ a.s.} \tag{3.30}$$

Then, (2.7)–(2.8) follow directly from (3.30) and Lemma 3.2.

Finally, to prove Theorem 3, we rewrite $n^{-1}T_n$ as

$$n^{-1}T_n=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} J_n(nF_n(x))\,L_n(nG_n(y))\,dH_n(x,y). \tag{3.31}$$

We define $n^{-1}T_n^{**}=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} J(nF_n(x)/(n+1))\,L(nG_n(y)/(n+1))\,dH_n(x,y)$. We shall show that

$$n^{-1}(T_n-T_n^{**})\to 0 \text{ a.s., and } n^{-1}T_n^{**}\to\mu \text{ a.s., as } n\to\infty. \tag{3.32}$$

We can write

$$n^{-1}(T_n-T_n^{**})=I_{n1}+I_{n2}; \tag{3.33}$$

$$I_{n1}=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} J_n(nF_n(x))\,[L_n(nG_n(y))-L(nG_n(y)/(n+1))]\,dH_n(x,y), \tag{3.34}$$

$$I_{n2}=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} L(nG_n(y)/(n+1))\,[J_n(nF_n(x))-J(nF_n(x)/(n+1))]\,dH_n(x,y). \tag{3.35}$$

By the Schwarz inequality and the definitions of F_n, G_n and H_n,

$$I_{n1}^2\le\left[n^{-1}\sum_{i=1}^{n}J_n^2(i)\right]\left[n^{-1}\sum_{i=1}^{n}\{L_n(i)-L(i/(n+1))\}^2\right]. \tag{3.36}$$

By (3.10), the first factor on the right hand side of (3.36) is bounded by 1, while by Proposition 1 of Hoeffding (1973), the second factor goes to 0 as $n\to\infty$. Thus $\lim_{n\to\infty}I_{n1}=0$. *Also, since*

$$\int_{-\infty}^{\infty} L^2\left(\frac{n}{n+1}G_n(y)\right)dG_n(y)=n^{-1}\sum_{i=1}^{n}L^2(i/(n+1))\to\int_0^1 L^2(u)\,du=1, \quad \text{as } n\to\infty,$$

by the same technique, it follows that $\lim_{n\to\infty}I_{n2}=0$. Thus, $n^{-1}(T_n-T_n^{**})\to 0$ as $n\to\infty$.

Now, by the Kintchine strong law of large numbers,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} J(F(x))\,L(G(y))\,dH_n(x,y)=n^{-1}\sum_{i=1}^{n}J(F(X_i))\,L(G(Y_i))\to\mu \text{ a.s.,} \quad \text{as } n\to\infty.$$

Hence, to prove (3.32), one needs to show only that as $n\to\infty$, with probability one,

$$I_n=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[J\left(\frac{n}{n+1}F_n(x)\right)L\left(\frac{n}{n+1}G_n(y)\right)-J(F(x))L(G(y))\right]dH_n(x,y)\to 0. \tag{3.37}$$

But, $I_n=I_{n3}+I_{n4}$, *where*

$$I_{n3}=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}J(nF_n(x)/(n+1))\left[L(nG_n(y)/(n+1))-L(G(y))\right]dH_n(x,y), \tag{3.38}$$

$$I_{n4}=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}L(G(y))\left[J(nF_n(x)/(n+1))-J(F(x))\right]dH_n(x,y). \tag{3.39}$$

Again, on using the Schwarz inequality and (1.4), we have

$$I_{n3}^2\leqq\int_{-\infty}^{\infty}\left[L(nG_n(y)/(n+1))-L(G(y))\right]^2dG_n(y). \tag{3.40}$$

Now, (1.4) insures that for every $\varepsilon>0$, there exists a $\delta_1(0<\delta_1<1/2)$, such that $\int_0^{\delta_1}+\int_{1-\delta_1}^{1}L^2(u)\,du<\frac{1}{8}\varepsilon$. Let a and b such that $G(a)=1-G(b)=\frac{1}{2}\delta_1$. Then,

$$\begin{aligned}&\int_a^b\left[L(nG_n(y)/(n+1))-L(G(y))\right]^2dG_n(y)\\&\quad=\int_{\delta_1/2}^{1-\delta_1/2}\left[L(nG_n(G^{-1}(u))/(n+1))-L(u)\right]^2dG_n(G^{-1}(u)).\end{aligned} \tag{3.41}$$

Note that $L(u)$ is continuous in the open interval (0, 1), and hence, it is uniformly continuous in u in any closed interval $[\eta,1-\eta]$, $0<\eta\leqq 1/2$. Also, by the Glivenko-Cantelli Theorem, $\sup_{u\in[0,1]}|nG_n(G^{-1}(u))/(n+1)-u|\to 0$ a.s., as $n\to\infty$. Consequently, (3.41) converges to 0 a.s., as $n\to\infty$. Again,

$$\begin{aligned}&\int_b^{\infty}\left[L(nG_n(y)/(n+1))-L(G(y))\right]^2dG_n(y)\\&\quad\leqq 2\left[\int_b^{\infty}L^2(nG_n(y)/(n+1))\,dG_n(y)+\int_b^{\infty}L^2(G(y))\,dG_n(y).\right.\end{aligned} \tag{3.42}$$

Note that $nG_n(b)/(n+1)\to G(b)=1-\delta_1/2$ a.s., as $n\to\infty$. Thus, $G_n(b)\geqq 1-\delta_1$ a.s. as $n\to\infty$. Hence, on writing $n^*=[nG_n(b)]+1$, we obtain that $\int_b^{\infty}L^2(nG_n(y)/(n+1))\,dG_n(y)=n^{-1}\sum_{i=n^*}^{n}L^2(i/(n+1))$ is a.s. bounded by $n^{-1}\sum_{i=[n(1-\delta_1)]+1}^{n}L^2(i/(n+1))$ which converges to $\int_{1-\delta_1}^{1}L^2(u)\,du<\varepsilon/8$, as $n\to\infty$. Also, by the strong law of large numbers, $\int_b^{\infty}L^2(G(y))\,dG_n(y)$ a.s. converges to $\int_{1-\delta_1/2}^{1}L^2(u)\,du\,(<\varepsilon/8)$ as $n\to\infty$. Consequently, (3.42) can be made smaller than $\varepsilon/2$ with probability 1, as $n\to\infty$. A similar treatment follows for I_{n4}. Hence the proof is complete.

4. Some Sequential Nonparametric Tests for Independence

For the one sample univariate goodness of fit problem, Darling and Robbins (1968) have considered a sequential test based on the classical Kolmogorov-Smirnov statistic. Their test has power 1 and an arbitrarily small type I error. For this purpose, they used the law of iterated logarithm for the Kolmogorov-Smirnov statistics. In view of our Theorems 2 and 3, a parallel class of nonparametric sequential tests for independence may be posed as follows.

Integrating by parts (1.5) and using (1.4), we have

$$\mu=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[H(x,y)-F(x)G(y)]\,dJ(F(x))\,dL(G(y)). \tag{4.1}$$

Thus, under (1.1), $\mu=0$, while by the assumed monotonicity of $J(u)$ and $L(u)$ (in u), for positively (or negatively) quadrant dependence [cf. Lehmann (1966)], viz., $H(x,y)-F(x)G(y)\geqq 0$ (or $\leqq 0$), for all x, y, with the strict inequality for at least a set of points with measure non-zero, μ will be positive (or negative). Suppose that we want to test H_0: $\mu=0$ vs. H_1: $\mu>0$. In the same spirit as in Darling and Robbins (1968), we consider a sequential procedure which consists in starting with an initial sample of size n_0 (moderately large) taking observations until for the first time $T_n\geqq c_n$, where $\{c_n\}$ is an appropriate sequence of positive numbers; the null hypothesis is rejected when $T_n\geqq c_n$ for some $n\geqq n_0$. It follows from Theorem 3 that if $n^{-1}c_n\to 0$ with $n\to\infty$, then for $\mu>0$, T_n eventually exceeds c_n with probability 1, so that the test has power 1, for every $\mu>0$. Also, if we let $c_n^2\geqq 2n\log\log n$, it follows from (2.7) that $P\{T_n\geqq c_n$ for some $m\geqq n_0\,|\,\mu=0\}\to 0$ as $n_0\to\infty$, so that if n_0 is chosen adequately large, the type I error can also be made arbitrarily small. A similar procedure follows for H_0 vs. H_2: $\mu<0$ or H_0 vs. H^*: $\mu\neq 0$.

Our theorems are also useful for the study of the asymptotic properties of an alternative class of sequential tests for independence, which may be posed as follows. Suppose, we want to test H_0: $\mu=0$ against H_1: $\mu=\Delta>0$, where Δ is small. Corresponding to preassigned (α,β) (where $0<\alpha,\beta<1/2$), we let $a=\log[(1-\beta)/\alpha]$ and $b=\log[\beta/(1-\alpha)]$. Then, starting with an initial sample of size $n_0(\Delta)$ (moderately large) we continue drawing observations one by one as long as

$$b<\Delta[T_m-m\Delta/2]<a\big(m\geqq n_0(\Delta)\big), \tag{4.2}$$

where T_m is defined by (1.2). If, for the first time, (4.2) is violated for $m=n$ and $\Delta[T_n-n\Delta/2]$ is $\leqq b$ (or $\geqq a$), accept H_0 (or H_1); the corresponding stopping variable is denoted by $N(\Delta)$. By virtue of our Theorem 3, it follows precisely on the same line as in the proof of Theorem 3.1 of Sen (1973) that for every (fixed) Δ (>0), under (1.4), the sequential procedure terminates with probability 1, i.e.,

$$P\{N(\Delta)>n\,|\,\Delta\}\to 0 \quad \text{as} \quad n\to\infty.$$

As in Sen (1973) and Sen and Ghosh (1974), we consider now the asymptotic properties of the procedure where we let $\Delta\to 0$. We assume that as $\Delta\to 0$, $n_0(\Delta)\to\infty$ but $\Delta^2 n_0(\Delta)\to 0$, and further, on writing $\mu=\phi\Delta$, we frame our H_0: $\phi=0$ and H_1: $\phi=1$, and we allow ϕ to lie in $I^*=\{t\colon |t|\leqq K\}$ where $K(>1)$ is a positive number. Then, by virtue of the Brownian motion approximations studied in Theorems 1 and 2, we can show on preceeding along the lines of the proof of Theorem 3.2 of Sen and Ghosh (1974) that as $\Delta \to 0$, the OC function of the above test approaches the asymptotic limit

$$P(\phi)=\begin{cases}(e^{(1-2\phi)}-1)/(e^{(1-2\phi)}-e^{(1-2\phi)}), & \phi \neq 1/2,\\ a/(a-b), & \phi=1/2.\end{cases} \tag{4.3}$$

Thus, asymptotically (as $\Delta \to 0$), the proposed test is distribution-free [for all J and L satisfying (2.5)], and $P(0)=1-\alpha$, $P(1)=\beta$. Hence, the asymptotic strength of the test is (α, β).

It remains to study the ASN function of the procedure, namely, to show that for $\phi \in \Gamma^*$, $\Delta^2 E[N(\Delta)|\,\mu=\phi\Delta]$ tends (as $\Delta \to 0$) to

$$\begin{matrix}[P(\phi)\,b+\{1-P(\phi)\}\,a]/(\phi-1/2), & \phi \neq 1/2,\\ -P'(1/2)/(a-b), & \phi=1/2,\end{matrix} \tag{4.4}$$

where $P'(1/2)$ stands for the derivative of $P(\phi)$ at $\phi=1/2$. For $\phi=0$, i.e., under the null hypothesis, (4.4) can be proved by using the elegant result of Chow, Robbins and Teicher (1965), our martingale result (Lemma 3.1) and some standard steps. But, for $\phi \neq 0$, (1.1) does not hold, and our Lemma 3.2 does not apply. This invalidates the application of the martingale stopping time theorems of Chow et al. (1965). However, (4.4) can be proved along the lines of the proof of Theorem 3.3 of Sen and Ghosh (1974), provided (2.5) is replaced by the more stringent condition that

$$|J^{(r)}(u)| \leqq K[u(1-u)]^{-\gamma-r+\delta}, \quad r=0,1, \quad \text{for some } \delta>0 \text{ and } 0 \leqq \gamma \leqq 1/4, \tag{4.5}$$

and a similar condition on $L(u)$. Since the proof is based on certain moment inequalities on T_n when (1.1) is not necessarily true, and these are quite space consuming, for intended brevity, the details are not included here.

Theorem 1 also provides another sequential test for independence. Suppose, we want to test H_0 in (1.1) vs. $H_1: \mu>0$. Instead of basing our test on a fixed sample size n, we may consider the following scheme which allows for a termination at an early stage depending on the accumulated evidence upto that stage. Continue sampling so long as T_m lies below some $C_{n,\alpha}$, where $C_{n,\alpha}$ is some positive number, and α is the size of the test. If N is the smallest positive integer ($\leqq n$) for which $T_N \geqq C_{n,\alpha}$, we reject H_0 and accept H_1; if $N>n$, we make the terminal decision based on T_n, rejecting H_1 and accepting H_0. By virtue of our Theorem 1, under H_0, $n^{-1/2}[\max_{1\leqq k\leqq n} T_k]$ converges in distribution to $M=\sup_{0\leqq t\leqq 1} W(t)$, where the distribution of M is wellknown [cf. Billingsley (1968, p. 79)]. Thus, if M_α be the upper 100% point of the distribution of M, we may approximate $n^{-1/2}C_{n,\alpha}$ by M_α when n is large. The case of two-sided alternatives follows on parallel lines. The overall level of significance of the test remains asymptotically equal to α.

We conclude this paper by some additional remarks. First, we have defined the scores in (1.2) by $J_n(i)=EJ(U_{ni})$, $i=1, \ldots, n$. In practice, accasionally, we take $J_n(i)=J(i/(n+1))$, $i=1, \ldots, n$, and similarly for the $L_n(i)$. In that case, the martingale result in Lemma 3.2 may not hold generally, as (3.4) may not hold. However, if in addition to (1.4), we assume that $n^{-1}\sum_{i=1}^{n}|J_n(i)-J(i/(n+1))|=o(n^{-1/2})$, then Theorem 1 holds. Since this assumption holds under (2.5) [viz., Puri and Sen (1971, pp. 408–413)], for Theorem 2, we need no additional assumption. Theorem 3 has been shown (see T_n^{**}) to be true for such scores. Second, the class of rank order statistics considered in this paper does not include some other nonparametric test statistics for independence. Kendall's (1938) tau statistic and Hoeffding's (1948) statistic are not members of this class, but are both U-statistics. For U-statistics, the weak convergence to Wiener processes is studied by Loynes (1970) and Miller and Sen (1972), while the a.s. convergence by Berk (1966); a.s. invariance principles are studied by Sen (1974). As such, similar results for these statistics hold. Blum, Kiefer and Rosenblatt (1961) considered distribution-free tests for independence based on the empirical df's. For multi-dimensional empirical processes, weak convergence has been studied by a host of workers; we may refer to a recent paper of Neuhaus (1971) where other references are cited. Kiefer (1972), Wichura (1973) and Sen (1973) have considered a.s. invariance principles for such processes, and these in turn imply similar results for the Blum-Kiefer-Rosenblatt statistic. However, instead of the Wiener process, it will correspond to a functional of a multi-dimensional Gaussian process. The a.s. convergence of the statistic follows directly from the Glivenko-Cantelli Lemma.

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