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Strong Limit Theorems for General Supercritical Branching Processes with Applications to Branching Diffusions

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Introduction

The limit theory for supercritical, positively regular Markov branching processes with a finite set of types has long been known in its sharpest form. See [18] and [2] for the discrete and continuous time case, respectively. For processes with a general set of types the situation is less satisfactory. The theory is comparatively incomplete, and the techniques used to prove the fundamental convergence results depend on second moment assumptions. See [12, 13, 14] for the general case and [9, 10, 21, 22] for diffusion examples.

In this paper we develop the general theory with conditions as weak as those for a finite set of types. In particular, we obtain almost sure convergence without assumptions beyond positive regularity, and we solve the problem of finding the proper generalization of the $x \log x$ condition which is necessary and sufficient for the non-degeneracy of the limit variable. Some results are extensions or sharpened versions of known results, others are completely new. Also, many of our proofs, when specialized, are simpler than those in the literature for a finite set of types, which often do not admit a generalization to the infinite case.

The formal basis of our theory is an asymptotic representation of the first moment semigroup, which we adopt as definition of positive regularity in case of a general set of types. The concept of positive regularity is not unambigous in the infinite case, and the motivation for our specific assumptions derives from branching diffusions. For a large class of such processes the representation can be derived by exploiting asymptotic spectral properties of the generator. The idea first occurred in connection with the limit theory for critical processes ([15]).

Branching diffusions not only are of heuristic value for the development of a general theory, they also serve as models for various biological and physical phenomena, thus providing a testing ground for any general theory. We considered it to be crucial that, when applying the theory to a branching diffusion, all our conditions could be expressed in terms of the quantities from which the process is actually constructed.

In Section 1 we give the preliminaries and state the results in the general setting. Section 2 contains the corresponding proofs. Sections 3 and 4 deal with a class of branching diffusions: We define the model, derive the first moment representation, and reformulate our limit theory in terms of natural model parameters.

§ 1. General Model: Preliminaries and Statement of Results.

Let $(X, \mathfrak{A})$ be a measurable space, $\mathscr{B}$ the Banach algebra of all bounded, $\mathfrak{A}$-measurable functions ξ on X with norm $\|\xi\| = \sup_x |\xi(x)|$, and denote by $\mathscr{B}_+$ the nonnegative cone in $\mathscr{B}$.

Write $X^{(n)}$ for the symmetrized n-fold direct product of X, let θ be some extra point, $X^{(0)} = \{\theta\}$, and

$$\hat{X} = \bigoplus_{n=0}^{\infty} X^{(n)}.$$

Define $\hat{\mathfrak{A}}$ as the σ-algebra induced on $\hat{X}$ by $\mathfrak{A}$. Every element $\hat{x} \in \hat{X}$ defines a counting measure

$$\hat{x}[A] = \begin{cases} 0; & \hat{x} = \theta, \\ \sum_{i=1}^{n} 1_A(x_i); & \hat{x} = \langle x_1, \dots, x_n \rangle, \quad n > 0, \end{cases}$$

on X, where 1_A is the indicator function of $A \subset X$, and we write

$$\hat{x}[\xi] = \int_X \xi(x)\, \hat{x}[dx].$$

Take $\mathbb{T} = \mathbb{N} = \{0, 1, 2, \dots\}$ or $\mathbb{T} = \mathbb{N}_+ = [0, \infty[$, and suppose to be given a Markov process $\{\hat{x}_t, \mathbf{P}^{\hat{x}}\}$ in $(\hat{X}, \hat{\mathfrak{A}})$ with parameter set $\mathbb{T}$ and stationary transition probabilities satisfying the *branching condition*

$$\mathbf{P}^{\theta}(\hat{x}_t[X] = 0) = 1,$$

$$\mathbf{P}^{\hat{x}}(\hat{x}_t[A_i] = n_i;\ i = 1, \dots, m) = \sum_{\substack{n_{i1} + \dots + n_{ik} = n_i \\ i = 1, \dots, m}} \prod_{j=1}^{k} \mathbf{P}^{\langle x_j \rangle}(\hat{x}_t[A_i] = n_{ij},\ i = 1, \dots, m) \tag{1.1}$$

for all $t \in \mathbb{T}$, $\hat{x} = \langle x_1, \dots, x_k \rangle \in \hat{X}$, $k > 0$, $n_i \in \mathbb{N}$, and every decomposition $\{A_1, \dots, A_m\}$ of X with $A_i \in \mathfrak{A}$, $i = 1, \dots, m$, $m > 0$. Such a process is called a *Markov branching process*. For questions of existence and construction see [16, 20].

If for some $\hat{x} \in \hat{X}$ and $t \in \mathbb{T}$, $\mathbf{E}^{\hat{x}} \hat{x}_t[\mathbf{1}] < \infty$, where $\mathbf{1}(\cdot) \equiv 1$, then $\mathbf{E}^{\hat{x}} \hat{x}_t[\cdot]$ is a bounded linear functional on $\mathscr{B}$. If furthermore

$$\sup_{x \in X} \mathbf{E}^{\langle x \rangle} \hat{x}_t[\mathbf{1}] < \infty \tag{1.2}$$

then $\mathbf{E}^{\langle \cdot \rangle} \hat{x}_t[\cdot]: \mathscr{B} \to \mathscr{B}$ is a bounded linear operator, and if (1.2) holds for $s, t \in \mathbb{T}$, then

$$\mathbf{E}^{\langle x \rangle} \hat{x}_{t+s}[\eta] = \mathbf{E}^{\langle x \rangle} \hat{x}_t[\mathbf{E}^{\langle \cdot \rangle} \hat{x}_s[\eta]] \qquad (\eta \in \mathscr{B})$$

is an immediate consequence of (1.1) and the Markov property.

We now define our general model by the following additional structure:

(M) *The first moment semigroup* $\{\mathbf{E}^{\langle\cdot\rangle}\hat{x}_t[\cdot]\}_{t\in\mathbb{T}}$ *exists and can be represented in the form*

$$\mathbf{E}^{\langle x\rangle}\hat{x}_t[\eta]=\rho^t\varphi^*[\eta]\,\varphi(x)+Q_t^{\langle x\rangle}[\eta],\qquad x\in X,\quad t\in\mathbb{T},\quad \eta\in\mathscr{B}$$

with $\rho\in]0,\infty[$, $\varphi\in\mathscr{B}_+$ *and* φ^* *a non-negative bounded linear functional on* $\mathscr{B}$ *such that*

$$\varphi^*[\varphi]=1,$$
$$\varphi^*[Q_t^{\langle\cdot\rangle}[\cdot]]\equiv 0,\qquad Q_t^{\langle\cdot\rangle}[\varphi]\equiv 0,$$
$$|Q_t^{\langle x\rangle}[\eta]|\leqq\alpha_t\,\varphi^*(\eta)\,\varphi(x),\qquad x\in X,\quad \eta\in\mathscr{B}_+,\quad t>0,$$

for some $\alpha_\cdot:\mathbb{T}\to[0,\infty[$ *satisfying*

$$\rho^{-t}\alpha_t\to 0,\qquad t\to\infty.$$

Notice that (M) implies that φ^* is a measure. For convenience we take $\varphi^*[\mathbf{1}]=1$. Also, φ^* and φ are the left and right eigenvectors, respectively, of $\mathbf{E}^{\langle\cdot\rangle}\hat{x}_t[\cdot]$ corresponding to the eigenvalues ρ^t. In particular, φ^* is the invariant distribution of the types and $W_t=\rho^{-t}\hat{x}_t[\varphi]$, $t\in\mathbb{T}$, is a martingale with respect to

$$\mathfrak{F}_t=\sigma(\hat{x}_s;\ 0\leqq s\leqq t).$$

We restrict ourselves to the investigation of the *supercritical case* $\rho>1$. We first state the main results for $\mathbb{T}=\mathbb{N}$ and then give the extension to $\mathbb{T}=\mathbb{R}_+$. In discrete time the a.s. existence of $W=\lim_{n\to\infty}W_n$ is immediate without further assumptions, appealing to the martingale theorem.

Theorem 1. *Given* (M) *with* $\rho>1$,

$$\rho^{-n}\hat{x}_n[\eta]\xrightarrow{\text{a.s.}}\varphi^*[\eta]\,W,\qquad n\to\infty,$$

for all $\eta\in\mathscr{L}^1_{\varphi^*}$.

The limit variable W may be degenerate at 0. The question of how properly to generalize the $x\log x$ condition known to be necessary and sufficient for non-degeneracy in the finite case presents a problem. The answer is provided by the next theorem. Let

$$I_n=\varphi^*[\mathbf{E}^{\langle\cdot\rangle}\hat{x}_n[\varphi]\log\hat{x}_n[\varphi]],\tag{1.3}$$

so that I_n is the $x\log x$ moment of $\hat{x}_n[\varphi]$ given that the type of the original particle is distributed according to φ^*.

Theorem 2. *Either* $I_n<\infty$ *for all* $n>0$, *or* $I_n=+\infty$ *for all* $n>0$. *If* $I_1<\infty$, *then* $\mathbf{E}^{\langle x\rangle}W=\varphi(x)\ \forall\, x\in X$. *If* $I_1=+\infty$, *then* $W\equiv 0$ a.s. $[\mathbf{P}^{\langle x\rangle}]\ \forall\, x\in X$.

Let us now turn to $\mathbb{T}=\mathbb{R}_+$. In order to ensure the a.s. existence of $W=\lim_{t\to\infty}W_t$ we have to assume separability of $\{\hat{x}_t[\varphi],\mathbf{P}^{\hat{x}}\}$. The continuous time version of Theorem 2 follows immediately from the discrete time version. As regards Theorem 1, the proof for discrete time is easily adapted to show that

$$\rho^{-t}\hat{x}_t[\eta]\xrightarrow{\mathbf{P}}\varphi^*[\eta]\,W,\qquad t\to\infty,\ \eta\in\mathscr{L}^1_{\varphi^*}.\tag{1.4}$$

Also,

$$\rho^{-n\varepsilon}\,\hat{x}_{n\varepsilon}[\eta] \xrightarrow{\text{a.s.}} \varphi^*[\eta]\,W, \qquad N\ni n\to\infty, \quad \varepsilon>0, \quad \eta\in\mathscr{L}^1_{\varphi^*} \tag{1.5}$$

is immediate without further assumptions. But the passage from this a.s. convergence of skeletons to a.s. convergence as $t\to\infty$ continuously is a non-trivial problem, which has been considered before in various settings ([2, 14[1], 17]). A simple and natural situation is the following:

Theorem 1′. *Let X be a separable metric space, $\mathfrak{A}$ the topological Borel algebra, and $\{\hat{x}_t, \mathbf{P}^{\hat{x}}\}$ right-continuous, satisfying* (M) *with $\rho>1$. If $\vartheta\in\mathscr{B}_+$ is lower semicontinuous a.e. $[\varphi^*]$ and*

$$\rho^{-t}\,\hat{x}_t[\vartheta] \xrightarrow{\text{a.s.}} \varphi^*[\vartheta]\,W, \qquad t\to\infty, \tag{1.6}$$

then

$$\rho^{-t}\,\hat{x}_t[\vartheta\eta] \xrightarrow{\text{a.s.}} \varphi^*[\vartheta\eta]\,W, \qquad t\to\infty, \tag{1.7}$$

for every $\eta\in\mathscr{B}$ which is continuous a.e. $[\varphi^*]$.

The role of the various assumptions will become transparent from the proofs in § 2. We always have (1.6) for $\vartheta=\varphi$, of course, but if $\inf_x \varphi(x)=0$, as is the case for branching diffusions with absorbing barriers, this is unsatisfactory since one would like to describe also, for example, the asymptotic behaviour of $\hat{x}_t[\mathbf{1}]$, the size of the population at time t. To deal with this case we need additional structure and we return to the problem in connection with our branching diffusion model in § 4.

We conclude with a theorem on the existence of moments of W and rates of convergence. Theorem 2 suggests that corresponding results from the finite case ([5, 7, 1]) can be generalized to the present context by carrying conditions on the offspring distribution into conditions on

$$F^\delta(y)=\varphi^*[\mathbf{P}^{\langle\cdot\rangle}(\hat{x}_\delta[\varphi]\leqq y)], \qquad \delta\in\mathbb{T}\setminus\{0\}. \tag{1.8}$$

Theorem 3. *Let $\delta>0$. If*

$$\int_0^\infty y^p\,dF^\delta(y)<\infty$$

for some p with $1<p<2$, then

$$\mathbf{E}^{\hat{x}}\,W^p<\infty \quad \textit{and} \quad W-W_n=o(\rho^{-n/q}) \quad [\mathbf{P}^{\hat{x}}]$$

where $1/p+1/q=1$. Also, if

$$\int_0^\infty y(\log^+ y)^{\alpha+1}\,dF^\delta(y)<\infty$$

for some $\alpha>0$, then

$$\mathbf{E}^{\hat{x}}\,W(\log^+ W)^\alpha<\infty \quad \textit{and} \quad W-W_n=o(n^{-\alpha}) \quad [\mathbf{P}^{\hat{x}}].$$

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[1] In Theorem 2 and Corollary 2.1 of [14] it should read $\xi=\vartheta\zeta$ and $\xi=\varphi\zeta$, respectively, where $\zeta\in\mathscr{B}$

We shall omit the proof. The convergence rates are obtained by combining methods of the present paper and of [1]. For the existence of moments our method is different from the approach in the literature ([5, 7]) and will appear elsewhere.

§ 2. General Model: Proofs

The branching property (1.1) implies that for every $\delta \in \mathbb{T} \setminus \{0\}$ there exists a process $\{\hat{y}_t, \mathbf{P}_\delta^{\hat{y}}\}$ in $(\hat{X}, \hat{\mathfrak{A}})$ which is equivalent to $\{\hat{x}_t, \mathbf{P}^{\hat{x}}\}$ and has the following property: There exists an increasing family $\{\mathfrak{G}_k\}_{k \in \mathbb{N}}$ of σ-algebras such that for every $\mathfrak{A}$-measurable η and all $n, m \in \mathbb{N}$

$$\hat{y}_{(n+m)\delta}[\eta] = \sum_{i=1}^{\hat{y}_{n\delta}[1]} \hat{y}_{(n+m)\delta}^{n\delta, i}[\eta] \quad [\mathbf{P}_\delta^{\hat{y}}]$$

where the $\hat{y}_{(n+m)\delta}^{n\delta,i}$; $i = 1, \ldots, \hat{y}_{n\delta}[1]$, are $\mathfrak{G}_{n+m}$-measurable, independent conditioned upon $\mathfrak{G}_n$, and satisfy

$$\mathbf{P}_\delta^{\hat{y}}(\hat{y}_{(n+m)\delta}^{n\delta,i} \in \hat{A} \mid \mathfrak{G}_n) = \mathbf{P}_\delta^{\langle y_{n\delta}^i \rangle}(\hat{y}_{m\delta} \in \hat{A}), \qquad \hat{A} \in \hat{\mathfrak{A}},$$

with $\hat{y}_{n\delta} = \langle \ldots, y_{n\delta}^i, \ldots \rangle$. Hence it does not lead to a loss in generality if for any fixed δ such a representation is used for $\{\hat{x}_t, \mathbf{P}^{\hat{x}}\}$ itself.

Where it is unambiguous, we shall write $\mathbf{P}, \mathbf{E}$ instead of $\mathbf{P}^{\hat{x}}, \mathbf{E}^{\hat{x}}$.

Our plan for the proofs is motivated by the fact that if the $x \log x$ condition fails to hold, we need $W=0$ a.s. in our proof of Theorem 1. Hence Theorem 2 will be proved first.

Here and in § 4 it will be convenient to work with the function

$$\log^* x = \begin{cases} x/e, & 0 \leqq x \leqq e, \\ \log x, & e < x < \infty. \end{cases}$$

We summarize some of the properties of $\log^*$:

Lemma 1. *The function $x \log^* x$ is non-negative, non-decreasing, and convex. If $S = X_1 + \cdots + X_N$ is the sum of N independent non-negative random variables,*

$$\mathbf{E}\, S \log^* S \leqq \mathbf{E}\, S \log^* \mathbf{E}\, S + \sum_{i=1}^{N} \mathbf{E}\, X_i \log^* X_i. \tag{2.1}$$

Proof. The first part of the lemma is immediate from $\log^*(a+b) \leqq \log^* a + \log^* b$, $a, b \geqq 0$, and Jensen's inequality

$$\begin{aligned} \mathbf{E}\, S \log^* S &= \mathbf{E} \sum_{i=1}^{N} X_i \log^* \sum_{j=1}^{N} X_j \\ &\leqq \sum_{i=1}^{N} \{\mathbf{E}\, X_i \log^* \sum_{j \neq i} X_j + \mathbf{E}\, X_i \log^* X_i\} \\ &\leqq \sum_{i=1} \{\mathbf{E}\, X_i \log^* \sum_{j \neq i} \mathbf{E}\, X_j + \mathbf{E}\, X_i \log^* X_i\} \\ &\leqq \mathbf{E}\, S \log^* \mathbf{E}\, S + \sum_{i=1}^{N} \mathbf{E}\, X_i \log^* X_i. \quad \square \end{aligned}$$

We shall use (M) in the form

$$c_m^- \rho^m \varphi^*[\eta]\, \varphi(x) \leqq \mathbf{E}^{\langle x\rangle} \hat{x}_m[\eta] \leqq c_m^+ \rho^m \varphi^*[\eta]\, \varphi(x),$$
$$m=1,2,\ldots, \quad 0 \leqq \eta \in \mathscr{L}^1_{\varphi^*}, \quad c_m^- \to 1, \quad c_m^+ \to 1 \quad (m \to \infty) \tag{2.2}$$

with c_m^-, c_m^+ independent of x, η. This is immediate for η bounded, and the extension to $\mathscr{L}^1_{\varphi^*}$ follows by monotone convergence.

Proof of the first assertion of Theorem 2. Let

$$\iota_n^*(x) = \mathbf{E}^{\langle x\rangle} \hat{x}_n[\varphi] \log^* \hat{x}_n[\varphi], \qquad I_n^* = \varphi^*[\iota_n^*]. \tag{2.3}$$

We may replace I_n by I_n^*. By convexity and the martingale property I_n^* is non-decreasing in n, so that it suffices to prove that $I_n^* < \infty$, $n > 0$, implies that $I_{2n}^* < \infty$. Letting $N = \hat{x}_n[\mathbf{1}]$, $X_i = \hat{x}_{2n}^{n,i}[\varphi]$, $S = \hat{x}_{2n}[\varphi]$ it follows from (2.1), (2.2) that

$$\iota_{2n}^*(x) = \mathbf{E}^{\langle x\rangle} \mathbf{E}(\hat{x}_{2n}[\varphi] \log^* \hat{x}_{2n}[\varphi] | \mathfrak{F}_n)$$
$$\leqq \mathbf{E}^{\langle x\rangle} \{\rho^n \hat{x}_n[\varphi] \log^* \rho^n \hat{x}_n[\varphi] + \hat{x}_n[\iota_n^*]\} \leqq c_1 + c_2 \iota_n^*(x) + c_3 I_n^*$$

for suitable constants c_1, c_2, c_3. Integration with respect to φ^* completes the proof. □

The next lemma presents a key step in the proofs of Theorem 1 and Theorem 2 as well as in the transition from discrete to continuous time here and in § 4.

Lemma 2. *Let* $\delta \in \mathbb{T} \setminus \{0\}$, *and let* $Y_{n,i}^\delta, Z_{n,i}^\delta$; $n=0,1,\ldots, i=1,\ldots,\hat{x}_{n\delta}[\mathbf{1}]$ *be random variables such that* $0 \leqq Y_{n,i}^\delta \leqq Z_{n,i}^\delta$. *Suppose that the* $Y_{n,i}^\delta$*'s are independent conditioned upon* $\mathfrak{F}_{n\delta}$, *that the same is true for the random variables*

$$\tilde{Y}_{n,i}^\delta = Y_{n,i}^\delta 1_{\{Z_{n,i}^\delta \leqq \rho^{n\delta}\}}; \qquad i=1,\ldots,\hat{x}_{n\delta}[\mathbf{1}],$$

and that the distribution $G_{\langle x_i\rangle}^\delta$ *of* $Z_{n,i}^\delta$ *depends only on the type* x_i *of the* ith *particle alive at time* $n\delta$. *Define*

$$S_n^\delta = \rho^{-n\delta} \sum_{i=1}^{\hat{x}_{n\delta}[\mathbf{1}]} Y_{n,i}^\delta, \qquad \tilde{S}_n^\delta = \rho^{-n\delta} \sum_{i=1}^{\hat{x}_{n\delta}[\mathbf{1}]} \tilde{Y}_{n,i}^\delta.$$

Then the assumptions (M), $\rho > 1$, *and* $\varphi^*\left[\int_0^\infty y\, dG_{\langle\cdot\rangle}^\delta(y)\right] < \infty$ *imply that*

$$\sum_{n=0}^\infty \mathbf{P}(S_n^\delta \neq \tilde{S}_n^\delta) < \infty, \tag{2.4}$$

$$\sum_{n=0}^\infty \mathbf{Var}\{\tilde{S}_n^\delta - \mathbf{E}(\tilde{S}_n^\delta | \mathfrak{F}_{n\delta})\} < \infty. \tag{2.5}$$

In particular

$$S_n^\delta - \mathbf{E}(\tilde{S}_n^\delta | \mathfrak{F}_{n\delta}) \xrightarrow{\text{a.s.}} 0, \qquad n \to \infty. \tag{2.6}$$

Proof. Let

$$\xi_n^0(x) = \int_{\rho^{n\delta}}^\infty dG_{\langle x\rangle}^\delta(y), \qquad \xi_n^2(x) = \int_0^{\rho^{n\delta}} y^2\, dG_{\langle x\rangle}^\delta(y), \qquad G^\delta(y) = \varphi^*[G_{\langle\cdot\rangle}^\delta(y)].$$

By assumption the distribution G^δ has finite mean, and we get

$$\sum_{n=0}^{\infty} \mathbf{P}(S_n^\delta \neq \tilde{S}_n^\delta) = \sum_{n=0}^{\infty} \mathbf{E}\left(\sum_{i=1}^{\hat{x}_{n\delta}[1]} \mathbf{P}(Z_{n,i}^\delta > \rho^{n\delta} | \mathfrak{F}_{n\delta})\right) = \sum_{n=0}^{\infty} \mathbf{E}\,\hat{x}_{n\delta}[\xi_n^0]$$
$$\leqq c \sum_{n=0}^{\infty} \rho^{n\delta} \varphi^*[\xi_n^0] = c \sum_{n=0}^{\infty} \rho^{n\delta} \int_{\rho^{n\delta}}^{\infty} dG^\delta(y) \leqq c' \int_0^\infty y\, dG^\delta(y) + c'' < \infty,$$

$$\sum_{n=0}^{\infty} \mathbf{Var}\{\tilde{S}_n^\delta - \mathbf{E}(\tilde{S}_n^\delta | \mathfrak{F}_{n\delta})\} = \sum_{n=0}^{\infty} \mathbf{E}\,\mathbf{Var}(\tilde{S}_n^\delta | \mathfrak{F}_{n\delta})$$
$$\leqq \sum_{n=0}^{\infty} \rho^{-2n\delta}\left(\mathbf{E} \sum_{i=1}^{\hat{x}_{n\delta}[1]} \mathbf{E}(\tilde{Y}_{n,i}^{\delta\,2} | \mathfrak{F}_{n\delta})\right) \leqq \sum_{n=0}^{\infty} \rho^{-2n\delta} \mathbf{E}\,\hat{x}_{n\delta}[\xi_n^2]$$
$$\leqq C \sum_{n=0}^{\infty} \rho^{-n\delta} \varphi^*[\xi_n^2] = C \sum_{n=0}^{\infty} \rho^{-n\delta} \int_0^{\rho^{n\delta}} y^2\, dG^\delta(y)$$
$$\leqq C' \int_0^\infty y\, dG^\delta(y) + C'' < \infty.$$

By standard estimates (2.5) implies that $\tilde{S}_n^\delta - \mathbf{E}(\tilde{S}_n^\delta | \mathfrak{F}_{n\delta}) \xrightarrow{\text{a.s.}} 0$, so that (2.6) follows by (2.4). □

Remark. If we drop the summation over n and replace $n \in \mathbb{N}$ with $t \in [0, \infty[$, the estimates above show that at least $\mathbf{P}(S_t^\delta \neq \tilde{S}_t^\delta) \to 0$, $\mathbf{Var}(\tilde{S}_t^\delta - \mathbf{E}(\tilde{S}_t^\delta | \mathfrak{F}_{t\delta})) \to 0$, $S_t^\delta - \mathbf{E}(\tilde{S}_t^\delta | \mathfrak{F}_{t\delta}) \xrightarrow{\mathbf{P}} 0$, $t \to \infty$.

We are now prepared to complete the proof of Theorem 2. The idea is to exploit systematically the martingale property of $\{W_n\}_{n\in\mathbb{N}}$ or rather of $\{W_{n\delta}\}_{n\in\mathbb{N}}$, where δ is a large integer to be determined later. Our method is different from that used for a finite set of types in [18, 4]. We shall use the set-up of Lemma 2 with

$$Y_{n,i}^\delta = Z_{n,i}^\delta = \hat{x}_{(n+1)\delta}^{n\delta,i}[\varphi].$$

Then

$$S_n^\delta = \frac{1}{\rho^{n\delta}} \sum_{i=1}^{\hat{x}_{n\delta}[1]} \hat{x}_{(n+1)\delta}^{n\delta,i}[\varphi] = \rho^\delta W_{(n+1)\delta},$$

and we let

$$\begin{aligned} \tilde{W}_{(n+1)\delta} &= \rho^{-\delta} \tilde{S}_n^\delta = \frac{1}{\rho^{(n+1)\delta}} \sum_{i=1}^{\hat{x}_{n\delta}[1]} Y_{n,i}^\delta 1_{\{Y_{n,i}^\delta \leqq \rho^{n\delta}\}}, \\ \xi_{n,\delta}(x) &= \mathbf{E}^{\langle x\rangle}\, \hat{x}_\delta[\varphi]\, 1_{\{\hat{x}_\delta[\varphi] > \rho^n\}}, \\ \varepsilon_{n,\delta} &= \frac{1}{\rho^{(n+1)\delta}} \hat{x}_{n\delta}[\xi_{n\delta,\delta}] \end{aligned} \tag{2.7}$$

so that

$$\begin{aligned} \mathbf{E}(\tilde{W}_{(n+1)\delta} | \mathfrak{F}_{n\delta}) &= W_{n\delta} - \mathbf{E}(W_{(n+1)\delta} - \tilde{W}_{(n+1)\delta} | \mathfrak{F}_{n\delta}) \\ &= W_{n\delta} - \frac{1}{\rho^{(n+1)\delta}} \sum_{i=1}^{\hat{x}_{n\delta}[1]} \mathbf{E}(Y_{n,i}^\delta 1_{\{Y_{n,i}^\delta > \rho^{n\delta}\}} | \mathfrak{F}_{n\delta}) = W_{n\delta} - \varepsilon_{n,\delta}. \end{aligned} \tag{2.8}$$

From Lemma 2 we immediately get

Lemma 3. (M), $\rho>1$, *implies that*

$$\sum_{n=0}^{\infty} \mathbf{P}(W_{(n+1)\delta} \neq \tilde{W}_{(n+1)\delta}) < \infty \tag{2.9}$$

and that

$$\sum_{n=0}^{\infty} \{\tilde{W}_{(n+1)\delta} - W_{n\delta} + \varepsilon_{n,\delta}\} \text{ converges a.s. and in } \mathscr{L}^1. \tag{2.10}$$

In fact, (2.9) is clear from (2.4), and (2.10) follows from (2.5) and the convergence theorem for $\mathscr{L}^2$-bounded martingales, if we observe that the terms of the series in (2.10) are martingale increments by (2.8).

Lemma 4. *Let $\xi_{n,\delta}$ be defined by (2.7). Then for any $m, \delta>0$, $I_\delta<\infty$ if and only if $\sum_{n=0}^{\infty} \varphi^*[\xi_{nm,\delta}]<\infty$.*

Proof. Note that (cf. (1.8))

$$\varphi^*[\xi_{nm,\delta}] = \int_{\rho^{nm}}^{\infty} y\, dF^\delta(y), \qquad I_\delta = \int_0^{\infty} y \log y\, dF^\delta(y). \quad \square$$

Lemma 5. *For an appropriate choice of δ there are constants $c_1, c_2>0$ such that $\xi_{n,\delta}(x) \geqq c_1\, \varphi^*[\xi_{n,1}]\, \varphi(x)$ and thus*

$$\varepsilon_{n,\delta} \geqq c_2\, W_{n\delta}\, \varphi^*[\xi_{n\delta,1}] \tag{2.11}$$

Proof. Let $A=\{\hat{x}_\delta[\varphi]>\rho^n\}$, $A_i=\{\hat{x}_\delta^{\delta-1,i}[\varphi]>\rho^n\}$. Since $A_i \subseteq A$,

$$\begin{aligned}\xi_{n,\delta}(x) &= \mathbf{E}^{\langle x\rangle}\, \mathbf{E}(\hat{x}_\delta[\varphi]\, 1_A | \mathfrak{F}_{\delta-1})\\ &= \mathbf{E}^{\langle x\rangle}\, \mathbf{E}\left(\sum_{i=1}^{\hat{x}_{\delta-1}[1]} \hat{x}_\delta^{\delta-1,i}[\varphi]\, 1_A \middle| \mathfrak{F}_{\delta-1}\right) \geqq \mathbf{E}^{\langle x\rangle}\, \mathbf{E}\left(\sum_{i=1}^{\hat{x}_{\delta-1}[1]} \hat{x}_\delta^{\delta-1,i}[\varphi]\, 1_{A_i} \middle| \mathfrak{F}_{\delta-1}\right)\\ &= \mathbf{E}^{\langle x\rangle}\, \hat{x}_{\delta-1}[\xi_{n,1}] \geqq \rho^{\delta-1}\, c_{\delta-1}^-\, \varphi^*[\xi_{n,1}]\, \varphi(x) \quad \text{(cf. (2.2))},\end{aligned}$$

and we need only to take δ with $c_{\delta-1}^->0$. The estimate (2.11) is now immediate from the definitions. $\square$

Proof of Theorem 2, completed. Suppose first $I_1<\infty$. From (2.2), Lemma 4, and the definition of $\varepsilon_{n,\delta}$ we get

$$\sum_{n=0}^{\infty} \mathbf{E}\, \varepsilon_{n,1} \leqq c \sum_{n=0}^{\infty} \varphi^*[\xi_{n,1}] < \infty.$$

Therefore we have $\mathscr{L}^1$-convergence of $\sum \varepsilon_{n,1}$ by positivity and of $\sum\{\tilde{W}_{n+1}-W_n\}$ by (2.10) with $\delta=1$. Since $\tilde{W}_{n+1} \leqq W_{n+1}$, we get for any N

$$\begin{aligned}\mathbf{E}^{\langle x\rangle} W &= \mathbf{E}^{\langle x\rangle}\left(W_0 + \sum_{n=0}^{\infty} \{W_{n+1}-W_n\}\right)\\ &\geqq \mathbf{E}^{\langle x\rangle}\left(W_0 + \sum_{n=0}^{N} \{W_{n+1}-W_n\} + \sum_{n=N+1}^{\infty} \{\tilde{W}_{n+1}-W_n\}\right)\\ &= \varphi(x) + 0 + \mathbf{E}^{\langle x\rangle}\left(\sum_{n=N+1}^{\infty} \{\tilde{W}_{n+1}-W_n\}\right).\end{aligned}$$

As $N \to \infty$, the last term tends to 0, so that $\mathbf{E}^{\langle x \rangle} W \geqq \varphi(x)$. The converse inequality is immediate from Fatou's lemma, since $\mathbf{E}^{\langle x \rangle} W_n = \varphi(x)$.

Next suppose that $I_1 = +\infty$. By (2.9), the Borel-Cantelli lemma, and (2.10) we have a.s. convergence of $\sum \{W_{(n+1)\delta} - W_{n\delta} + \varepsilon_{n,\delta}\}$ and therefore, since W exists, of $\sum \varepsilon_{n,\delta}$. Let $W^- = \inf_n W_{n\delta}$.

Then by Lemmata 4 and 5

$$\infty > \sum_{n=0}^{\infty} \varepsilon_{n,\delta} \geqq c_2 W^- \sum_{n=0}^{\infty} \varphi^*[\xi_{n\delta,1}] = +\infty \text{ a.s.} \quad \text{on} \quad \{W^- > 0\}$$

which is only possible if $\mathbf{P}^{\langle x \rangle}(W > 0) = \mathbf{P}^{\langle x \rangle}(W^- > 0) = 0$. □

We now proceed to the proof of Theorem 1. The idea is quite simple and becomes transparent if we set $\eta_n = \eta$ in the next two lemmata. However, this identification would lead us to Theorem 1 for $|\eta| \leqq c\varphi$ only. The added generality of Lemma 6 is needed in order to deal with $\eta \in \mathscr{L}^1_{\varphi^*}$.

Lemma 6. *Let $\{\eta_n\}$ be a sequence of averaging functions such that $0 \leqq \eta_n \leqq \eta$ for some $\eta \in \mathscr{L}^1_{\varphi^*}$. Define for any m*

$$Y^1_{n,i} = \hat{x}^{n,i}_{n+m}[\eta_{n+m}], \qquad Z^1_{n,i} = \hat{x}^{n,i}_{n+m}[\eta].$$

Then (M), $\rho > 1$, *and the assumption*

$$\delta_{n,m} = \frac{1}{\rho^n} \sum_{i=1}^{\hat{x}_n[1]} \mathbf{E}(Y^1_{n,i} 1_{\{Z^1_{n,i} > \rho^n\}} | \mathfrak{F}_n) \xrightarrow{\text{a.s.}} 0, \qquad n \to \infty, \quad \forall m$$

imply

$$\hat{x}_n[\eta_n]/\rho^n - \varphi^*[\eta_n] W \xrightarrow{\text{a.s.}} 0, \qquad n \to \infty. \tag{2.13}$$

Proof. In the notation of Lemma 2 with $\delta = 1$,

$$\hat{x}_{n+m}[\eta_{n+m}]/\rho^{n+m} = S^1_n/\rho^m, \qquad \delta_{n,m} = \rho^m \mathbf{E}(S^1_n - \tilde{S}^1_n | \mathfrak{F}_n).$$

Also, by (2.2)

$$c^-_m \rho^m \varphi^*[\eta_{n+m}] W_n \leqq \mathbf{E}(S^1_n | \mathfrak{F}_n) \leqq c^+_m \rho^m \varphi^*[\eta_{n+m}] W_n.$$

Using (2.6) and (2.12) we get

$$\begin{aligned}
&\limsup_n \{\hat{x}_n[\eta_n]/\rho^n - \varphi^*[\eta_n] W\} \\
&\quad = \limsup_n \{\hat{x}_{n+m}[\eta_{n+m}]/\rho^{n+m} - \varphi^*[\eta_{n+m}] W\} \\
&\quad = \limsup_n \{S^1_n/\rho^m - \varphi^*[\eta_{n+m}] W\} \\
&\quad = \limsup_n \{\mathbf{E}(S^1_n | \mathfrak{F}_n)/\rho^m - \varphi^*[\eta_{n+m}] W\} \\
&\quad \leqq \limsup_n \varphi^*[\eta_{n+m}] \{c^+_m W_n - W\} \leqq \varphi^*[\eta] \{c^+_m - 1\} W.
\end{aligned}$$

Letting $m \to \infty$, it follows that the lim sup of the left-hand side of (2.13) is $\leqq 0$. The inequality for lim inf is obtained similarly. □

Remark. If we replace $n \in \mathbb{N}$ with $t \in [0, \infty[$, then $\mathbf{E}\,\delta_{t,m} \leqq \int_{\rho^t}^{\infty} y\, dG^1(y)$, where G^1 is as in the proof of Lemma 2. This shows $\delta_{t,m} \xrightarrow{\mathbf{P}} 0$, $t \to \infty$. Taking $\eta_n = \eta$ and using the remark following the proof of Lemma 2, we can repeat the argument with a.s. convergence replaced by convergence in probability to get (1.4).

Lemma 7. *If we can take* $\eta \leqq c\,\varphi$, $c \geqq 0$, *in Lemma 6, then* (2.13) *holds, assuming only* (M), $\rho > 1$.

Proof. We can assume $c=1$. Then by (2.7)

$$0 \leqq \delta_{n,m} \leqq \rho^{-n}\,\hat{x}_n[\xi_{n,m}] \leqq \rho^m\, W_n.$$

Thus by Theorem 2, (2.12) holds if $I_m = +\infty$, and otherwise Lemma 4 implies

$$\sum_{n=0}^{\infty} \mathbf{E}\,\delta_{n,m} \leqq c^* \sum_{n=0}^{\infty} \varphi^*[\xi_{n,m}] < \infty \qquad \text{and thus (2.12).} \quad \square$$

Proof of Theorem 1. We take $\eta_n = \eta$ in Lemma 6 and have to prove (2.12), that is $\delta_{n,m} = \rho^{-n}\,\hat{x}_n[\tilde{\eta}_n] \xrightarrow{\text{a.s.}} 0$, where

$$\tilde{\eta}_n(x) = \mathbf{E}^{\langle x\rangle}\,\hat{x}_m[\eta]\,1_{\{\hat{x}_m[\eta] > \rho^n\}}.$$

But by (2.2) $\{\tilde{\eta}_n\}$ satisfies the assumption of Lemma 7 with $c = c_m^+\,\rho^m\,\varphi^*[\eta]$ so that by (2.13)

$$\limsup_n \delta_{n,m} = \limsup_n \varphi^*[\tilde{\eta}_n]\, W = 0,$$

where we have used the dominated convergence theorem for the last equality. $\square$

We conclude this section by giving the transition from discrete to continuous time, i.e. the proof of Theorem 1′. Define for $\varepsilon, \delta > 0$, $U \subseteq X$, $\vartheta \in \mathscr{B}_+$

$$U^\varepsilon(x) = \{y \in U : \vartheta(y) \geqq (1+\varepsilon)^{-1}\,\vartheta(x)\},$$

$$\xi_U^{\delta,\varepsilon}(x) = \mathbf{P}^{\langle x\rangle}(\hat{x}_t \in U^\varepsilon(x)\,\forall\, t \in [0,\delta]).$$

Lemma 8. *Suppose that for every* $\varepsilon > 0$

$$\xi_U^{\delta,\varepsilon} \uparrow 1_U \quad \text{a.s.} \quad [\varphi^*], \quad \delta \downarrow 0. \tag{2.14}$$

Then

$$\liminf_{t\to\infty} \rho^{-t}\,\hat{x}_t[\vartheta\, 1_U] \geqq \varphi^*[\vartheta\, 1_U]\, W. \tag{2.15}$$

Proof. Consider the ith particle alive at time $n\delta$ with type x_i, and let

$$Y_{n,i}^{\delta,\varepsilon} = (1+\varepsilon)^{-1}\,\vartheta(x_i)\,1_{\{\hat{x}_t^{n\delta,i} \in U^\varepsilon(x_i)\forall t\in[n\delta,(n+1)\delta]\}}$$

Then

$$\hat{x}_t[\vartheta\, 1_U] \geqq \sum_{i=1}^{\hat{x}_{n\delta}[1]} Y_{n,i}^{\delta,\varepsilon} \quad \forall\, t \in [n\delta, (n+1)\delta].$$

With $Z_{n,i}^{\delta,\varepsilon} = \vartheta(x_i)$, (2.6) and (1.5) give

$$\begin{aligned}
\liminf_{t\to\infty} \rho^{-t}\,\hat{x}_t[\vartheta\, 1_U] &\geqq \rho^{-\delta} \liminf_{t\to\infty} \rho^{-n\delta} \sum_{i=1}^{\hat{x}_{n\delta}[1]} Y_{n,i}^{\delta,\varepsilon} \\
&= \rho^{-\delta} \liminf_{n\to\infty} \rho^{-n\delta} \sum_{i=1}^{\hat{x}_{n\delta}[1]} \mathbf{E}(Y_{n,i}^{\delta,\varepsilon} \mid \mathfrak{F}_{n\delta}) \\
&= \rho^{-\delta} \liminf_{n\to\infty} \rho^{-n\delta}\,\hat{x}_{n\delta}[(1+\varepsilon)^{-1}\,\vartheta\,\xi_U^{\delta,\varepsilon}] = \rho^{-\delta}\,\varphi^*[(1+\varepsilon)^{-1}\,\vartheta\,\xi_U^{\delta,\varepsilon}]\, W.
\end{aligned}$$

(2.15) follows from (2.14) by letting $\delta \to 0$, $\varepsilon \to 0$ in that order. $\square$

Lemma 9. *Suppose X is metric, $\mathfrak{A}$ the Borel σ-algebra, $(\hat{x}_t, \mathbf{P}^{\hat{x}})$ right-continuous. Also, let $\rho^{-t}\hat{x}_t[\vartheta] \xrightarrow{\text{a.s.}} \varphi^*[\vartheta]\, W$ for some $\vartheta \in \mathscr{B}_+$, l.s.c. a.e. $[\varphi^*]$. Then for any $U \subseteq X$ whose boundary ∂U has φ^*-measure* 0,

$$\rho^{-t}\hat{x}_t[\vartheta 1_U] \xrightarrow{\text{a.s.}} \varphi^*[\vartheta 1_U]\, W, \qquad t \to \infty. \tag{2.16}$$

Proof. If x is in the interior of U, our assumptions imply that $\xi_U^{\delta,\varepsilon}(x) \uparrow 1$ as $\delta \downarrow 0$, so that (2.15) holds. Since $\partial U = \partial U^c$, also

$$\limsup_{t\to\infty} \rho^{-t}\hat{x}_t[\vartheta 1_U] = \varphi^*[\vartheta]\, W - \liminf_{t\to\infty} \rho^{-t}\hat{x}_t[\vartheta 1_{U^c}] \leqq \varphi^*[\vartheta 1_U],$$

completing the proof. □

Proof of Theorem 1'. The case $\varphi^*[\vartheta]=0$ is trivial, and (1.7) is also obvious on $\{W=0\}$. On $\{W>0\}$ the random probability measure $\mu_t[\eta] = \hat{x}_t[\vartheta\eta]/\hat{x}_t[\vartheta]$ is well-defined and for each continuity set U of $\mu[\eta] = \varphi^*[\vartheta\eta]/\varphi^*[\vartheta]$, we have $\mu_t[1_U] \xrightarrow{\text{a.s.}} \mu[1_U]$ by (2.16). Taking an appropriate denumerable class of such U's, Theorem 2.2 of [6] shows that μ_t converges weakly to μ for almost all realizations of the process, completing the proof ([6]). □

§ 3. Branching Diffusions: Preliminaries and Representation of the First Moment Semi-Group

We now discuss our theory in terms of branching diffusions. The principal mathematical difficulties that arise in our context are present already with a one-dimensional diffusion. For greater clarity we therefore restrict ourselves to this case. However, all results and proofs of this and the following section can be formulated with n-dimensional diffusions, and we shall do this in the more comprehensive framework of a future publication.

Let $X \subset \mathbb{R}$ be a bounded interval with endpoints α, β. The interval may be closed, half-open, or open. Denote by $\mathscr{C}^n$ the set of real-valued functions on X which are restrictions of n-times continuously differentiable functions on $[\alpha, \beta]$. Let $\{\hat{x}_t, \mathbf{P}^{\hat{x}}\}$ be the Markov branching process determined ([16, 20]) by the following data:

(a) The *diffusion process* (x_t, P^x) on X defined by the differential generator of its transition semigroup,

$$A u = a \frac{d^2}{dx^2} u + b \frac{d}{dx} u, \qquad u \in \mathscr{D}(A),$$

where

$$a \in \mathscr{C}^2, \qquad b \in \mathscr{C}^1, \qquad \inf_{x\in X} a(x) > 0, \tag{3.1}$$

and $\mathscr{D}(A)$ is the set of all $u \in \mathscr{C}^2$ satisfying the separated endpoint conditions

$$\gamma_\alpha u(\alpha+) - \gamma'_\alpha u'(\alpha+) = 0, \qquad \gamma_\beta u(\beta-) + \gamma'_\beta u'(\beta-) = 0,$$
$$\gamma_\alpha, \gamma'_\alpha, \gamma_\beta, \gamma'_\beta \in \mathbb{R}_+, \qquad (\gamma_\alpha, \gamma'_\alpha) \neq (0,0), \qquad (\gamma_\beta, \gamma'_\beta) \neq (0,0).$$

Here $\gamma_\alpha \neq 0$, $\gamma'_\alpha = 0$ corresponds to complete absorption at α and $\gamma_\alpha = 0$, $\gamma'_\alpha \neq 0$ to complete reflection at α and correspondingly for β.

(b) The *termination density*

$$k \in \mathscr{C}^2, \quad k \geqq 0. \tag{3.2}$$

(c) The local *branching law*

$$\pi(x, \hat{A}) = p_0(x)\, 1_{\hat{A}}(\theta) + \sum_{n \geqq 2} 1_{\hat{A}}(\langle \overbrace{x, \ldots, x}^{n} \rangle)\, p_n(x) \qquad (\hat{A} \in \hat{\mathfrak{A}})$$

$$p_n \in \mathscr{B}_+, \qquad n \in \mathbb{N} \setminus \{1\}, \qquad \sum_{n \neq 1} p_n = \mathbf{1}$$

with

$$m = \sum_{n \neq 1} n p_n \in \mathscr{C}^2. \tag{3.3}$$

The process $\{\hat{x}_t, \mathbf{P}^{\hat{x}}\}$ is constructed according to the following intuitive picture: All particles move independently, each according to (x_t, P^x). Branching of a particle which is at the point $x \in X$ at time t occurs in $[t, t+\Delta]$ with probability $k(x)\, \Delta + o(\Delta)$. At the point x and time of branching a particle is replaced according to $\pi(x, \cdot)$ by a (possibly empty) population of new particles.

Condition (3.3) is more than enough to guarantee that $(\hat{x}_t, \mathbf{P}^x)$ is conservative ([20]).

Theorem 4. *The process* $\{\hat{x}_t, \mathbf{P}^{\hat{x}}\}$ *constructed from* (a), (b), (c) *has the property* (M) *with*

$$\varphi \in \mathscr{D}(A), \quad \varphi^*[\eta] = \int_\alpha^\beta \eta(x)\, \varphi(x)\, r(x)\, dx, \qquad 0 < c \leqq r(x) \in \mathscr{C}^2.$$

Proof. The diffusion process is constructable as a conservative process either on X, when X is compact and no absorption occurs, or on $X' = X \cup \{\partial\}$ with ∂ serving as trap. If X is non-compact, we take X' as the one-point compactification of X. Define $\mathscr{C}_0 = \mathscr{C}^0$ if X is compact and $\mathscr{C}_0 = \{\xi \in \mathscr{C}^0 \colon \xi(x) \to 0 \text{ as } x \to \partial\}$ otherwise. Then the transition semigroup $\{T_t\}$ of (x_t, P^x) satisfies $T_t \mathscr{C}_0 \subseteq \mathscr{C}_0$, $t \geqq 0$, and is strongly continuous on $\mathscr{C}_0$. In conjunction with $k, m \in \mathscr{C}^0$ this implies not only $\mathbf{E}^{\langle \cdot \rangle} \hat{x}_t[\mathbf{1}] \in \mathscr{B}_+$ $\forall t \geqq 0$, but also that $\{\mathbf{E}^{\langle \cdot \rangle} \hat{x}_t[\cdot]\}_{t \geqq 0}$ is a strongly continuous semigroup on $\mathscr{C}_0$ with differential generator

$$Lv = Av + k(m - \mathbf{1})\, v, \qquad v \in \mathscr{D}(L) = \mathscr{D}(A),$$

cf. [16], slightly adapted.

Consider now the eigenvalue problem

$$Lv = \lambda v, \qquad v \in \mathscr{D}(A).$$

Since $a, b, k, m \in \mathscr{C}^0$ and $\inf_x a(x) > 0$, multiplication by

$$p(x) = \exp\left\{\int_\alpha^x \frac{b(y)}{a(y)}\, dy\right\}$$

leads to the regular Sturm-Liouville problem

$$(p v')' + (q - \lambda r) v = 0, \qquad v \in \mathscr{D}(A),$$
$$q = k(m-1) p/a, \qquad r = p/a.$$

As is well-known, see e.g. [8] Chapters 7 and 8, there exists a sequence of real eigenvalues

$$\lambda_0 > \lambda_1 > \lambda_2 > \cdots \downarrow -\infty \tag{3.4}$$

and a complete system of eigenfunctions $\{v_\nu\}$, v_ν corresponding to λ_ν, such that

$$\int_\alpha^\beta v_\nu(x)\, v_\mu(x)\, r(x)\, dx = \delta_{\nu\mu}.$$

That is, for $\xi \in \mathscr{D}(A)$

$$\mathbf{E}^{\langle x \rangle}\, \hat{x}_t[\xi] = \sum_{\nu=0}^{\infty} e^{\lambda_\nu t}\, v_\nu^*[\xi]\, v_\nu(x), \qquad t > 0, \tag{3.5}$$

where

$$v_\nu^*[\xi] = \int_\alpha^\beta \xi(x)\, v_\nu(x)\, r(x)\, dx.$$

Moreover, v_ν has exactly ν zeros in $]\alpha, \beta[$, i.e. we can choose

$$v_0(x) > 0, \qquad x \in]\alpha, \beta[, \tag{3.6}$$

and know by unicity that

$$(v_0(\alpha+),\, v_0'(\alpha+)) \neq (0,0), \qquad (v_0(\beta-),\, v_0'(\beta-)) \neq (0,0). \tag{3.7}$$

By (3.1)–(3.3) we have $p, r \in \mathscr{C}^2$, and we can use Liouville's substitution

$$v(x) = (p(x)\, r(x))^{-1/4}\, w(y), \qquad y = \int_\alpha^x (r(z)/p(z))^{1/2}\, dz \in [0, y(\beta)]$$

to obtain the normal form

$$w'' + (\tilde{q} - \lambda)\, w = 0,$$

where

$$\tilde{q} = q/r + (p r)^{-1/4} \frac{d^2}{dy^2} (p r)^{1/4},$$

with regular separated endpoint conditions, generally with changed coefficients. The eigenvalues remain the same as before.

Now applying the theory of asymptotic spectral behavior of differential operators ([19]),

$$\lambda_\nu = -\left(\frac{\pi \nu}{y(\beta)}\right)^2 \left[1 + O\left(\frac{1}{\nu}\right)\right], \qquad \nu > 0, \tag{3.8}$$

$$w_\nu(y) = \left(\frac{2}{y(\beta)}\right)^{1/2} \sin\left[\sqrt{|\lambda_\nu|}\, y + c_\nu\right] + \delta_\nu(y), \qquad \nu > 0, \tag{3.9}$$

$$\sup_{y,\nu} |\nu\, \delta_\nu(y)| < \infty, \tag{3.10}$$

$$\sup_{y,\nu} |\delta'_\nu(y)| < \infty. \tag{3.11}$$

Given (3.8) to (3.10), we can extend (3.5) first to $\xi \in \mathscr{C}_0$, using the Stone-Weierstrass theorem, and then to $\xi \in \mathscr{B}$, using the fact that $\mathbf{E}^{\langle x \rangle} \hat{x}_t[1_A]$ and $v_\nu^*[1_A]$ are σ-additive in $A \in \mathfrak{A}$. With (3.6), (3.8), (3.9), (3.10) and, if $w_0(0)=0$ or $w_0(y(\beta))=0$, also (3.7), (3.11), and L'Hospital's rule

$$\sup_x \left|\frac{v_\nu(x)}{v_0(x)}\right| = \sup_y \left|\frac{w_\nu(y)}{w_0(y)}\right| = O(\nu). \tag{3.12}$$

Since $v_0 \geqq 0$, $r \geqq 0$, (3.12) implies

$$|v_\nu^*[\xi]| \leqq O(\nu)\, v_0^*[|\xi|]$$

uniformly in $\xi \in \mathscr{B}$. Hence

$$|\mathbf{E}^{\langle x \rangle} \hat{x}_t[\xi] - e^{\lambda_0 t} v_0^*[\xi]\, v_0(x)| \leqq \alpha_t\, v_0^*[|\xi|]\, v_0(x),$$

$$\alpha_t = \sum_{\nu=1}^{\infty} O(\nu^2)\, e^{\lambda_\nu t},$$

and by (3.4), (3.7), $e^{-\lambda_0 t} \alpha_t \to 0$, $t \to \infty$. Thus (M) with the convention $\varphi^*[\mathbf{1}]=1$ is satisfied with $\varphi = v_0^*[\mathbf{1}]\, v_0$, $\varphi^* = (v_0^*[\mathbf{1}])^{-1}\, v_0^*$, $\rho = e^{\lambda_0}$. □

§ 4. Branching Diffusions: Limit Results

The set-up is that of § 3 with $\rho = e^{\lambda_0} > 1$, $W_t = \rho^{-t} \hat{x}_t[\varphi]$, $W = \lim_t W_t$.

Theorem 1″. *Assume in addition to the assumptions of § 3 that $\rho > 1$. Then for any $\eta \in \mathscr{L}^1_{\varphi^*}$*

$$\rho^{-t} \hat{x}_t[\eta] \xrightarrow{\mathbf{P}} \varphi^*[\eta]\, W, \qquad t \to \infty,$$

$$\rho^{-n\varepsilon} \hat{x}_{n\varepsilon}[\eta] \xrightarrow{\text{a.s.}} \varphi^*[\eta]\, W, \qquad \mathbb{N} \ni n \to \infty, \qquad \forall \varepsilon > 0.$$

If η is bounded and a.e. continuous, then

$$\rho^{-t} \hat{x}_t[\eta] \xrightarrow{\text{a.s.}} \varphi^*[\eta]\, W, \qquad t \to \infty.$$

Proof. The two first assertions are immediate from (1.4) and (1.5). For the third we have only to prove $\rho^{-t} \hat{x}_t[\mathbf{1}] \xrightarrow{\text{a.s.}} W$, $t \to \infty$, in view of the absolute continuity of φ^*, $\varphi^*[\mathbf{1}]=1$, and Theorem 1′. However, the assumption (2.14) of Lemma 8 is fulfilled for $U = X$, so our problem reduces to proving

$$\limsup_{t \to \infty} \rho^{-t} \hat{x}_t[\mathbf{1}] \leqq W \quad \text{a.s.}$$

Define the auxiliary process $(y_t, \mathbf{P}^{\hat{x}})$, $y_t = \hat{x}_t[\mathbf{1}] + N_t$, where N_t is the number of particles absorbed up to time t plus the number of branching events up to time t. Clearly, y_t is non-decreasing and

$$\hat{x}_s[\mathbf{1}] \leqq y_t, \qquad 0 \leqq s \leqq t, \tag{4.1}$$

$$\mathbf{E}^{\langle x \rangle} y_t \leqq e^{\alpha t}, \qquad 0 < \alpha = \|k\|\,(\|m\|+1) < \infty. \tag{4.2}$$

Now let $\delta>0$ be fixed and let $Y^{\delta}_{n,i}=Z^{\delta}_{n,i}=y^{n\delta,i}_{(n+1)\delta}$ be defined the obvious way. Using (2.6), we get in the notation of Lemma 2

$$\begin{aligned}\limsup_{t\to\infty} \rho^{-t}\,\hat{x}_t[\mathbf{1}] &\leqq \limsup_{n\to\infty} \rho^{-n\delta} \sum_{i=1}^{\hat{x}_{n\delta}[\mathbf{1}]} Y^{\delta}_{n,i}\\ &=\limsup_{n\to\infty} S^{\delta}_n = \limsup_{n\to\infty} \mathbf{E}(\tilde{S}^{\delta}_n|\mathfrak{F}_{n\delta})\\ &\leqq \limsup_{n\to\infty} \mathbf{E}(S^{\delta}_n|\mathfrak{F}_{n\delta}) \leqq e^{\alpha\delta} \limsup_{n\to\infty} \rho^{-n\delta}\,\hat{x}_{n\delta}[\mathbf{1}] = e^{\alpha\delta}\,W\end{aligned}$$

As $\delta\downarrow 0$, the proof is complete. □

We recall the definition (1.3) of I_t. Degeneracy of W at 0 is, of course, equivalent to $I_t=+\infty$ for some $t>0$. But the relation $I_t<\infty$ is difficult to verify and it is therefore desirable to have a criterion in terms of the natural model parameters A, k, π of the branching diffusion. A heuristic consideration of I_t for small t leads to the condition

$$\varphi^*[k\,\varphi\,\kappa]<\infty, \tag{4.3}$$

where

$$\kappa(x)=\sum_{n=2}^{\infty} p_n(x)\, n\log n.$$

Theorem 2′. *For any $t>0$, $I_t<\infty$ if and only if* (4.3) *holds. That is,*

$$\mathbf{E}^{\langle x\rangle}\,W=\varphi(x) \qquad \forall x\in X$$

if and only if (4.3) *holds, and $W=0$ a.s. $[\mathbf{P}^{\langle x\rangle}]$ $\forall x\in X$ otherwise.*

Remark. The properties of φ and φ^* obtained in § 3 make it possible to verify (4.3) without knowing φ or φ^* explicitly.

E.g. in the case of absorption at α and β, (4.3) is equivalent to

$$\int_{\alpha}^{\beta} (x-\alpha)^2\,(\beta-x)^2\,k(x)\,\kappa(x)\,dx<\infty.$$

Of course the heuristic derivation of (4.3) does not constitute a rigorous proof. To see the problem, note that the situation is somewhat similar to that of [3], where despite the simplicity of the model the proof is non-trivial. In addition to the times of branching, or split times, $\tau_1, \tau_2, \ldots$, our proof also has to take into account the stopped diffusion, i.e. the transition semigroup $\{T^0_t\}$ of the $\exp\left\{-\int_0^t k(x_s)\,ds\right\}$-subprocess of (x_t, P^x).

For convenience let $c_1, c_2, \ldots$ denote constants with $0<c_v<\infty$.

Lemma 10. *There are $t_0>0$ and c_1, c_2 such that*

$$c_1\,\varphi^* \leqq \varphi^*\,T^0_t \leqq c_2\,\varphi^*, \qquad 0\leqq t\leqq t_0. \tag{4.4}$$

Proof. We have

$$\varphi^*\,T^0_t = T^{0*}_t\,\varphi^* = \varphi^* + \int_0^t T^{0*}_s\,A^{0*}\,\varphi^*\,ds, \tag{4.5}$$

where A^{0*} is the adjoint of the generator A^0 of $\{T_t^0\}$, see [11]. But

$$(A^{0*}+k\,m)\,\varphi^* = \lambda_0\,\varphi^*, \tag{4.6}$$

so that

$$\varphi^*\,T_t^0 \leqq \varphi^* + \lambda_0 \int_0^t T_s^{0*}\,\varphi^*\,ds,$$

from which the second inequality in (4.4) follows by iteration (take $c_2 = e^{\lambda_0 t_0}$). Also, from (4.6), $A^{0*}\,\varphi^* \geqq -c_3\,\varphi^*$. Inserting this in (4.5), we get

$$\varphi^*\,T_t^0 \geqq \varphi^* - c_3 \int_0^t T_s^{0*}\,\varphi^*\,ds \geqq (1-c_2\,c_3\,t_0)\,\varphi^*$$

for $t \leqq t_0$. Choose t_0 such that $e^{\lambda_0 t_0}\,c_3\,t_0 < 1$. □

Proof of the Sufficiency of (4.3). For $\hat{x}_0 \in \hat{X}$, $x \in X$ define (cf. (2.3))

$$\tilde{\iota}^*_{t,n}(\hat{x}_0) = \mathbf{E}^{\hat{x}_0}\,\hat{x}_t[\varphi]\,\log^*\,\hat{x}_t[\varphi]\,1_{\{\tau_{n+1}>t\}}, \qquad \iota^*_{t,n}(x) = \tilde{\iota}^*_{t,n}(\langle x\rangle)$$

Then

$$\iota^*_{t,n+1} = \int_0^t T_s^0 \left[k \int_{\hat{X}} \tilde{\iota}^*_{t-s,n}(\hat{x})\,\pi(\cdot, d\hat{x})\right] ds + \iota^*_{t,0}. \tag{4.7}$$

For $j = 1, \ldots, \hat{x}_0[\mathbf{1}]$, define $B(j,n)$ as the event that there are at most n splits in the j th line of descent up to time t. Then

$$\hat{x}_t[\varphi]\,1_{\{\tau_{n+1}>t\}} \leqq \sum_{j=1}^{\hat{x}_0[\mathbf{1}]} \hat{x}_t^{0,j}[\varphi]\,1_{B(j,n)},$$

and by (2.1),

$$\tilde{\iota}^*_{t,n}(\hat{x}_0) \leqq \rho^t\,\hat{x}_0[\varphi]\,\log^*(\rho^t\,\hat{x}_0[\varphi]) + \hat{x}_0[\iota^*_{t,n}].$$

Thus for $0 \leqq t \leqq t_0$,

$$\begin{aligned} &\int_{\hat{X}} \tilde{\iota}^*_{t,n}(\hat{x})\,\pi(x, d\hat{x}) \\ &\leqq \sum_{\nu=0}^{\infty} p_\nu(x)\,\{\rho^t\,\nu\,\varphi(x)\,\log^*(\rho^t\,\nu\,\varphi(x)) + \nu\,\iota^*_{t,n}(x)\} \\ &\leqq c_4 + c_5\,\varphi(x)\,\kappa(x) + c_6\,\iota^*_{t,n}(x). \end{aligned} \tag{4.8}$$

Inserting (4.4) and (4.8) in (4.7) leads to

$$\varphi^*[\iota^*_{t,n+1}] \leqq c_7 + c_8\,\varphi^*[k\,\varphi\,\kappa] + c_9\,t \sup_{0\leqq s\leqq t} \varphi^*[\iota^*_{s,n}]$$

for $0 \leqq t \leqq t_0$. Since $\|\iota^*_{t,0}\| \leqq c_{10}\ \forall t>0$, it follows from our assumption (4.3) that

$$I_t^* = \varphi^*[\iota_t^*] \leqq \sup_{n\in\mathbb{N}} \sup_{0\leqq s\leqq t} \varphi^*[\iota^*_{s,n}] < \infty$$

for $0 \leqq t \leqq t_0$, $c_9\,t < 1$. □

Proof of the Necessity of (4.3). Let

$$\tilde{\iota}_t^*(\hat{x}) = \mathbf{E}^{\hat{x}}\,\hat{x}_t[\varphi]\,\log^*\,\hat{x}_t[\varphi],$$

so that $\iota_t^*(x) = \tilde{\iota}_t^*(\langle x \rangle)$ (cf. (2.3)). Then

$$\iota_t^* \geqq \mathbf{E}^{\langle \cdot \rangle} \hat{x}_t[\varphi] \log^* \hat{x}_t[\varphi] \, 1_{\{\tau_1 \leqq t\}} = \int_0^t T_s^0 [k \int_{\hat{X}} \tilde{\iota}_{t-s}^*(\hat{x}) \, \pi(\cdot, d\hat{x})] \, ds. \tag{4.9}$$

Since $\hat{x}_t[\varphi]$ is a submartingale, it follows by convexity that

$$\tilde{\iota}_t^*(\hat{x}) \geqq \hat{x}[\varphi] \log^* \hat{x}[\varphi],$$

$$\int_{\hat{X}} \tilde{\iota}_t^*(\hat{x}) \, \pi(x, d\hat{x}) \geqq \sum_{n=2}^{\infty} p_n(x) \, n \, \varphi(x) \log^*(n \, \varphi(x)) \geqq \varphi(x) \, \kappa(x) - c_{11}. \tag{4.10}$$

Thus, if $\varphi^*[k \varphi \kappa] = +\infty$, it follows by inserting (4.4) and (4.10) in (4.9) that

$$I_t^* = \varphi^*[\iota_t^*] \geqq c_1 \int_0^t \varphi^*[k \varphi \kappa - k c_{11}] \, ds = +\infty$$

for $0 \leqq t \leqq t_0$. □

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