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Markov Additive Processes. I*

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1. Introduction

Our notation and terminology follows Blumenthal and Getoor [1]. When referring to it we write, for example, BG II.7.4 to mean expression (or theorem or statement) (7.4) in chapter II of [1]. We recall some of the basic particulars as follows. If $(F, \mathscr{F})$ and $(G, \mathscr{G})$ are measurable spaces and a function $f: F \to G$ is measurable relative to $\mathscr{F}$ and $\mathscr{G}$ then we write $f \in \mathscr{F}/\mathscr{G}$; if $G = \overline{\mathbb{R}}$, $\mathscr{G} = \overline{\mathscr{R}}$ then we abbreviate this and write $f \in \mathscr{F}$; if $f \in \mathscr{F}$ is also bounded then we write $f \in b\mathscr{F}$. By $\sigma(\cdot)$ is meant the σ-algebra generated by $(\cdot)$. Let $(F, \mathscr{F})$ and $(G, \mathscr{G})$ be two measurable spaces. A mapping $N: F \times \mathscr{G} \to [0, 1]$ is called a transition probability from $(F, \mathscr{F})$ into $(G, \mathscr{G})$ if a) $A \to N(x, A)$ is a measure on $\mathscr{G}$ for fixed $x \in F$, and b) $x \to N(x, A)$ is in $b\mathscr{F}$ for any fixed $A \in \mathscr{G}$. If N is a transition probability from $(F, \mathscr{F})$ into itself and $f \in b\mathscr{F}$, then we write

$$(1.1) \qquad Nf(x) = N(x, f) = \int N(x, dy) f(y), \qquad x \in F.$$

Then, $Nf \in b\mathscr{F}$ and $f \to Nf$ is a positive linear contraction on the Banach space $(F, \mathscr{F})$ with the supremum norm.

Let $(E, \mathscr{E})$ be a measurable space, T a subset of $[0, +\infty]$ containing the origin. Let Ω be an arbitrary set, $\mathscr{M}$ a σ-algebra of subsets of Ω, $\{\mathscr{M}_t; t \in T\}$ an increasing family of sub-σ-algebras of $\mathscr{M}$, and P a probability measure on $\mathscr{M}$. Let $\{X_t; t \in T\}$ be a stochastic process over $(\Omega, \mathscr{M}, P)$ with values in $(E, \mathscr{E})$ and let $\{Y_t; t \in T\}$ be a stochastic process over $(\Omega, \mathscr{M}, P)$ with values in $(F, \mathscr{F}) = (\mathbb{R}^m, \mathscr{R}^m)$ for some $m \geqq 1$. We define $\mathscr{K}_{s,t} = \sigma(X_u; u \in T, s \leqq u \leqq t)$ for $s, t \in T$; $\mathscr{K}_t = \mathscr{K}_{0,t}$ for $t \in T$; $\mathscr{K} = \sigma(X_u; u \in T)$; and similarly, $\mathscr{L}_t = \sigma(X_u, Y_u; 0 \leqq u \leqq t, u \in T)$ for $t \in T$, and put $\mathscr{L} = \sigma(X_u, Y_u; u \in T)$.

(1.2) *Definition.* A family $\{Q_{s,t}; s < t, s, t \in T\}$ of transition probabilities from $(E, \mathscr{E})$ into $(E \times F, \mathscr{E} \times \mathscr{F})$ is called a semi-Markov transition function on $(E, \mathscr{E}, \mathscr{F})$ provided that

$$Q_{s,t}(x, A \times B) = \int_{E \times F} Q_{s,u}(x, dy \times dz)\, Q_{u,t}(y, A \times (B - z))$$

for any $s < u < t$, $s, u, t \in T$, $x \in E$, $A \in \mathscr{E}$, $B \in \mathscr{F}$ where $B + a = \{b + a: b \in B\}$ for any $a \in F$.

Given a semi-Markov transition function $Q_{s,t}$ on $(E, \mathscr{E}, \mathscr{F})$ the formula

$$(1.3) \qquad P_{s,t}(x, y; A \times B) = Q_{s,t}(x, A \times (B - y))$$

defines a Markov transition function $P_{s,t}$ on $(E \times F, \mathscr{E} \times \mathscr{F})$, which is translation invariant in the second variable and vice versa.

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(1.4) *Definition.* $(X, Y)=\{X_t, Y_t; t\in T\}$ is a Markov additive process with respect to $\{\mathcal{M}_t; t\in T\}$ and with semi-Markov transition function $Q_{s,t}$ provided that (X, Y) be a Markov process with respect to $\{\mathcal{M}_t; t\in T\}$ (in the sense of BG I.1.1) whose transition function $P_{s,t}$ satisfies (1.3).

Let (X, Y) be a Markov additive process with respect to $\{\mathcal{M}_t; t\in T\}$; then (X, Y) is a Markov additive process with respect to $\{\mathcal{L}_t; t\in T\}$ also. Following are some of the immediate consequences of the Definition (1.4). We omit the proofs (most of which may be found in [5]). Throughout we take $T=[0, +\infty]$; the results below need only trivial alterations for general T, and if $T=\{0, 1, \ldots\}$ most of these results are well known.

(1.5) The process $X=\{X_t; t\in T\}$ is a Markov process with respect to $\{\mathcal{M}_t\}$ with state space $(E, \mathcal{E})$ and transition function

$$K_{s,t}(x, A)=Q_{s,t}(x, A\times F);$$

$s<t$, $x\in E$, $A\in\mathcal{E}$.

(1.6) For any $t\geqq 0$, if $W\in b\,\sigma(X_u, Y_u-Y_t; u\geqq t)$, then

$$E[W|\mathcal{M}_t]=E[W|X_t].$$

(1.7) For any $s<t$ and $f\in b\mathcal{F}$ we have

$$E[f(Y_t-Y_s)|\mathcal{K}]=E[f(Y_t-Y_s)|\mathcal{K}_{s,t}].$$

(1.8) For any integer $n\geqq 1$, $0\leqq t_0<t_1<\cdots<t_n$, and $h_1, \ldots, h_n\in b\mathcal{F}$ we have

$$E\left[\prod_{i=1}^n h_i(Y_{t_i}-Y_{t_{i-1}})|\mathcal{K}\right]=\prod_{i=1}^n E[h_i(Y_{t_i}-Y_{t_{i-1}})|\mathcal{K}],$$

that is, given the process X, Y is a process with independent increments.

(1.9) Let $A_t=E[Y_t-Y_0|\mathcal{K}]$. Then $A=\{A_t; t\geqq 0\}$ is an additive functional of X.

(1.10) Let $M_t^y=E[\exp[i(Y_t-Y_0, y)]|\mathcal{K}]$ where (y', y) is the usual inner product in F. Then, for any fixed $y\in F$, $\{M_t^y; t\geqq 0\}$ is a multiplicative functional of X.

The name Markov additive process is supposed to suggest its two most important properties: (1.5) and (1.8). Ezhov and Skorohod [5] called it a Markov process with homogeneous second component and characterized the Y process by using characteristic functions under the assumption of continuity for $(y, t)\to M_t^y$ (cf. [5] Theorem 1). They also give a complete characterization of M_t^y, and therefore of $\{Y_t; t\geqq 0\}$, in the case when X is a regular step process (with all states holding).

A more detailed account is given in [3] in a more modern setting; [3] can be read independent of this one. If the state space E is finite, what we have becomes what Neveu [12] called an F-process. In the discrete parameter case certain specific problems (such as central limit theorems, hitting times) associated with Markov additive processes were discussed by Volkov [18], Miller [10, 11], Keilson and Wishart [7, 8], Pyke [14], Pyke and Schaufele [15, 16], and Çinlar [2]. In the case of continuous time parameter, Fukushima and Hitsuda [6], Keilson and Wishart [7], and Pinsky [13] have given central limit theorems for the second component in the case where E is finite. In a different direction, where $F=\mathbb{R}$ and Y non-decreasing, a random time change (with Y as time) gives us semi-Markov processes. These were introduced by Lévy [9], Smith [17].

In the next section we show that, given a semi-Markov transition function $Q_{s,t}$, a Markov additive process (X, Y) exists if and only if the Markov process X exists. In Section 3 we show the existence of a regular version of the conditional probability $P\{\cdot|\mathscr{X}\}$; and using it the process Y is characterized completely. In Section 4 we consider the construction of the conditional distribution of $Y_t - Y_s$ given $\mathscr{X}$ directly from the transition function $Q_{s,t}$. When this construction is possible we have a nice canonical construction for the (X, Y) process.

2. Existence of Markov Additive Processes

Let $\{Q_{s,t}; s, t\in T\}$ be a semi-Markov transition function on $(E, \mathscr{E}, \mathscr{F})$ and define $P_{s,t}$ and $K_{s,t}$ as in (1.3) and (1.5) respectively.

We let $\mathscr{T}$ be the class of all finite subsets of the parameter set T; and define φ_J: $E^T\to E^J$, ψ_J: $F^T\to F^J$, π_J: $E^T\times F^T\to E^J\times F^J$ to be the natural projections defined for $J\in\mathscr{T}$.

(2.1) **Theorem.** *Let μ be a probability measure on $\mathscr{E}\times\mathscr{F}$ and put $v(A)=\mu(A\times F)$, $A\in\mathscr{E}$. Then, there exists a Markov additive process with the initial distribution μ and semi-Markov transition function $Q_{s,t}$ if and only if there exists a Markov process with the initial distribution v and transition function $K_{s,t}$.*

Proof. Necessity follows from the statement (1.5). To prove the sufficiency, suppose there exists a Markov process with the initial distribution v and transition function $K_{s,t}$. Then there is a Markov process of the function space type equivalent to it; that is, a Markov process $\hat{X}=\{\hat{X}_t; t\in T\}$ over the probability space $(E^T, \mathscr{E}^T, \hat{P})$ so that $\hat{X}_t=\varphi_{\{t\}}$ and $\hat{P}$ is the limit of the projective system of probability measures $\{\hat{P}_J; J\in\mathscr{T}\}$ where

$$(2.2)\qquad \hat{P}_J(A)=\int v(dx_0)\int K_{0,t_1}(x_0, dx_1)\int\cdots\int K_{t_{n-1},t_n}(x_{n-1}, dx_n)\, I_A(x_1, \ldots, x_n)$$

for any $A\in\mathscr{E}^J$ if $J=\{t_1, \ldots, t_n\}\in\mathscr{T}$.

Let us define, for $A\in\mathscr{E}^J\times\mathscr{F}^J$, $J=\{t_1, \ldots, t_n\}\in\mathscr{T}$,

$$(2.3)\qquad P_J(A)=\int \mu(dx_0)\int P_{0,t_1}(x_0, dx_1)\int\cdots\int P_{t_{n-1},t_n}(x_{n-1}, dx_n)\, I_A(x_1, \ldots, x_n).$$

This defines a projective system of probability measures $\{P_J; J\in\mathscr{T}\}$. Let P be the finitely additive set function on the (finite dimensional) cylinder sets C of $\mathscr{M}=\mathscr{E}^T\times\mathscr{F}^T$ defined by putting

$$(2.4)\qquad P(C)=P_J(C_0)\quad \text{if } C=\pi_J^{-1}(C_0),\quad C_0\in\mathscr{E}^J\times\mathscr{F}^J.$$

For any fixed cylinder set B of $\mathscr{F}$, the mapping $A\to P(A\times B)$ is finitely additive on the cylinder sets of $\mathscr{E}^T$. Further, since $K_{s,t}(x, A)=P_{s,t}(x, y; A\times F)=Q_{s,t}(x, A\times F)$ for all $s, t\in T$, $x\in E$, $A\in\mathscr{E}$, we have

$$P(A\times B)\leqq P(A\times F^T)=P(A)$$

for any cylinder A of $\mathscr{E}^T$ by (2.2) and (2.3). Hence, if the cylinders $A_n \downarrow \emptyset$ then $P(A_n \times B) \downarrow 0$ since $\hat{P}(A_n) \downarrow 0$. Thus, $A \to P(A \times B)$ is countably additive on the algebra of finite dimensional cylinders of $\mathscr{E}^T$, and therefore there is a unique measure $A \to Q(A, B)$ on $\mathscr{E}^T$ which coincides with $A \to P(A \times B)$ on the cylinders of $\mathscr{E}^T$.

Now $B \to Q(A, B)$ is finitely additive for any cylinder A in $\mathscr{E}^T$. This implies, through the monotone class theorem, that $B \to Q(A, B)$ is finitely additive for any A in $\mathscr{E}^T$. By the special nature of $(\mathbb{R}^m, \mathscr{R}^m)$, Kolmogorov extension theorem applies to show the existence of a measure $B \to P(A, B)$ on $\mathscr{F}^T$ wich is the unique extension of $B \to Q(A, B)$ for each fixed $A \in \mathscr{E}^T$. Another application of the monotone class theorem shows that $A \to P(A, B)$ is a measure on $\mathscr{E}^T$ for any $B \in \mathscr{F}^T$. Thus, by standard theorems, there is a unique measure $\bar{P}$ on $\mathscr{M} = \mathscr{E}^T \times \mathscr{F}^T$ so that $\bar{P}(A \times B) = P(A, B)$ for all $A \in \mathscr{E}^T$, $B \in \mathscr{F}^T$. Obviously $\bar{P}(C) = P(C)$ for any cylinder $C \in \mathscr{M}$. Therefore, $\bar{P}$ is the unique limit of the projective system $\{P_J; J \in \mathscr{T}\}$ defined by (2.3).

Finally, let $\Omega = E^T \times F^T$ and for each $\omega = (\omega_1, \omega_2) \in \Omega$ define

$$X_t(\omega) = \varphi_{\{t\}}(\omega_1), \qquad Y_t(\omega) = \psi_{\{t\}}(\omega_2).$$

Then, $(X, Y) = \{X_t, Y_t; t \in T\}$ is a Markov process over $(\Omega, \mathscr{M}, \bar{P})$ with state space $(E \times F, \mathscr{E} \times \mathscr{F})$ and transition function $P_{s,t}$. That (X, Y) is a Markov process over $(\Omega, \mathscr{M}, \bar{P})$ adapted to $\{\sigma(X_u, Y_u; u \leq t); t \in T\}$ with semi-Markov transition function $Q_{s,t}$ follows from Definition (1.4). This completes the proof.

A second canonical construction will be given in Section 4; it will be a more intuitive one but is possible only under certain, however slight, restrictions.

3. Characterization of the Additive Part Y

Let (X, Y) be a Markov additive process over $(\Omega, \mathscr{M}, P)$ with X taking values in $(E, \mathscr{E})$ and Y in $(F, \mathscr{F})$, with parameter set $T = [0, +\infty]$. We assume that, almost surely, $t \to Y_t$ is right continuous. We will first show that there exists a regular version Q^w of $P\{\cdot | \mathscr{K}\}(w)$ on $\mathscr{L}$. Then, by (1.8), Y is a process with independent increments on the probability space $(\Omega, \mathscr{L}, Q^w)$ for any fixed $w \in \Omega$. This enables us to use the well-known results about such processes to obtain a complete characterization of Y, thus generalizing Theorem 1 of [5]. Of course, the existence of a regular version Q is also of independent interest.

Let $\hat{\Omega} = E^T \times F^T$ and define a mapping $\pi: \Omega \to \hat{\Omega}$ by putting

(3.1) $$\pi\,\omega(t) = (X_t(\omega), Y_t(\omega)), \qquad t \in T, \ \omega \in \Omega.$$

For $\hat{\omega} = (w, z) \in \hat{\Omega}$ define $\hat{X}_t(\hat{\omega}) = w(t)$, $\hat{Y}_t(\hat{\omega}) = z(t)$ for all $t \in T$. Define $\hat{\mathscr{L}} = \sigma(\hat{X}_t, \hat{Y}_t; t \in T)$, $\hat{\mathscr{K}} = \sigma(\hat{X}_t; t \in T)$, and $\hat{P} = P\pi^{-1}$.

(3.2) **Lemma.** *There exists a regular version $\hat{Q}$ of $\hat{P}\{\cdot | \hat{\mathscr{K}}\}$ on $\hat{\mathscr{L}}$.*

Proof. For $s < t$, $B \in \mathscr{F}$, $w \in \hat{\Omega}$ let

(3.3) $$\mu(w, B) = (\hat{P}\{Y_0 \in B | \hat{\mathscr{K}}\})(w),$$

(3.4) $$G_{s,t}(w, B) = (\hat{P}\{\hat{Y}_t - \hat{Y}_s \in B \ \hat{\mathscr{K}}\})(w)$$

be selected so that $B \to \mu(w, B)$ and $B \to G_{s,t}(w, B)$ be both probability measures for any fixed w; (this is possible by the special nature of $(\mathbb{R}^m, \mathscr{R}^m)$). Let

$$M_{s,t}^y = \hat{E}\{\exp[i(Y_t - Y_s, y)] | \mathscr{K}\}$$

for $0 \leqq s \leqq t$, $y \in F$. Then, for any $y \in F$, $s < t < u$

$$M_{s,t}^y(w) M_{t,u}^y(w) = M_{s,u}^y(w) \tag{3.5}$$

for all w except in some null set $N(s, t, u, y)$. By the right continuity of Y, $t \to M_{s,t}^y$ is right continuous. Then, the results of Walsh [19] apply (though our $M_{s,t}^y$ are complex valued instead of his real-valued case, all the proofs go through with obvious modifications) to show that the exceptional set can be taken to be independent of s, t, u. So, there is a null set N_y such that (3.5) holds for all s, t, u whenever $w \notin N_y$. Repeating this for each y in a suitably chosen countable set which is dense in F, and using the continuity properties of characteristic functions, we conclude that (3.5) holds for all s, t, u, y provided that w is not in a certain null set N.

We have thus shown that there is a null set N such that, for $w \notin N$,

$$\int G_{s,t}(w, dy)\, G_{t,u}(w, B - y) = G_{s,u}(w, B)$$

for all s, t, u, B. We now re-define $G_{s,t}(w, B)$ for $w \in N$ so that $G_{s,t}(w, \{0\}) = 1$ for all s, t.

Then, if for fixed $w \in \hat{\Omega}$ we define

$$Q_J(w, A) = \int_\mu (w, dx_0) \int G_{0,t_1}(w, dx_1 - x_0) \int \cdots \int G_{t_{n-1},t_n}(w, dx_n - x_{n-1})\, I_A(x_1, \ldots, x_n)$$

for each finite subset $J = \{t_1, \ldots, t_n\}$ of $T = [0, +\infty]$ and set $A \in \mathscr{F}^J$, it follows that $\{Q_J(w, \cdot);\ J$ is finite subset of $T\}$ is a projective system of measures over $(F, \mathscr{F})$, and by the Kolmogorov extension theorem, has a projective limit $\bar{Q}(w, \cdot)$ on $\mathscr{F}^T$. For $A \in \mathscr{E}^T$ and $B \in \mathscr{F}^T$ and $\hat{\omega} = (w, z) \in \hat{\Omega}$ we put

$$\tilde{Q}(\hat{\omega}, A \times B) = \varepsilon_w(A)\, \bar{Q}(\hat{\omega}, B) \tag{3.6}$$

where ε_w is the Dirac measure concentrated at w. Finally, let $\hat{Q}(\hat{\omega}, \cdot)$ be the restriction to $\hat{\mathscr{L}}$ of the unique extension of $\tilde{Q}(\hat{\omega}, \cdot)$ onto $\mathscr{E}^T \times \mathscr{F}^T$. There remains only to show that $\hat{Q}$ is the desired version.

By their definitions (3.3), (3.4) the functions $\mu(\cdot, B)$ and $G_{s,t}(\cdot, B)$ are both in $\hat{\mathscr{K}}$ for any $B \in \mathscr{F}$. Hence, being essentially a product of measurable functions, $Q_J(\cdot, A)$ is in $\hat{\mathscr{K}}$ for any $A \in \mathscr{F}^J$, J finite. By the monotone class theorem, since cylinders with base in $\mathscr{F}^J$ generate $\mathscr{F}^T$, the function $\bar{Q}(\cdot, A)$ is in $\hat{\mathscr{K}}$ for all $A \in \mathscr{F}^T$. Then, (3.6) and a second application of the monotone class theorem give the $\hat{\mathscr{K}}$-measurability of $\hat{Q}(\cdot, A)$ for any $A \in \hat{\mathscr{L}}$.

Applying (1.8) to the Markov additive process $(\hat{X}, \hat{Y})$ over $(\hat{\Omega}, \hat{\mathscr{L}}, \hat{P})$ we have, in view of (3.3), (3.4),

$$\bar{Q}(\cdot, B) = \hat{P}\{(\hat{Y}_s, \ldots, \hat{Y}_t) \in B_0 | \hat{\mathscr{K}}\}$$

for any finite $J = \{s, \ldots, t\}$ and cylinder set $B \in \mathscr{F}^T$ with base $B_0 \in \mathscr{F}^J$. Applying the monotone class theorem twice and noting (3.6) we see that

$$\hat{Q}(\cdot, A) = \hat{P}\{A | \hat{\mathscr{K}}\}, \quad A \in \hat{\mathscr{L}}.$$

(3.7) **Theorem.** *There exists a regular version Q of $P\{\cdot|\mathcal{K}\}$ on $\mathcal{L}$.*

Proof. It is easy to show that the mapping π defined by (3.1) is in both $\mathcal{K}/\hat{\mathcal{K}}$ and $\mathcal{L}/\hat{\mathcal{L}}$; and further that, for any $f\in\mathcal{K}$ (respectively $f\in\mathcal{L}$), there is $g\in\hat{\mathcal{K}}$ (respectively $g\in\hat{\mathcal{L}}$) such that $f=g\circ\pi$.

For $f\in b\mathcal{L}$, $g\in b\hat{\mathcal{L}}$, $f=g\circ\pi$, define

$$Q(\omega,f)=\hat{Q}(\pi\,\omega,g) \tag{3.8}$$

where $\hat{Q}$ is the regular version of $\hat{P}\{\cdot|\hat{\mathcal{K}}\}$ on $\hat{\mathcal{L}}$ constructed in Lemma (3.2) above. Noting that $\omega\to Q(\omega,f)$ is the composition of $\hat{\omega}\to\hat{Q}(\hat{\omega},g)$ and $\omega\to\hat{\omega}=\pi\omega$ which are in $\hat{\mathcal{K}}$ and $\mathcal{K}/\hat{\mathcal{K}}$ respectively, we see that $Q(\cdot,f)\in\mathcal{K}$. For fixed $\omega\in\Omega$, $Q(\omega,\cdot)$ is non-negative, and finitely additive. Further, if $\{f_n\}$ is a non-decreasing sequence of functions in $b\mathcal{L}$ and if $f_n=g_n\circ\pi$, $f=\lim_n f_n\in b\mathcal{L}$, $f=g\circ\pi$, then

$$\lim g_n=g \quad \text{on } \pi\Omega, \quad \hat{P}(\pi\Omega)=1;$$

and

$$\lim Q(\omega,f_n)=\lim\hat{Q}(\pi\,\omega,g_n)=\hat{Q}(\pi\,\omega,g)=Q(\omega,f).$$

Therefore, $Q(\omega,\cdot)$ is a measure on $\mathcal{L}$.

To complete the proof we need to show that, for any $f\in b\mathcal{K}$ and $h\in b\mathcal{L}$,

$$\int f(\omega)\,Q(\omega,h)\,P(d\omega)=\int f(\omega)\,h(\omega)\,P(d\omega). \tag{3.9}$$

Let $f=g\circ\pi$, $h=k\circ\pi$ for $g\in b\hat{\mathcal{K}}$, $k\in b\hat{\mathcal{L}}$. Then

$$\begin{aligned}\int f\,Q h\,dP&=\int(g\circ\pi)(\hat{Q}k\circ\pi)\,dP\\&=\int g\,\hat{Q}k\,d(P\pi^{-1})=\int g\,\hat{Q}k\,d\hat{P}\end{aligned} \tag{3.10}$$

since $Qh(\omega)=Q(\omega,h)=\hat{Q}(\pi\,\omega,k)=\hat{Q}k\circ\pi(\omega)$ and $\hat{P}=P\pi^{-1}$. Since $g\in\hat{\mathcal{K}}$,

$$\int g\,\hat{Q}k\,d\hat{P}=\int g\,k\,d\hat{P}=\int(g\circ\pi)(k\circ\pi)\,dP=\int f h\,dP. \tag{3.11}$$

Now (3.10) and (3.11) give (3.9).

For fixed $w\in\Omega$, $Y=\{Y_t;\,t\in T\}$ is a process with independent increments on the probability space $(\Omega,\mathcal{L},Q(w,\cdot))$. From well-known results we get the following characterization. We omit the proof (cf. Doob [4], Chapter 8 and [5], Theorem 1).

(3.12) **Theorem.** *Suppose $Y=\{Y_t;\,t\geqq 0\}$ is right continuous. Then,*

$$Y_t-Y_0=A_t+Y_t^f+Y_t^c+Y_t^d, \qquad t\geqq 0$$

where $\sigma(A_t;\,t\geqq 0)$, $\sigma(Y_t^f;\,t\geqq 0)$, $\sigma(Y_t^c;\,t\geqq 0)$, $\sigma(Y_t^d;\,t\geqq 0)$ are conditionally independent given $\mathcal{K}=\sigma(X_t;\,t\geqq 0)$ and the following hold:

a) $A=\{A_t;\,t\geqq 0\}$ is an additive functional of the Markov process X.

b) $Y^f=\{Y_t^f;\,t\geqq 0\}$ is a purely discontinuous process whose jump times are fixed given $\mathcal{K}$, or more precisely,

$$Y_t^f=\sum_i V_i\,1_{[0,t]}(T_i)$$

where $V_1,V_2,V_3,\ldots$, are conditionally independent given $\mathcal{K}$, and for each i, T_i is either a constant or a stopping time for X.

c) $Y^c = \{Y_t^c; t \geqq 0\}$ *is a Gaussian process on* $(\mathbb{R}^m, \mathscr{R}^m)$ *over* $(\Omega, \mathscr{L}, Q(\omega, \cdot))$ *for any* $\omega \in \Omega$.

d) $Y^d = \{Y_t^d; t \geqq 0\}$ *is a stochastically continuous Lévy process with independent increments without a Gaussian component over* $(\Omega, \mathscr{L}, Q(\omega, \cdot))$.

(3.13) **Corollary.** *Let* M_t^y *be defined as in* (1.10).

a) *The "fixed" discontinuities* $\{T_i\}$ *described in* (3.12b) *are the points of discontinuity of the non-decreasing function*

$$t \to \int_{\|y\| \in [0,1]} |M_t^y| \, dy.$$

b) *If* $N_t^y = E[\exp[i(y, A_t + Y_t^c + Y_t^d)] | \mathscr{K}]$, $y \in F$, *then*

$$N_t^y(\omega) = \exp\left[i(A_t(\omega), y) - \tfrac{1}{2}(C_t(\omega)\, y, y) + \int \left(e^{i(x,y)} - 1 - \frac{i(x,y)}{1+\|x\|^2}\right) B_t(\omega, dx)\right]$$

where $C_t(\omega)$ *is a non-negative definite symmetric operator on* F *and* $B_t(\omega, \cdot)$ *is a measure on* $(F, \mathscr{F})$ *satisfying*

$$\int \frac{\|x\|^2}{1+\|x\|^2} B_t(\omega, dx) < \infty.$$

Furthermore, $\{C_t; t > 0\}$ *is a continuous additive functional of* X *for which* $C_t - C_u$ *is non-negative for* $t \geqq u$; *and* $\{B_t(A); t > 0\}$ *is a non-decreasing continuous additive functional of* X *for each* $A \in \mathscr{F}$.

We shall give a finer analysis of the structure of Y later after we introduce the proper machinery.

4. A Canonical Construction

Suppose we are given an integer m, a measurable space $(E, \mathscr{E})$, and a semi-Markov transition function $\{Q_{s,t}; 0 \leqq s < t \leqq +\infty\}$ on $(E, \mathscr{E}, \mathscr{F})$. Our object is to construct a Markov additive process (X, Y) with these elements so as to render the structure presented in Section 3 clear. This we will be able to do under certain conditions.

Define

(4.1) $$K_{s,t}(x, A) = Q_{s,t}(x, A \times F),$$

(4.2) $$L_{s,t}^y(x, A) = \int e^{i(y', y)} Q_{s,t}(x, A \times dy'), \qquad y \in F.$$

We assume the following hold:

(4.3) a) There exists a probability space $(\Omega^0, \mathscr{K}^0, P^0)$ and functions $X_t \in \mathscr{K}^0/\mathscr{E}$ such that $X = \{X_t; t \geqq 0\}$ is a Markov process over $(\Omega^0, \mathscr{K}^0, P^0)$ with transition function $K_{s,t}$;

b) For each $x \in E$, $\{x\} \in \mathscr{E}$; and there is a countable family $\mathscr{B} \subset \mathscr{E}$ such that $\sigma(\mathscr{B}) = \mathscr{E}$;

c) For any countable set T which is dense in $[s, t]$ we have $\sigma(X_u; s \leqq u \leqq t) = \sigma(X_u; u \in T)$.

For any fixed $y \in F$, $\{L^y_{s,t}; 0 \leqq s < t \leqq +\infty\}$ is a semi-group of transition operators which is subordinate to $\{K_{s,t}; 0 \leqq s < t \leqq +\infty\}$. By (4.3a) we have a Markov process X, and conditions (4.3b) and (4.3c) insure the existence of a multiplicative functional $\{M^y_t; t \geqq 0\}$ of X which generates $\{Q^y_{s,t}\}$ (cf. BG III.2.3 for the detailed description whose adaptation to the present case we omit).

For fixed $\omega^0 \in \Omega^0$, construct a probability measure $Q(\omega^0, \cdot)$ on $\Omega^1 = F^{[0,+\infty]}$, $\mathscr{K}^1 = \mathscr{F}^{[0,+\infty]}$ and functions $Y_t \colon \Omega^1 \to F$, $Y_t \in \mathscr{K}^1/\mathscr{F}$, so that

$$\int Q(\omega^0, d\omega^1) \exp\left[i\left(Y_t(\omega^1) - Y_0(\omega^1), y\right)\right] = M^y_t(\omega^0)$$

for each $y \in F$; $t \geqq 0$. Since M^y_t is a multiplicative functional which is a characteristic function in y, this is possible.

Finally let

$$\Omega = \Omega^0 \times \Omega^1, \qquad \mathscr{M} = \mathscr{K}^0 \times \mathscr{K}^1,$$

and define P as the unique probability measure on $\mathscr{M}$ which satisfies

$$P(A \times B) = \int_A P^0(d\omega^0)\, Q(\omega^0, B)$$

for each $A \in \mathscr{K}^0$, $B \in \mathscr{K}^1$. Extend the definitions of X_t and Y_t onto Ω in the natural manner. Then, $(X, Y) = \{X_t, Y_t; t \geqq 0\}$ is a Markov additive process over $(\Omega, \mathscr{M}, P)$ with Q as the semi-Markov transition function. (We omit the proof.)

We close this account with some examples.

(4.4) *Example*. Let X be a regular step process over $(\Omega, \mathscr{M}, P)$ with state space $(E, \mathscr{E})$ and suppose $t \to Y_t$ is right-continuous, non-decreasing. Then, in the decomposition given in (3.12) the Y^c_t term is missing and each of $t \to A_t$, $t \to Y^f_t$, $t \to Y^d_t$ is non-decreasing. Define

$$\tau_t = \inf\{s \colon Y_s > t\}.$$

Then, if we define $\hat{X}_t = X_{\tau_t}$, we obtain a process which is in general non-Markovian. Such a process is called a semi-Markov process (cf. [3, 9, 14, 17]).

(4.5) *Example*. Let Y be a Brownian motion over $(\Omega, \mathscr{M}, P)$ with state space $(\mathbb{R}, \mathscr{R})$. Let X be a diffusion on $(\mathbb{R}, \mathscr{R})$ obtained from Y through a stochastic integral. Then X is a Markov process and (X, Y) is a Markov additive process. If the diffusion X is observed, then Y can be written

$$Y_t = A_t + Y^c_t$$

where $\{A_t\}$ is an additive functional of X and Y^c_t is obtained from a Brownian motion $\tilde{Y}$ independent of X via the time change

$$Y^c_t = \tilde{Y}_{B_t}$$

where $\{B_t\}$ is a continuous non-decreasing additive functional of the diffusion X.

This example may be helpful if the same Brownian motion Y is used to define two diffusions X^1, X^2 and X^1 can be observed. Then, given X^1, we first consider the conditional structure of Y and then use this to make inferences about X^2.

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