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On the Barrier Problem for Sine Series with Independent Gaussian Coefficients

E.M. Cabaña

The barrier problem consisting in estimating the probability (1) is solved when the process $u(z)$ belongs to a class of sine series with independent coefficients. The solution is obtained by identifying the process u with the positions of a vibrating string forced by white noise, for which the same barrier problem has been solved in a previous paper ([2]).

0. Introduction

The present paper gives an estimate for the probability

$$P\{\max_{0\leqq z\leqq\pi} u(z)>a\} \tag{1}$$

that the process $u(z)$, defined as the sum of the series

$$u(z)=\sum_{n=1}^{\infty} X_n \sin nz \tag{2}$$

with independent centered Gaussian coefficients X_n, overpasses the barrier $a\,(a>0)$ on the interval $(0,\pi)$.

It is assumed that there exists a constant M such that the variances $\sigma_n^2=\operatorname{Var}(X_n)$ satisfy the inequalities

$$\sigma_n^2\leqq\frac{M}{n^2}\qquad (n=1,2,\ldots). \tag{3}$$

It is well known that this condition implies the a.s. convergence of the series in (2), and also (by Billard's Theorem, cf. [3]) the a.s. continuity of the paths of the process $u(z)$, hence the probability in (1) is well defined.

Further assumptions are also imposed on the variances in order to obtain the required estimate; in fact, the process $u(z)$ is identified with the position $u(\pi,z)$ of a vibrating string forced by white noise, and the remaining assumptions are introduced in order to render the identification possible.

1. The Barrier Problem for the Vibrating String

Let $u(t,z)$ be the position at time t and abscissa z of a vibrating string of length L which starts from rest $\big(u(0,z)=u_t(0,z)=0,\ 0\leqq z\leqq L\big)$, is tied at both ends $\big(u(t,0)=u(t,L)=0,\ t\geqq 0\big)$, and satisfies the equation

$$u_{tt}(t,z)=u_{zz}(t,z)+\frac{\partial^2}{\partial t\,\partial z}\beta(t,z)$$

where β represents an ordinary plane Brownian motion (and hence the forcing term is what we call a plane white noise). (See [1].)

We have obtained in [2] an estimate for the probability

$$P\{\max_{0\leqq z\leqq L} u(t,z)>a\}$$

which can be modified with no additional trouble to cover the case in which β is related to a canonical measure $d\mu$ ([1], §1.2.), that is, when $d\mu$ takes the place of the Lebesgue measure along the length of the string, and the covariance of $\beta(A), \beta(B)$ for two given plane sets is consequently equal to $\iint_{A\cap B} dt\, d\mu(z)$.

The length L of the string and the time t in which the position $u(t,z)$ is observed will both be set equal to π, and the measure $d\mu$ will be assumed to be symmetric with respect to $\pi/2$.

The formulation of the results in [2] adapted to a μ-Brownian motion is now the following, assuming the a.s. continuity (or at least the separability) of $u(t,\cdot)$:

(i) if $\phi_a(\sigma^2)=\frac{1}{\sqrt{2\pi}\sigma}\int_0^\infty e^{-\frac{1}{2}\frac{y^2}{\sigma^2}}dy$ denotes the probability that a centered Gaussian variable be greater than a, and $\mu=\int_0^L d\mu(z)$, then

$$P\{\sup_{0\leqq z\leqq L} u(t,z)>a\}\leqq 4\phi_a(t\mu),$$

and

(ii) given any $\delta>0$, there exists a constant A_δ such that

$$P\{\sup_{0\leqq z\leqq L} u(t,z)>a\}\leqq A_\delta\phi_a(g(t)+\delta), \tag{4}$$

where $g(t)=\max_{0\leqq z\leqq L} \operatorname{Var} u(t,z)$.

When the particular assumptions regarding L, t and $d\mu$ are applied, the inequality in (i) reduces to

$$P\{\max_{0\leqq z\leqq \pi} u(\pi,z)>a\}\leqq 4\phi_a(\pi\mu). \tag{5}$$

2. Representation of Sine Series

The process $u(\pi,z)$ can be expanded in a sine series

$$u(\pi,z)\sim\sum_{n=1}^\infty b_n \sin nz \tag{6}$$

with

$$b_n=\frac{2}{\pi}\int_0^\pi u(\pi,z)\sin nz\, dz,$$

and since $u(\pi,z)$ can be written in the integral form

$$u(\pi,z)=\iint_{\substack{\pi-z\leqq\zeta+\tau\leqq\pi+z\\ z-\pi\leqq\zeta-\tau\leqq\pi-z}} d\beta(\tau,\zeta),$$

then

$$b_n=\frac{2}{\pi}\iint d\beta(\tau,\zeta)\int_{|\pi-(\zeta+\tau)|}^{\pi-|\zeta-\tau|}\sin nz\,dz=\frac{4}{n\pi}\iint \sin n\zeta\,\sin n\tau\,d\beta(\tau,\zeta)$$

thus the Fourier coefficients b_n are centered Gaussian variables with covariances

$$\operatorname{Cov}(b_m,b_n)=\frac{16}{mn\pi^2}\int_0^\pi \sin mt\,\sin nt\,dt\int_0^\pi \sin mz\,\sin nz\,d\mu(z)$$

$$=\begin{cases} 0 & \text{if } m\neq n \\ \dfrac{8}{n^2\pi}\displaystyle\int_0^\pi \sin^2 nz\,d\mu(z) & \text{if } m=n. \end{cases} \tag{7}$$

We conclude that the Fourier series (6) has independent centered Gaussian coefficients with variances given by (7). In order to represent $u(z)$ by means of $u(\pi,z)$, that is, in order that the Gaussian process defined by (2) have the same distribution as $u(\pi,z)$, it is necessary and sufficient that

$$\sigma_n^2=\frac{8}{n^2\pi}\int_0^\pi \sin^2 nz\,d\mu(z). \tag{8}$$

If a finite measure $d\mu$ such that (8) holds does exist, the condition (3) is fulfilled, and this justifies its assumption. Moreover, if we set

$$h(z)=\int_z^{\pi/2} d\mu(\zeta)$$

and

$$H(z)=\int_0^z h(\zeta)\,d\zeta,$$

a plain calculation shows that (8) implies

$$\sigma_n^2=-\frac{16}{\pi}\int_0^\pi H(z)\cos 2nz\,dz \qquad (n=1,2,\ldots). \tag{9}$$

On the other hand, the variance function of the process $u(z)$ is

$$V(z)=\operatorname{Var}(u(z))=\sum_{n=1}^\infty \sigma_n^2\sin^2 nz=\tfrac{1}{2}\sum_{n=1}^\infty \sigma_n^2-\tfrac{1}{2}\sum_{n=1}^\infty \sigma_n^2\cos 2nz,$$

thus

$$\sigma_n^2=-\frac{4}{\pi}\int_0^\pi V(z)\cos 2nz\,dz \qquad (n=1,2,\ldots) \tag{10}$$

and it follows that if (8) holds, then $V(z)-4H(z)$ vanish a.e. because $V(0)=H(0)=0$, and for all z in $(0,\pi/2)$

$$V\left(\frac{\pi}{2}-z\right)=V\left(\frac{\pi}{2}+z\right),$$

$$H\left(\frac{\pi}{2}-z\right)=H\left(\frac{\pi}{2}+z\right),$$

and this joined with (9) and (10) implies that all the Fourier coefficients of $V(z)-4H(z)$ vanish.

Conversely, if $V(z)$ has a.e. a non-increasing derivative $v(z)$, then the measure $d\mu=-dv/4$ provides the representation (8).

3. Conclusions

Theorem 1. *Let $u(z)$ be defined by (2) with independent centered Gaussian coefficients X_n having variances $\sigma_n^2=\mathrm{Var}(X_n)$ which satisfy (3). Furthermore the variance function $V(z)=\sum_{n=1}^{\infty}\sigma_n^2\sin^2 nz$ has a.e. a bounded non-increasing derivative $v(z)$. Then*

$$P\{\max_{0\leqq z\leqq\pi} u(z)>a\}\leqq 4\phi_a\left(\frac{\pi v(0)}{2}\right)$$

(where the function ϕ_a is defined above) and for each $\delta>0$ there exists a constant A_δ such that the same probability is also bounded by

$$A_\delta\phi_a(V(\pi/2)+\delta).$$

The proof follows readily from the preceding context. The former assertion is a consequence of (5), with $d\mu=-dv/4$, hence $\mu=\dfrac{v(0)-v(\pi)}{4}=\dfrac{v(0)}{2}$. For the latter one use (4) and notice that $V(\pi/2)=\max_{0\leqq z\leqq\pi}V(z)$.

Theorem 2. *Let $u(z)$ and $V(z)$ be defined as in Theorem 1 and let $V(z)$ have a.e. a derivative $v(z)$ of bounded variation with canonical decomposition*

$$v(z)=\bar{v}(z)-\underline{v}(z)$$

as a difference of two non-increasing functions. The symmetry of V with respect to $\pi/2$ implies that for $0\leqq z\leqq\pi/2$

$$v\left(\frac{\pi}{2}-z\right)+v\left(\frac{\pi}{2}+z\right)=0$$

and $\bar{v},\underline{v}$ will be chosen satisfying the analogous property. Then for non-negative $\bar{a},\underline{a}$ such that $\bar{a}+\underline{a}=a$,

$$P\{\max_{0\leqq z\leqq\pi} u(z)>a\}\leqq 4\phi_{\bar{a}}\left(\frac{\pi\bar{v}(0)}{2}\right)+4\phi_{\underline{a}}\left(\frac{\pi\underline{v}(0)}{2}\right)$$

and for each $\delta>0$ there exists a constant A_δ such that the same probability is also bounded by

$$A_\delta\left[\phi_{\bar{a}}\left(\bar{V}\left(\frac{\pi}{2}\right)+\delta\right)+\phi_{\underline{a}}\left(\underline{V}\left(\frac{\pi}{2}\right)+\delta\right)\right]$$

with

$$\bar{V}(z)=\int_0^z\bar{v}(\zeta)\,d\zeta,\qquad \underline{V}(z)=\int_0^z\underline{v}(\zeta)\,d\zeta.$$

Let us define the sequences

$$\bar{\sigma}_n^2 = -\frac{2}{n^2\pi}\int_0^\pi \sin^2 nz\, d\bar{v}(z),$$

$$\underline{\sigma}_n^2 = -\frac{2}{n^2\pi}\int_0^\pi \sin^2 nz\, d\underline{v}(z),$$

and notice that, since the equalities

$$\sigma_n^2 = -\frac{2}{n^2\pi}\int_0^\pi \sin^2 nz\, dv(z)$$

hold, then

$$\sigma_n^2 = \bar{\sigma}_n^2 - \underline{\sigma}_n^2.$$

Let us consider now a new sequence $\underline{X}_n$ of independent centered Gaussian variables, also independent of the sequence X_n, with variances $\operatorname{Var}(\underline{X}_n) = \underline{\sigma}_n^2$, and let us set

$$\underline{u}(z) = \sum_{n=1}^{\infty} \underline{X}_n \sin nz,$$

therefore

$$\bar{u}(z) = u(z) + \underline{u}(z) = \sum_{n=1}^{\infty} (X_n + \underline{X}_n) \sin nz$$

has coefficients $\bar{X}_n = X_n + \underline{X}_n$ with variances $\bar{\sigma}_n^2$.

We finally notice that

$$\{\max_{0\leqq z\leqq\pi} u(z) > a\} \subset \{\max_{0\leqq z\leqq\pi} \bar{u}(z) > \bar{a}\} \cup \{\max_{0\leqq z\leqq\pi} [-\underline{u}(z)] > a\}$$

and use the estimates of Theorem 1 for the probabilities of the events at the right hand side of the inclusion.

References

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E.M. Cabaña
Instituto de Matematíca y Estadística
Universidad de la República
Av. J. Herrera y Reissig 565
Montevideo, Uruguay

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