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Deviations in the Skorohod-Strassen Approximation Scheme

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1. Introduction

Recent proofs of the law of the iterated logarithm rely on the Skorohod embedding into Brownian motion $\xi(t)$ of the sum S_n of n independent, identically distributed random variables X_i with mean zero and variance one (see [3, pp. 276 to 278] and (4) in the Appendix). By this embedding S_n has the same distribution as $\xi\left(\sum_{i=1}^{n} T_i\right)$ where the sequence $\{T_i\}_{i\geqq 1}$ are independent, non-negative, identically distributed with mean one. Using the strong law of large numbers on $\sum_{i=1}^{n} T_i$ it is shown (see [3, pp. 291–292]) that

$$(1.1)\qquad P\left[\omega:\ \overline{\lim_{n\to\infty}}\frac{\xi\left(\sum_{i=1}^{n} T_i\right)-\xi(n)}{\sqrt{n\lg\lg n}}=0\right]=1$$

where $\lg n=\log_e n$.

More recent work has involved the situation where the X_i have moments higher than the second and thus the T_i have moments higher than the first. For this case better convergence rates than given by the strong law of large numbers may be used (see [2]). In this section assuming $\overline{\lim_{n\to\infty}}\left|\sum_{i=1}^{n}(T_i-1)\right|=O(c_n)$ a.s. for a sequence of positive numbers $\{c_n\}$, it is shown that

$$(1.2)\qquad P\left[\omega:\ \overline{\lim_{n\to\infty}}\frac{\xi\left(\sum_{i=1}^{n} T_i\right)-\xi(n)}{(c_n\lg n)^{\frac{1}{2}}}<\infty\right]=1.$$

This is an upper class result for the $\overline{\lim}$.

Kiefer [9] has considered a more specialized case. Assuming $E(T_i-1)^2=\beta<\infty$ he shows that

$$(1.3)\qquad P\left[\omega:\ \overline{\lim_{n\to\infty}}\frac{\xi\left(\sum_{i=1}^{n} T_i\right)-\xi(n)}{((2\beta n\lg\lg n)^{\frac{1}{2}}\lg n)^{\frac{1}{2}}}=1\right]=1.$$

Notice that he finds the exact constant for the $\overline{\lim}$. His proof relies on the law of the iterated logarithm applied to the sequence $\{T_i\}$. Namely,

$$(1.4)\qquad P\left[\omega:\ \overline{\lim_{n\to\infty}}\frac{\sum_{i=1}^{n}(T_i-1)}{\sqrt{2\beta n\lg\lg n}}=1\right]=1.$$

In this section Kiefer's result is generalized. Essentially it is shown that if there exists a sequence of numbers $\{c_n\}_{n\geq 1}$, where c_n is regularly varying with exponent $\frac{1}{2}$, $c_n/n^{\frac{1}{2}}\uparrow$ as $n\to\infty$, and

$$(1.5)\qquad P\left[\omega\colon \overline{\lim_{n\to\infty}}\frac{\sum_{i=1}^{n}(T_i-1)}{c_n}=1\right]=1$$

and

$$(1.6)\qquad P\left[\omega\colon \overline{\lim_{n\to\infty}}\frac{\left|\sum_{i=1}^{n}(T_i-1)\right|}{c_n}=1\right]=1$$

then

$$(1.7)\qquad P\left[\omega\colon \overline{\lim_{n\to\infty}}\frac{\xi\left(\sum_{i=1}^{n}T_i\right)-\xi(n)}{(c_n\lg n)^{\frac{1}{2}}}=1\right]=1.$$

This is both an upper and lower class result for the $\overline{\lim}$. The method of proof follows that employed by Kiefer but is substantially simpler.

This section concludes with a specific example of a sequence of random variables $\{T_i\}$ such that $E(T_i-1)^2=\infty$ and a sequence of numbers $\{c_n\}$ such that (1.5) and (1.6) and thus Eq. (1.7) are satisfied. The example is interesting since the usual law of the iterated logarithm does not apply (see [14]) to this sequence of variables.

2. Proof of Eq. (1.2)

Let $\{X_n\}_{n\geq 1}$ be independent random variables with the same distribution; make the normalizations $E(X_n)=0$, $E(X_n^2)=1$; let $S_n=X_1+\cdots+X_n$; and let $\xi\left(\sum_{i=1}^{n}T_i\right)$ be the Skorohod representation of S_n where $\xi(t)$ is standard Brownian motion (see (4) in the Appendix). We can prove an easy upper class result about the fluctuations in the Skorohod embedding versus the Brownian motion.

(2.1) **Theorem.** *Suppose* $\overline{\lim_{n\to\infty}}\frac{\left|\sum_{i=1}^{n}(T_i-1)\right|}{c_n}\leq K$ *with probability one for some* $K>0$ *where* $\{c_n\}$ *is a sequence of positive numbers, then*

$$P\left[\omega\colon \overline{\lim_{n\to\infty}}\frac{\xi\left(\sum_{i=1}^{n}T_i\right)-\xi(n)}{(c_n\lg n)^{\frac{1}{2}}}<\infty\right]=1$$

where $\lg n=\log_e n$.

Proof. For $\theta>0$ let

$$A_n=\left\{\omega\colon \xi\left(\sum_{i=1}^{n}T_i\right)-\xi(n)>\theta(c_n\lg n)^{\frac{1}{2}}\right\}\cap\left\{\omega\colon \left|\sum_{i=1}^{n}T_i-n\right|\leq 2Kc_n\right\}.$$

It is sufficient to show that with probability one only finitely many A_n occur.

$$(2.2)\qquad A_n\subset\{\omega\colon \sup_{|t-n|\leq 2Kc_n}\xi(t)-\xi(n)>\theta(c_n\lg n)^{\frac{1}{2}}\}\equiv A_n'.$$

In the following C stands for various positive constants. By Lemma (3) in the Appendix

$$\begin{aligned}P(A_n') &\leqq 4P(\xi(n+2Kc_n)-\xi(n)>\theta(c_n \lg n)^{\frac{1}{2}})\\ &=4P\left(\frac{\xi(n+2Kc_n)-\xi(n)}{\sqrt{2Kc_n}}>\frac{\theta(c_n \lg n)^{\frac{1}{2}}}{\sqrt{2Kc_n}}\right)\\ &=4P\left(\xi(1)>\frac{\theta}{\sqrt{2K}}(\lg n)^{\frac{1}{2}}\right)\\ &\leqq C e^{-\theta^2 \lg n/4K} \text{ by Gaussian tail estimates}\\ &\leqq C n^{-\theta^2/4K}.\end{aligned}$$

Thus,

$$\begin{aligned}\sum_{n=1}^{\infty} P(A_n) &\leqq \sum_{n=1}^{\infty} P(A_n')\\ &\leqq C\sum_{n=1}^{\infty} n^{-\theta^2/4K}\\ &<\infty\end{aligned}$$

if $\theta^2/4K>1$. By Borel-Cantelli only finitely many A_n occur a.s. This completes the proof.

(2.3) **Proposition.** *Let* $\{c_n\}_{n\geqq 1}$ *be a sequence of positive numbers, then*

$$P\left[\omega:\ \overline{\lim_{n\to\infty}}\left(\sup_{|t-n|\leqq c_n}\frac{\xi(t)-\xi(n)}{(c_n \lg n)^{\frac{1}{2}}}\right)<\infty\right]=1.$$

Proof. Same as for Theorem (2.1).

The proofs of Theorem (2.1) and Proposition (2.3) do not require the usual and more difficult method (see [2]) of looking at the events in (2.2) along a subsequence. (However, the method will not yield Eq. (1.1).) That we do have the right order of magnitude for the $\overline{\lim}$ is indicated by later equations of the form (1.7) and by the next proposition.

(2.4) **Proposition.** *Let* $\{c_n\}_{n\geqq 1}$ *be a sequence of real numbers where* $c_n\uparrow\infty$ as $n\to\infty$ *and* $c_n\leqq n^{1-\delta}$ *for some* $\delta>0$, *then*

$$P\left[\omega:\ 0<\overline{\lim_{n\to\infty}}\left(\sup_{|t-n|\leqq c_n}\frac{\xi(t)-\xi(n)}{(c_n \lg n)^{\frac{1}{2}}}\right)\right]=1.$$

Proof. Set $\varepsilon>0$ and let

(2.5) $$n_{k+1}=n_k+[c_{n_k}].$$

(2.6) $$\begin{aligned}&\sum_{k=1}^{\infty} P(\xi(n_{k+1})-\xi(n_k)>\varepsilon(c_{n_k}\lg n_k)^{\frac{1}{2}})\\ &\quad=\sum_{k=1}^{\infty} P(\xi(1)>\varepsilon(\lg n_k)^{\frac{1}{2}})\end{aligned}$$

(2.7) $$\geqq\sum_{k=1}^{\infty}\frac{1}{n_k^{[\varepsilon^2/2+o(1)]}}$$

by Gaussian tail estimates.

Now note that

$$\sum_{n=n_k}^{n_{k+1}} \frac{1}{c_n n^{[\varepsilon^2/2+o(1)]}} \leqq (n_{k+1}-n_k) \frac{1}{c_{n_k} n_k^{[\varepsilon^2/2+o(1)]}} \tag{2.8}$$
$$= \frac{c_{n_k}}{c_{n_k} n_k^{[\varepsilon^2/2+o(1)]}} = \frac{1}{n_k^{[\varepsilon^2/2+o(1)]}}.$$

Using this in line (2.7) gives

$$\sum_{k=1}^{\infty} P(\xi(n_{k+1})-\xi(n_k)>\varepsilon(c_{n_k} \lg n_k)^{\frac{1}{2}})$$
$$\geqq \sum_{k=1}^{\infty} \sum_{n=n_k}^{n_{k+1}} \frac{1}{c_n n^{[\varepsilon^2/2+o(1)]}} \geqq \sum_{n=1}^{\infty} \frac{1}{c_n n^{[\varepsilon^2/2+o(1)]}} \tag{2.9}$$
$$\geqq \sum_{n=1}^{\infty} \frac{1}{n^{(1-\delta)} n^{[\varepsilon^2/2+o(1)]}} = \infty \quad \text{when } \frac{\varepsilon^2}{2} < \delta.$$

Thus, since the events in (2.9) are independent, the Borel-Cantelli lemma gives $\xi(n_{k+1})-\xi(n_k)>\varepsilon(c_{n_k} \lg n_k)^{\frac{1}{2}}$ infinitely often almost surely which yields the required result.

The following theorem due to Feller [6] is useful for finding sequences $\{c_n\}$ in Theorem (2.1) in terms of moment conditions on the random variables $\{X_n\}$.

(2.10) **Theorem (Feller).** *Let $\{X_n\}_{n\geqq 1}$ be a sequence of independent, identically distributed random variables such that $E(|X_1|)<\infty$, $E(X_1)=0$, and for some $0<\delta<1$, $E(|X_1|^{1+\delta})=\infty$. Let $\{c_n\}$ be a sequence of numbers for which there exists an ε with $0<\varepsilon<1$ such that $c_n n^{-1/1+\varepsilon}\uparrow$, $c_n/n\downarrow$, and let $S_n=X_1+\cdots+X_n$, then $|S_n|>c_n$ infinitely often* a.s. *if and only if $|X_n|>c_n$ infinitely often* a.s.

In the following there are two applications of Theorem (2.1).

(2.11) *Example.* Assume $E(X_i)=0$, $E(X_i^2)=1$, $E(|X_i|^3)<\infty$, and $E(|X_i|^{3+\delta})=\infty$ for any $\delta>0$. Then by the Skorohod embedding $\xi\left(\sum_{i=1}^{n} T_i\right)$ has the same distribution as $S_n=X_1+\cdots+X_n$, $E(T_i)=1$, and $E(T_i^{\frac{3}{2}})<\infty$. Assume $E(T_i^{\frac{3}{2}+\delta})=\infty$ for any $\delta>0$. By Feller's theorem (2.10) and Lemma (1) in the Appendix

$$\left|\sum_{i=1}^{n} T_i - n\right| = O(n^{\frac{2}{3}}) \text{ a.s.}$$

Thus by Theorem (2.1)

$$\xi\left(\sum_{i=1}^{n} T_i\right) - \xi(n) = O(n^{\frac{2}{3}} \lg n)^{\frac{1}{2}} \text{ a.s.}$$

(2.12) *Example.* Assume $E(X_i)=0$, $E(X_i^2)=1$, and $E(X_i^4)<\infty$. Then $\xi\left(\sum_{i=1}^{n} T_i\right)$ has the same distribution as S_n, $E(T_i)=1$, and $E(T_i^2)<\infty$. By the law of the iterated logarithm

$$\left|\sum_{i=1}^{n} T_i - n\right| = O(n \lg\lg n)^{\frac{1}{2}} \text{ a.s.}$$

Thus by Theorem (2.1)

$$\xi\left(\sum_{i=1}^{n} T_i\right)-\xi(n)=O[\lg n(n \lg\lg n)^{\frac{1}{2}}]^{\frac{1}{2}} \text{ a.s.}$$

Except for showing the exact constant, this is the upper class result of Kiefer [9].

3. Proof of Eq. (1.7)

Under more specialized conditions on the sequence of random variables $\{X_i\}_{i\geq 1}$ it is possible to find the exact deviations of the $\overline{\lim}$ and prove equations of the form (1.7).

(3.1) **Theorem.** *If $\{T_i\}_{i\geq 1}$ are the random times of the Skorohod embedding and $U_n=T_1+\cdots+T_n$ is such that there exists a sequence of real numbers $\{c_n\}$ where c_n is regularly varying with exponent $\frac{1}{2}$, $c_n/n^{\frac{1}{2}}\uparrow$, and*

(3.2) $$\overline{\lim_{n\to\infty}}\frac{|U_n-n|}{c_n}\leqq K \text{ a.s.,}\qquad \overline{\lim_{k\to\infty}}\frac{U_{n_k}-n_k}{c_{n_k}}\geqq\left(1-\frac{\varepsilon}{2}\right) \text{ a.s.}$$

for some $K>0$, $\frac{\varepsilon}{2}>0$ and subsequence $\{n_k\}_{k\geq 1}$ where $\sum_{i=1}^{r} n_i=O(n_r)$, $n_k\geqq\gamma^k$ for some $\gamma\geqq 2$, and

(3.3) $$\overline{\lim_{n\to\infty}}\frac{U_n-n}{c_n}=1 \text{ a.s.}$$

then

$$P\left[\omega:\ 1\leqq\overline{\lim_{n\to\infty}}\frac{\xi(U_n)-\xi(n)}{(c_n\lg n)^{\frac{1}{2}}}\right]=1.$$

(Note: In our examples condition (3.3) forces condition (3.2) to hold.)

Proof. We will show that

$$\xi(U_n)-\xi(n)>(1-\eta)(c_n\lg n)^{\frac{1}{2}}$$

infinitely often a.s. for any $\eta>0$. Let n_r be a subsequence of the integers increasing as fast as γ^r, $2\leqq\gamma$, so that

(3.4) $$\left(1-\frac{\varepsilon}{2}\right)c_{n_r}<U_{n_r}-n_r<\left(1+\frac{\varepsilon}{2}\right)c_{n_r}$$

occurs infinitely often a.s. for $\varepsilon>0$. Let

(3.5) $$F_{r,\delta}=\left\{\max_{1\leqq i\leqq\delta n_r}|U_{n_r+i}-U_{n_r}-i|<\frac{\varepsilon}{2}c_{n_r}\right\}$$

where $\delta<\gamma$. We will now show that there exists a $\delta>0$ such that $\overline{F_{r,\delta}}$, the complement of $F_{r,\delta}$, occur only finitely often a.s. as $r\to\infty$. By the assumption on the sequence $\{T_i\}$

(3.6) $$\left|\sum_{i=1}^{m}(T_i-1)\right|\leqq 2Kc_m$$

for all but a finite number of m a.s. Thus if (3.6) holds for two values $m=M$ and $m=k>M$, then

$$\left|\sum_{i=M+1}^{k}(T_i-1)\right| \leqq \left|\sum_{i=1}^{k}(T_i-1)\right| + \left|\sum_{i=1}^{M}(T_i-1)\right|$$
$$\leqq 2K\,c_k + 2K\,c_M.$$

In particular

(3.7) $$\max_{n_r<k\leqq(1+\delta)n_r}\left|\sum_{i=n_r}^{k}(T_i-1)\right| \leqq 4K\,c_{k_r}$$

for all but a finite number of n_r a.s. where $k_r=\sum_{i=1}^{r}\delta n_i$. But

(3.8) $$k_r=\sum_{i=1}^{r}\delta n_i \leqq C\,\delta(n_r)$$

where C, as usual, represents various positive constants. Since c_n is regularly varying with exponent $\frac{1}{2}$,

$$4K\,c_{k_r}=4K(k_r)^{\frac{1}{2}}L(k_r)$$

where L is a slowly varying function (see [5, pp. 268–269]). Thus using

(3.8) $$4K\,c_{k_r} \leqq 4K(C\,\delta\,n_r)^{\frac{1}{2}}L(C\,\delta\,n_r) \leqq C\,\delta\,c_{n_r} \leqq \frac{\varepsilon}{2}\,c_{n_r}$$

for $\delta=\varepsilon/2C$. Thus only finitely many $\overline{F_{r,\delta}}$ occur a.s. Now the event

(3.9) $$(1-\varepsilon)\,c_{n_r} < U_n - n < (1+\varepsilon)\,c_{n_r}$$

for all n such that $n_r \leqq n \leqq n_r+\delta n_r$ occurs for infinitely many r a.s.

Let $J_r=$ *integral part of* $(\delta n_r/4c_{n_r})$ and for $0\leqq i<J_r$ define

$$n'_{r,i}=n_r+\operatorname{int}(2i\,c_{n_r}), \qquad n''_{r,i}=n_r+\operatorname{int}\left((2i+1)\,c_{n_r}\right)$$
$$C'_{r,i}=\{\xi(n''_{r,i})-\xi(n'_{r,i})>(1-\eta)\,(c_{n_r}\lg n_r)^{\frac{1}{2}}\}$$
$$C''_{r,i}=\{\sup_{0\leqq x\leqq 3\varepsilon}|\xi(n''_{r,i}+x\,c_{n_r})-\xi(n''_{r,i})|<\eta\,(c_{n_r}\lg n_r)^{\frac{1}{2}}\}$$

$$Q'_r=\bigcup_{0\leqq i<J_r} C'_{r,i}, \qquad Q''_r=\bigcap_{0\leqq i<J_r} C''_{r,i}.$$

Suppose (3.9) holds for $n=n'_{r,i}$, $0\leqq i<J_r$, then

$$n''_{r,i}-\varepsilon\,c_{n_r} \leqq n'_{r,i}+(1-\varepsilon)\,c_{n_r}$$
$$\leqq U_{n'_{r,i}} \leqq n'_{r,i}+(1+\varepsilon)\,c_{n_r} \leqq n''_{r,i}+\varepsilon\,c_{n_r}.$$

This together with $C'_{r,i}\cap C''_{r,i}$ entails

$$\xi(U_{n'_{r,i}})-\xi(n'_{r,i})>(1-2\eta)\,(c_{n_r}\lg n_r)^{\frac{1}{2}}$$

which gives the desired result of the theorem. Thus it is sufficient to show that

$$P(\overline{Q''_r} \text{ occurs i.o.})=0$$

and

$$P(\overline{Q'_r} \text{ occurs i.o.})=0$$

which we do in that order.

$$\begin{aligned} P(\overline{C''_{r,i}}) &\leqq CP\{\xi(n''_{r,i}+3\varepsilon c_{n_r})-\xi(n''_{r,i})>\eta(c_{n_r}\lg n_r)^{\frac{1}{2}}\} \\ &\leqq CP\left\{\xi(1)>\frac{\eta(c_{n_r}\lg n_r)^{\frac{1}{2}}}{(3\varepsilon c_{n_r})^{\frac{1}{2}}}\right\} \\ &\leqq C e^{-\eta^2 \lg n_r/6\varepsilon} \leqq C\frac{1}{n_r^{\eta^2/6\varepsilon}}, \end{aligned}$$

$$P(\overline{Q''_r}) \leqq \sum_{i=0}^{J_r} P(\overline{C''_{r,i}}) \leqq C\frac{\delta n_r}{4c_{n_r} n_r^{\eta^2/6\varepsilon}}.$$

Therefore, $\sum_{r=1}^{\infty} P(\overline{Q''_r})<\infty$ for ε sufficiently small and thus by Borel-Cantelli $P(\overline{Q''_r}$ occurs i.o.$)=0$. Now,

$$\begin{aligned} P(C'_{r,i}) &= P(\xi(1)>(1-\eta)(\lg n_r)^{\frac{1}{2}}) \\ &= e^{-\frac{(1-\eta)^2}{2}(\lg n_r)(1+o(1))} = \left(\frac{1}{n_r}\right)^{\frac{(1-\eta)^2}{2}(1+o(1))}, \end{aligned}$$

$$\begin{aligned} \lg P(\overline{Q'_r}) &= \sum_{i=0}^{J_r} \lg P(\overline{C'_{r,i}}) \\ &\leqq \frac{\delta n_r}{4c_{n_r}} \lg\left[1-n_r^{-\frac{(1-\eta)^2}{2}(1+o(1))}\right] \\ &\leqq -C\frac{\delta n_r}{4c_{n_r}} n_r^{-\frac{(1-\eta)^2}{2}(1+o(1))} < -C\delta n_r^{\frac{1}{2}-\alpha} n_r^{-\frac{(1-\eta)^2}{2}(1+o(1))} \end{aligned}$$

since the regular variation with exponent $\frac{1}{2}$ of c_n implies $c_n<n^{\frac{1}{2}+\alpha}$ for $n \geqq N(\alpha)$. Thus $\sum_{r=1}^{\infty} P(\overline{Q'_r})<\infty$ since α is arbitrarily small and $P(\overline{Q'_r})$ occur i.o.$)=0$ by Borel-Cantelli. This completes the proof.

By a method more delicate than used in the proof of Theorem (2.1), we can find the exact upper class result. The method of proof follows that employed by Kiefer (but involves a change of his subsequence) and uses the following lemma which he proves.

(3.10) **Lemma.** *If ξ is standard Brownian motion and T, L, δ, c are positive values with $T<L$, then*

$$P\Big[\sup_{0\leqq t_1<t_2\leqq L;\, |t_1-t_2|\leqq T} |\xi(t_1)-\xi(t_2)| \geqq c\Big] \leqq \frac{8(L-T+\delta)(T+2\delta)^{\frac{1}{2}}}{\delta c(2\pi)^{\frac{1}{2}}} \exp[-c^2/2(T+2\delta)].$$

The upper class theorem is stated below.

(3.11) **Theorem.** *Suppose* $\overline{\lim}_{n\to\infty} \frac{\left|\sum_{i=1}^{n}(T_i-1)\right|}{c_n} \leqq 1$ *with probability one where* $c_n = n^{\frac{1}{2}} h(n)$ *and* $h(n)$ *is a non-decreasing slowly varying function, then*

$$P\left[\omega:\ \overline{\lim}_{n\to\infty} \frac{\left|\xi\left(\sum_{i=1}^{n} T_i\right) - \xi(n)\right|}{(c_n \lg n)^{\frac{1}{2}}} \leqq 1\right] = 1$$

where $\lg n = \log_e n$.

Proof. For $\varepsilon > 0$ let

$$q_n = (1+\varepsilon)\, c_n, \qquad d_n = (1+\varepsilon)(c_n \lg n)^{\frac{1}{2}}, \qquad n_r = \text{integral part of } r^2.$$

For real numbers t and integers n and r let

$$M_n = \{t:\ |t-n| < q_n\}$$

and

$$M_r^* = \{(t,n):\ |t-n| < q_{n_{r+1}};\ t, n \in [n_r - q_{n_{r+1}}, n_{r+1} + q_{n_{r+1}}]\}.$$

Define the events

$$A_n = \{\sup_{t\in M_n} |\xi(t) - \xi(n)| > d_n\}$$

and

$$A_r^* = \{\sup_{(t,n)\in M_r^*} |\xi(t) - \xi(n)| > d_{n_r}\}.$$

If A_n occurs for some n satisfying $n_r \leqq n \leqq n_{r+1}$ then A_r^* occurs. Thus if $\sum_{r=1}^{\infty} P(A_r^*) < \infty$, the Borel-Cantelli lemma will give the desired result that only finitely many A_n occur almost surely. Using the fact that $h(n)$ is a non-decreasing slowly varying function we have

$$c_{n_r} \sim r\, h(r^2) \sim c_{n_{r+1}}$$
$$q_{n_r} = (1+\varepsilon)\, c_{n_r} \sim (1+\varepsilon)\, r\, h(r^2) \sim q_{n_{r+1}}$$
$$d_{n_r} \sim (1+\varepsilon)(c_{n_r} 2\lg r)^{\frac{1}{2}} \sim (1+\varepsilon)\left(r\, h(r^2)\, 2\lg r\right)^{\frac{1}{2}}.$$

In the lemma above let

$$L = n_{r+1} - n_r + 2q_{n_{r+1}}, \qquad T = q_{n_{r+1}}, \qquad c = d_{n_r}, \qquad \delta = r.$$

This lemma yields the following where K represents various constants.

$$P(A_r^*) \leqq \frac{8(L-T+\delta)(T+2\delta)^{\frac{1}{2}}}{\delta c (2\pi)^{\frac{1}{2}}} \exp\left[-c^2/2(T+2\delta)\right]$$
$$\leqq K \frac{r\, h(r^2)\left(r\, h(r^2)\right)^{\frac{1}{2}}}{r\left(r\, h(r^2) \lg r\right)^{\frac{1}{2}}} \exp\left[\frac{-(1+\varepsilon)^2 (2c_{n_r} \lg r)}{2(1+\varepsilon)(c_{n_{r+1}} + 2r)}\right]$$
$$\leqq K \frac{h(r^2)}{(\lg r)^{\frac{1}{2}}} \exp\left[-(1+\varepsilon)(\lg r)\left(1+o(1)\right)\right]$$

since the slow variation of $h(n)$ implies $h(n)<n^\alpha$ for $n\geqq N(\alpha)$, $\alpha>0$. Thus $\sum_{r=1}^{\infty} P(A_r^*)<\infty$ and only finitely many A_r^* occur almost surely. This completes the proof.

(3.12) **Corollary.** *Under the assumptions of the above theorem and Theorem* (3.1)

$$P\left[\omega\colon \overline{\lim_{n\to\infty}} \frac{\left|\xi\left(\sum_{i=1}^{n} T_i\right)-\xi(n)\right|}{(c_n \lg n)^{\frac{1}{2}}}=1\right]=1.$$

Proof. This is a combination of the above theorem and Theorem (3.1).

In the following there are two examples of Corollary (3.12).

(3.13) *Example.* Assume $E(X_i)=0$, $E(X_i^2)=1$, and $E(X_i^4)<\infty$. Then $\xi\left(\sum_{i=1}^{n} T_i\right)$ has the same distribution as $S_n=X_1+\cdots+X_n$, $E(T_i)=1$, and $E(T_i-1)^2=\beta<\infty$. By the law of the iterated logarithm

$$\overline{\lim_{n\to\infty}} \frac{\sum_{i=1}^{n} T_i-n}{(2\beta n \lg\lg n)^{\frac{1}{2}}}=1 \quad \text{a.s.}$$

and

$$\overline{\lim_{n\to\infty}} \frac{\left|\sum_{i=1}^{n} T_i-n\right|}{(2\beta n \lg\lg n)^{\frac{1}{2}}}=1 \quad \text{a.s.}$$

By the Skorohod embedding (see (4) in the Appendix), $\left\{\xi\left(\sum_{i=1}^{n}\tau_i\right)\right\}_{n\geqq 1}$ has the same distribution as $\left\{\frac{\sum_{i=1}^{n} T_i-n}{\sqrt{\beta}}\right\}_{n\geqq 1}$ and

$$P\left[\omega\colon \overline{\lim_{n\to\infty}} \frac{\left|\xi\left(\sum_{i=1}^{n}\tau_i\right)-\xi(n)\right|}{\sqrt{n\lg\lg n}}=0\right]=1$$

(see [3, pp. 291-292]). It is also known (see [10, pp. 41–49]) that

$$\overline{\lim_{k\to\infty}} \frac{\xi(n_k)}{\sqrt{2n_k \lg\lg n_k}}\geqq\left(1-\frac{\varepsilon}{4}\right) \quad \text{a.s.}$$

where $n_k\sim\gamma^k$ and γ is large. Thus

$$\overline{\lim_{k\to\infty}} \frac{\sum_{i=1}^{n_k} T_i-n_k}{(2\beta n_k \lg\lg n_k)^{\frac{1}{2}}}\geqq\left(1-\frac{\varepsilon}{2}\right) \quad \text{a.s.}$$

Since $n_k \sim \gamma^k$ where γ is large, the second part of condition (3.2) is satisfied. Thus by Corollary (3.12)

$$P\left[\omega\colon \overline{\lim_{n\to\infty}} \frac{\xi\left(\sum_{i=1}^{n} T_i\right) - \xi(n)}{(\lg n(2\beta n \lg\lg n)^{\frac{1}{2}})^{\frac{1}{2}}} = 1\right] = 1.$$

This is Kiefer's result [9].

(3.14) *Example.* Assume the $\{T_i\}$ in the Skorohod embedding satisfy

$$\overline{\lim_{n\to\infty}} \frac{\sum_{i=1}^{n} T_i - n}{(2Kn(\lg\lg n)^2)^{\frac{1}{2}}} = 1 \quad \text{a.s.}$$

and

$$\overline{\lim_{n\to\infty}} \frac{\left|\sum_{i=1}^{n} T_i - n\right|}{(2Kn(\lg\lg n)^2)^{\frac{1}{2}}} = 1 \quad \text{a.s.}$$

and condition (3.2).

(Proposition (8) in the Appendix gives a proof of the existence of such non-negative i.i.d. random variables with infinite variance and their common distribution is explicitly shown.) The above sequence $\{T_i\}$ satisfies the conditions of Corollary (3.12) and thus

$$P\left[\omega\colon \overline{\lim_{n\to\infty}} \frac{\xi\left(\sum_{i=1}^{n} T_i\right) - \xi(n)}{(2Kn(\lg n \lg\lg n)^2)^{\frac{1}{4}}} = 1\right] = 1.$$

Appendix

(1) **Lemma.** *Let X be a random variable. Then $E(|X|)<\infty$ if and only if*

$$\sum_{n=1}^{\infty} P(|X|>n) < \infty. \tag{2}$$

Proof. Do an integration by parts on $E(|X|)$ and then approximate the integral with a series.

(3) **Lemma.** *If ξ is standard Brownian motion and T and b are positive values then*

$$P\left(\sup_{0\leq t\leq T} \xi(t) \geq b\right) = 2P(\xi(T)\geq b).$$

Proof. See [5, pp. 171–172].

(4) **Skorohod Embedding.** Let $\{X_n\}_{n\geq 1}$ be independent random variables with the same distribution; make the normalizations $E(X_n)=0$, $E(X_n^2)=1$; and let $S_n = X_1 + \cdots + X_n$; then the following theorem due to Skorohod holds (see [3, pp. 276–278] and [2]). There exists a probability space $(\Omega, \mathscr{B}, P)$ with a Brownian motion $\xi(t)$ (normalized so that $E[\xi(t)]=0$ and $E[\xi^2(t)]=t$) and a sequence of non-negative, independent, identically distributed random variables $\{T_i\}_{i\geq 1}$

defined on it such that the following conditions hold:

(5) i) $\left\{\xi\left(\sum_{i=1}^{n} T_i\right)\right\}_{n\geq 1}$ *has the same distribution as* $\{S_n\}_{n\geq 1}$.

(6) ii) $E(T_n)=E(X_n^2)=1$.

(7) iii) *if* $E(|X_n|^k)<\infty$ *then* $E(T_n^{k/2})<\infty$, $2\leq k$.

(8) **Proposition.** *Let* $\{T_n\}_{n\geq 1}$ *be independent random variables with the common (non-negative) distribution*

$$F(dx)=\frac{K\,dx}{x^3\lg x}, \qquad x\geq 2$$
$$=0, \qquad x<2$$

where $\lg x=\log_e x$. *Then*

$$\overline{\lim_{n\to\infty}}\frac{\sum_{i=1}^{n}(T_i-E(T_i))}{c_n}=1 \quad \text{a.s.}$$

and

$$\overline{\lim_{n\to\infty}}\frac{\left|\sum_{i=1}^{n}(T_i-E(T_i))\right|}{c_n}=1 \quad \text{a.s.}$$

where $c_n=[2Kn(\lg\lg n)^2]^{\frac{1}{2}}$.

Our proof makes use of the following result of Heyde [8]: Let $\{X_n\}_{n\geq 1}$ be a sequence of independent random variables, $\{a_n\}_{n\geq 1}$ a non-decreasing sequence of positive numbers, $a_n\to\infty$; let $V_n=X_n$ if $|X_n|<a_n$, while $V_n=0$ if $|X_n|\geq a_n$. If

$$\sum_{n=1}^{\infty}E\left(\frac{X_n^2}{X_n^2+a_n^2}\right)<\infty$$

then

$$\frac{1}{a_n}\sum_{k=1}^{n}[X_k-E(V_k)]\to 0$$

as $n\to\infty$ with probability one.

This allows us to reduce the problem to the law of the iterated logarithm for random variables which are bounded but whose distribution depends upon n. Then following Hartman and Wintner, we can use Kolmogorov's law of the iterated logarithm for bounded random variables to produce the required result.

Proof. Let $Y_n=T_n$ if $T_n\leq\varepsilon(n)\sqrt{n}$

$$=0 \quad \text{otherwise},$$

where $\lim_{n\to\infty}\varepsilon(n)=0$; and let $Z_n=T_n-Y_n$. We will exhibit a sequence $\{c_n\}$ such that

$$\overline{\lim_{n\to\infty}}\frac{Y_1+\cdots+Y_n-E(Y_1+\cdots+Y_n)}{c_n}=1 \quad \text{a.s.}$$

and

$$\overline{\lim_{n\to\infty}}\frac{|Y_1+\cdots+Y_n-E(Y_1+\cdots+Y_n)|}{c_n}=1 \quad \text{a.s.}$$

while

$$\lim_{n\to\infty} \frac{|Z_1+\cdots+Z_n-E(Z_1+\cdots+Z_n)|}{c_n}=0 \quad \text{a.s.}$$

Now,

$$E(Y_n^2)=\int_2^{\varepsilon(n)\sqrt{n}} \frac{K\,dx}{x\lg x}=K(\lg\lg x)|^{\varepsilon(n)\sqrt{n}}$$

$$\sim K[\lg(\lg\varepsilon(n)+\lg\sqrt{n})]\sim K\lg\lg n$$

for $\varepsilon(n)$ decreasing slowly enough.

$$\operatorname{var}(Y_n)=E(Y_n^2)-[E(Y_n)]^2\sim K\lg\lg n.$$

Define $B_n=\sum_{i=1}^n \operatorname{var}(Y_i)$, then $B_n\sim Kn\lg\lg n$ by the asymptotic properties of regularly varying functions (see [1, pp. 272–273]).

$$\left(\frac{B_n}{\lg\lg B_n}\right)^{\frac{1}{2}}=\left[\frac{Kn\lg\lg n}{\lg\lg(Kn\lg\lg n)}\right]^{\frac{1}{2}}\sim(Kn)^{\frac{1}{2}}.$$

Since $|Y_n|=o\left(\frac{B_n}{\lg\lg B_n}\right)^{\frac{1}{2}}$ as $n\to\infty$, Kolmogorov's law of the iterated logarithm (see [7, pp. 169–176]) gives

$$\overline{\lim_{n\to\infty}} \frac{Y_1+\cdots+Y_n-E(Y_1+\cdots+Y_n)}{(2B_n\lg\lg B_n)^{\frac{1}{2}}}=1 \quad \text{a.s.}$$

and

$$\overline{\lim_{n\to\infty}} \frac{|Y_1+\cdots+Y_n-E(Y_1+\cdots+Y_n)|}{(2B_n\lg\lg B_n)^{\frac{1}{2}}}=1 \quad \text{a.s.}$$

or

(9) $$\overline{\lim_{n\to\infty}} \frac{Y_1+\cdots+Y_n-E(Y_1+\cdots+Y_n)}{(2Kn(\lg\lg n)^2)^{\frac{1}{2}}}=1 \quad \text{a.s.}$$

and

(10) $$\overline{\lim_{n\to\infty}} \frac{|Y_1+\cdots+Y_n-E(Y_1+\cdots+Y_n)|}{(2Kn(\lg\lg n)^2)^{\frac{1}{2}}}=1 \quad \text{a.s.}$$

Now, define $V_n=Z_n<(2Kn(\lg\lg n)^2)^{\frac{1}{2}}$

$$=0 \quad \text{otherwise}$$

and let $W_n=Z_n-V_n$. Now if

$$\sum_{n=1}^{\infty}\int \frac{x^2}{x^2+c_n^2}\,G_n(dx)<\infty$$

where $c_n^2=2Kn(\lg\lg n)^2$ and G_n is the distribution function of Z_n then by Heyde (see [3, pp. 353–358])

(11) $$|Z_1+\cdots+Z_n-E(V_1+\cdots+V_n)|=o(c_n) \quad \text{a.s.}$$

Thus to prove (11) it is sufficient to show that

$$\sum_{n=1}^{\infty} \int_{x \geqq \varepsilon(n)\sqrt{n}} \left(\frac{x^2}{x^2+n(\lg\lg n)^2}\right) \frac{1}{x^3 \lg x}\, dx < \infty$$

which we proceed to do. In the following, C stands for various constants.

$$\begin{aligned}
&\sum_{n=1}^{\infty} \int_{x \geqq \varepsilon(n)\sqrt{n}} \left(\frac{x^2}{x^2+n(\lg\lg n)^2}\right) \frac{1}{x^3 \lg x}\, dx \\
&\quad = \sum_{n=1}^{\infty} \int_{x=\varepsilon(n)\sqrt{n}}^{\sqrt{n}\lg\lg n} \frac{1}{x\left(x^2+n(\lg\lg n)^2\right)\lg x}\, dx \\
&\quad\quad + \sum_{n=1}^{\infty} \int_{x=\sqrt{n}\lg\lg n}^{\infty} \frac{1}{x\left(x^2+n(\lg\lg n)^2\right)\lg x}\, dx \\
&\quad \leqq \sum_{n=1}^{\infty} \int_{x=\varepsilon(n)\sqrt{n}}^{\sqrt{n}\lg\lg n} \frac{1}{x\, n(\lg\lg n)^2 \lg x}\, dx \\
&\quad\quad + \sum_{n=1}^{\infty} \int_{x=\sqrt{n}\lg\lg n}^{\infty} \frac{1}{x^3 \lg x}\, dx \\
&\quad \leqq \sum_{n=1}^{\infty} \frac{1}{n(\lg\lg n)^2} [\lg\lg(\sqrt{n}\lg\lg n) - \lg\lg(\varepsilon(n)\sqrt{n})] \\
&\quad\quad + \sum_{n=1}^{\infty} \frac{C}{n(\lg\lg n)^2 \lg n} \\
&\quad \leqq C + \sum_{n=1}^{\infty} \frac{1}{n(\lg\lg n)^2} \left[\lg\left(\frac{\lg\sqrt{n}+\lg\lg\lg n}{\lg(\varepsilon(n)\sqrt{n})}\right)\right] \\
&\quad \leqq C + \sum_{n=1}^{\infty} \left[\frac{1}{n(\lg\lg n)^2}\right] \lg\left[1+\frac{\lg\sqrt{n}-\lg(\varepsilon(n)\sqrt{n})+\lg\lg\lg n}{\lg(\varepsilon(n)\sqrt{n})}\right] \\
&\quad \leqq C + C\sum_{n=1}^{\infty} \frac{1}{n(\lg\lg n)^2} \left[\frac{\lg\lg\lg n}{\lg n}\right] < \infty
\end{aligned}$$

if $\varepsilon(n)$ decrease slowly enough. For instance it suffices that $\varepsilon(n) \geqq 1/\lg\lg n$. Thus

(12) $$|Z_1+\cdots+Z_n - E(V_1+\cdots+V_n)| = o[n(\lg\lg n)^2]^{\frac{1}{2}}.$$

Now we will show that

$$E(W_1+\cdots+W_n) = o(c_n).$$

$$E(W_n) = \int_{x \geqq (2Kn(\lg\lg n)^2)^{\frac{1}{2}}} \frac{Kx}{x^3 \lg x}\, dx$$

$$\sim \frac{C}{n^{\frac{1}{2}} \lg\lg n \lg n} \quad \text{as } n \to \infty.$$

$$E(W_1+\cdots+W_n) \leqq C\sum_{i=1}^{n} \frac{1}{i^{\frac{1}{2}}} \leqq C n^{\frac{1}{2}}.$$

Thus

$$E(W_1+\cdots+W_n)=o\left(n(\lg\lg n)^2\right)^{\frac{1}{2}}. \tag{13}$$

Now,

$$\sum_{i=1}^{n}(T_i-E(T_i))=\sum_{i=1}^{n}(Y_i-E(Y_i))+\sum_{i=1}^{n}(Z_i-E(V_i))+\sum_{i=1}^{n}E(W_i). \tag{14}$$

Combining (9), (12) and (13) in Eq. (14) gives the desired result that

$$\overline{\lim_{n\to\infty}}\frac{\sum_{i=1}^{n}(T_i-E(T_i))}{\left(2Kn(\lg\lg n)^2\right)^{\frac{1}{2}}}=1 \quad \text{a.s.}$$

Combining (10), (12) and (13) in Eq. (14) gives the desired result that

$$\overline{\lim_{n\to\infty}}\frac{\left|\sum_{i=1}^{n}(T_i-E(T_i))\right|}{\left(2Kn(\lg\lg n)^2\right)^{\frac{1}{2}}}=1 \quad \text{a.s.}$$

Now, by Kolmogorov's lower class proof of the law of the iterated logarithm (see [15, pp. 260-263]), if $B_{n_k}^{\frac{1}{2}}\sim\gamma^k$ where γ is large then

$$\overline{\lim_{k\to\infty}}\frac{Y_1+\cdots+Y_{n_k}-E(Y_1+\cdots+Y_{n_k})}{c_{n_k}}\geqq\left(1-\frac{\varepsilon}{4}\right) \quad \text{a.s.}$$

For our case n_k is chosen so that $n_k\sim\gamma^{2k}/K\lg k$. Then the second part of condition (3.2) is satisfied and

$$\overline{\lim_{k\to\infty}}\frac{\sum_{i=1}^{n_k}(T_i-E(T_i))}{c_{n_k}}\geqq\left(1-\frac{\varepsilon}{2}\right) \quad \text{a.s.}$$

This completes the proof.

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