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On Strassen's Version of the Loglog Law for Some Classes of Dependent Random Variables*

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1. Preliminaries

Let $(\Omega, \mathscr{K}, \mathbf{P})$ be a probability space and for any two σ-algebras $\mathscr{K}_1$ and $\mathscr{K}_2$ contained in $\mathscr{K}$ define their dependence coefficient by (cf. [6], p. 1)

$$\phi(\mathscr{K}_1, \mathscr{K}_2) = \sup_{B \in \mathscr{K}_2} \left(\operatorname{ess\,sup}_{\omega \in \Omega} |\mathbf{P}(B|\mathscr{K}_1)(\omega) - \mathbf{P}(B)|\right).$$

Consider a strictly stationary sequence $(\xi_n)_{n \geq 1}$ of real valued random variables on Ω and assume that $\mathbf{E}\xi_1 = 0$, $\mathbf{E}|\xi_1|^{2+\delta} < \infty$ for some $\delta > 0$. For any $\Lambda \subset N^* = \{1, 2, \ldots\}$ denote by $\mathscr{K}_\Lambda$ the σ-algebra generated by the family $(\xi_n : n \in \Lambda)$ and put

$$\phi(n) = \sup_{r \in N^*} \phi(\mathscr{K}_{(1, \ldots, r)}, \mathscr{K}_{(r+n, \ldots)}).$$

It will be assumed that

$$\sum_{n \in N^*} \phi^{\frac{1}{2}}(n) < \infty.$$

It is known that under the conditions assumed, if we set

$$\sigma^2 = \mathbf{E}\xi_1^2 + 2 \sum_{n \in N^*} \mathbf{E}\xi_1 \xi_{n+1},$$

we have $0 \leqq \sigma^2 < \infty$ and $\mathbf{E}\left(\sum_{j=1}^{n} \xi_j\right) = n(\sigma^2 + \rho_n)$ with $\rho_n = o(1)$ as $n \to \infty$.

Such strictly stationary sequences have been first studied in [3] where, among other results, it has been proved that the central limit theorem holds for suitably normed consecutive partial sums. It has been shown in [4] and [5] that these sequences obey the loglog law, namely, if $\sigma > 0$, then

$$\mathbf{P}\left(\limsup_{n \to \infty} \frac{\sum_{j=1}^{n} \xi_j}{\sigma\sqrt{2n \log\log n}} = 1\right) = 1.$$

The same result has been later given in [7].

This paper shows that Strassen's version of the loglog law (cf. [8], Theorem 3) is still valid for strictly stationary sequences satisfying the above conditions. Moreover, it will be shown (§ 3) that some non-stationary sequences obey Strassen's theorem too.

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The proof of the main result is an adaptation of the proof Chover [1] has given to Strassen's theorem using the tools developed in [4] and [5].

2. The Main Result

Let C denote the Banach space of all continuous real valued functions defined on $[0, 1]$ endowed with the usual supremum norm. Let K be the set of absolutely continuous $h \in C$ such that $h(0)=0$ and $\int_0^1 [h'(t)]^2\, dt \leq 1$. Here h' denotes the derivative of h determined almost everywhere with respect to Lebesgue measure.

Assume that $\sigma > 0$ and for each fixed $\omega \in \Omega$ and $n \geqq 3$ define the function $f_n(\cdot, \omega)$ in C by

$$f_n(0, \omega) = 0, \qquad f_n\left(\frac{k}{n}, \omega\right) = \frac{\sum_{j=1}^{k} \xi_j(\omega)}{\sigma \sqrt{2n \operatorname{loglog} n}}, \qquad 1 \leqq k \leqq n,$$

and linear over the subintervals $k/n \leqq t \leqq (k+1)/n$, $0 \leqq k \leqq n-1$. Our main result is the following extension of Strassen's theorem.

Theorem 1. *The sequence $(f_n(\cdot, \omega))_{n \geqq 3}$ considered as a subset of C is precompact and its derived set coincides with K for almost every $\omega \in \Omega$.*

Proof. To prove the above theorem one has to follow Chover's proof of the corresponding theorem for independent, identically distributed random variables making the following changes.

1. The maximal inequality (4) in [1], p. 84 is to be replaced by the maximal inequality that can be deduced from Lemma 7 in [5] just as the former was obtained from Lemma (2) in [2], p. 192.

2. Esseen's estimate of the rapidity of convergence to the normal distribution is to be replaced by the estimate given in Theorem 1 of [4].

3. The arguments in Section 4 of [1] from line 10 of p. 88 on are now to be based on the fact that for every fixed $m \in N^*$ the distribution function

$$\mathbf{P}\left(\frac{m}{n\sigma^2} \sum_{v=0}^{m-1} \left(\sum_{k=[(v/m)n]+1}^{[((v+1)/m)n]} \xi_k \right)^2 < a \right)$$

is asymptotically $(n \to \infty)$ $\Psi_m(a) + o(n^{-c(\delta)})$ uniformly in $a \geqq 0$, where Ψ_m is the distribution function of $\chi^2(m, 1)$ and $c(\delta) > 0$ is a constant depending on δ. To prove this one has to immitate the computations in [4], pp. 307–309 after having written

$$\sum_{k=[(v/m)n]+1}^{[((v+1)/m)n]} \xi_k = u_v + v_v,$$

with

$$u_v = \sum_{k=[(v/m)n]+1}^{[((v+1)/m)n]-s} \xi_k, \qquad v_v = \sum_{k=[((v+1/m)n]-s+1}^{[((v+1)/m)n]} \xi_k,$$

$0 \leqq v \leqq m-1$, where $s = [(n/m)^\rho]$, $0 < \rho < 1$ suitably chosen.

4. The classical Borel-Cantelli lemma (for independent events) used in [1], p. 90 is to be replaced by Lemma 5′ in [4].

3. An Extension

We shall indicate a class of non-stationary sequences of random variables to which Theorem 1 still applies.

Let $(W, \mathscr{W})$ and $(X, \mathscr{X})$ be two measurable spaces, $u(\cdot;\cdot)$ a $\mathscr{W}\times\mathscr{X}$-measurable mapping of $W\times X$ into W, P a transition probability function from $(W,\mathscr{W})$ to $(X,\mathscr{X})$ (i.e. $P(w;\cdot)$ is a probability on $\mathscr{X}$ for any $w\in W$ and $P(\cdot;A)$ is a $\mathscr{W}$-measurable function for any $A\in\mathscr{X}$). These are the elements constituting a *homogeneous random system with complete connection* (for details see [6], Ch. 2).

Denote by $(X^n, \mathscr{X}^n)$ the n-fold product of the measurable space $(X,\mathscr{X})$ and set $u(\cdot;x^{(n)})=u(\cdot;x_n)\circ\cdots\circ u(\cdot;x_1)$ for $x^{(n)}=(x_1,\ldots,x_n)\in X^n$, $n\in N^*$, where $\circ$ denotes composition of mappings.

For any $w\in W$, consider the probability space $(\Omega,\mathscr{K},\mathbf{P}_w)$, the elements of which are defined as follows:[1] $\Omega=X^{N^*}$, $\mathscr{K}=\mathscr{X}^{N^*}$,

$$\mathbf{P}_w(\mathrm{pr}_{(1)}^{-1}A)=P(w;A),\qquad A\in\mathscr{X},$$

$$\mathbf{P}_w(\mathrm{pr}_{(1,l)}^{-1}A_1\times\cdots\times A_l)=\int_{A_1}P(w;dx_1)\int_{A_2}P(u(w;x_1);dx_2)\ldots\int_{A_l}P(u(w;x^{(l-1)});dx_l)$$

for $l\geqq 2$ and $A_1,\ldots,A_l\in\mathscr{X}$. Define on Ω the sequence of random variables $(\eta_n)_{n\in N^*}$ with values in X given by $\eta_n(\omega)=x_n$ if $\omega=(x_n)_{n\in N^*}$. Then we have

$$\mathbf{P}_w(\eta_1\in A)=P(w;A),$$
$$\mathbf{P}_w(\eta_{n+1}(\omega)\in A\,|\,\eta_j,1\leqq j\leqq n)=P(u(w;\eta^{(n)});A)$$

for all $n\in N^*$, $A\in\mathscr{X}$, where $\eta^{(n)}=(\eta_1,\ldots\eta_n)$.

We shall say that *uniform ergodicity* holds if for each $l\in N^*$ there exists a probability P_l^∞ on $\mathscr{X}^l$ such that

$$\lim_{n\to\infty}\mathbf{P}_w(\mathrm{pr}_{(n,n+l-1)}^{-1}A^{(l)})=P_l^\infty(A^{(l)})$$

uniformly with respect to $w\in W$, $l\in N^*$, $A^{(l)}\in\mathscr{X}^l$. Set

$$\varepsilon_n=\sup|\mathbf{P}_w(\mathrm{pr}_{(n,n+l-1)}^{-1}A^{(l)})-P_l^\infty(A^{(l)})|,\qquad n\in N^*,$$

the sup being taken over all $w\in W$, $l\in N^*$, $A^{(l)}\in\mathscr{X}^l$. For conditions ensuring uniform ergodicity and for estimates of the ε_n see [6], pp. 81–85.

When uniform ergodicity holds there exists a probability $\mathbf{P}_\infty$ on $\mathscr{K}$ such that the sequence $(\eta_n)_{n\in N^*}$ is a strictly stationary one on $(\Omega,\mathscr{K},\mathbf{P}_\infty)$. In particular, $\mathbf{P}_\infty(\mathrm{pr}_{(n,n+l-1)}^{-1}A^{(l)})=P_l^\infty(A^{(l)})$ for any $l,n\in N^*$, $A^{(l)}\in\mathscr{X}^l$. Further, $\phi_\infty(n)\leqq\varepsilon_n$ for all $n\in N^*$. Finally, if we consider the tail σ-algebra

$$\mathscr{T}=\bigcap_{n\in N^*}\mathscr{K}_{(n,\ldots)},$$

then $\mathbf{P}_\infty(T)=\mathbf{P}_w(T)$ for all $T\in\mathscr{T}$ and $w\in W$ (cf. [6], pp. 135–136).

Taking into account the fact that the event appearing in Theorem 1 lies in $\mathscr{T}$ we can state

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[1] For $\Lambda\subset N^*$ the mapping pr_Λ is the projection of X^{N^*} on X^Λ defined by $\mathrm{pr}_\Lambda\{(x_n)_{n\in N^*}\}=(x_\lambda)_{\lambda\in\Lambda}$. If $\Lambda=(m,m+1,\ldots,n)$ we shall write $\mathrm{pr}_\Lambda=\mathrm{pr}_{(m,n)}$.

Theorem 2. *Let f be an $\mathscr{X}$-measurable real valued function and set $\xi_n = f \circ \eta_n$. Assume that*

i) $\mathbf{E}_\infty \xi_1 = 0$ *and there exists a* $\delta > 0$ *such that* $\mathbf{E}_\infty |\xi_1|^{2+\delta} < \infty$;

ii) $\sum_{n \in N^*} \varepsilon_n^{\frac{1}{2}} < \infty$;

iii) $\sigma^2 = \mathbf{E}_\infty \xi_1^2 + 2 \sum_{n \in N^*} \mathbf{E}_\infty \xi_1 \xi_{n+1} \neq 0$.

Under these conditions the sequence $(f_n(\cdot, \omega))_{n \geqq 3}$ considered as a subset of C is precompact and its derived set coincides with K both $\mathbf{P}_\infty$- and $\mathbf{P}_w$-almost surely whatever $w \in W$.

Applications to the continued fraction expansion will be given elsewhere.

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