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A Decomposition of Square Integrable Martingales

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§ 1. Introduction

It is well known that if $(M_t, \mathscr{F}_t)_{t \geqq 0}$ is a continuous, square integrable martingale, then M_t can be written as an integral with respect to some Brownian motion if and only if $d\langle M\rangle_t$ is absolutely continuous with respect to Lebesgue measure [2]. It is also known that if $(M_t, \mathscr{F}_t)_{t \geqq 0}$ and $(N_t, \mathscr{F}_t)_{t \geqq 0}$ are continuous, square integrable martingales then M_t can be written uniquely as

$$M_t = \int_0^t \phi_s \, dN_s + \alpha_t \tag{1.1}$$

where $\langle \alpha, N\rangle_t \equiv 0$ [3]. Now if $(M_t, \mathscr{F}_t)_{t \geqq 0}$ is a continuous, square integrable martingale with $\langle M\rangle_t$ absolutely continuous and if $(X_t, \mathscr{F}_t)_{t \geqq 0}$ is a Brownian motion, one might ask whether

$$M_t = \int_0^t \phi_s \, dX_s.$$

That is, can M_t be written as an indefinite integral with respect to *the given* Brownian motion, X_t? It is known that if M_t is a martingale with respect to the σ-fields, $\sigma(X_s : s \leqq t)$, generated by the Brownian motion, X_t, then one can write $M_t = \int_0^t \phi_s \, dX_s$ [3]. However, the following example shows it is not true in general. Let X_t and Y_t be independent Brownian motions defined on $(\Omega_1 \times \Omega_2, \mathscr{F}_t^1 \times \mathscr{F}_t^2)$. Then $\langle X, Y\rangle_t \equiv 0$ and $\langle Y\rangle_t = \langle X\rangle_t = t$. We know by (1.1) that $Y_t = \int_0^t \phi_s \, dX_s + \alpha_t$ with $\langle \alpha, X\rangle_t \equiv 0$ is a unique decomposition of Y_t so we have $\alpha_t = Y_t$ and $\phi_s \equiv 0$. Hence Y_t can be written as an indefinite integral with respect to some Brownian motion (namely Y_t itself) but not with respect to X_t. In this paper we will use this sort of freedom in choosing the integrator to get a variation of the decomposition, (1.1).

§ 2. The Decomposition Theorem

Let $(M_t, \mathscr{F}_t)_{t \geqq 0}$ and $(N_t, \mathscr{F}_t)_{t \geqq 0}$ be continuous square integrable martingales. Let $\langle M\rangle_t$ be the natural increasing process arising in the Doob decomposition of M_t^2. In this paper we will consider the following questions.

(i) Under what conditions can M_t be written as an indefinite integral with respect to N_t? If one cannot use N_t itself as the integrator, what processes related to N_t are appropriate?

(ii) What is the relationship between $\langle M,N\rangle_t \equiv 0$ and $d\langle M\rangle \perp d\langle N\rangle$? (Notation: $d\langle M\rangle \perp d\langle N\rangle$ if the measures are singular on $[0,\infty)$ for a.e. ω).

(iii) Is there an analogue to (1.1) with the α_t having the property $d\langle\alpha\rangle \perp d\langle N\rangle$?

In the remainder of this paper we will write $M_t = N_t$ when $P[M_t = N_t] = 1$ for all t.

Definition 1. We say $(M_t, \mathscr{F}_t)$ and $(M_t^*, \mathscr{F}_t)$ are *similar* square integrable martingales if $\langle M\rangle_t = \langle M^*\rangle_t$.

Definition 2. Two square integrable martingales $(M_t, \mathscr{F}_t)$ and $(N_t, \mathscr{F}_t)$ are said to be *orthogonal* if $\langle M,N\rangle_t \equiv 0$ a.s.

Definition 3. Two square integrable martingales $(M_t, \mathscr{F}_t)$ and $(N_t, \mathscr{F}_t)$ are said to be *strongly orthogonal* if $d\langle M\rangle \perp d\langle N\rangle$ a.s.

Lemma 1. *If $(M_t, \mathscr{F}_t)_{t\geqq 0}$ and $(N_t, \mathscr{F}_t)_{t\geqq 0}$ are strongly orthogonal, then they are orthogonal.*

Proof. For fixed ω use $d\langle M\rangle$ and $d\langle N\rangle$ to partition $[0,\infty)$ into disjoint sets A_ω and B_ω such that $d\langle M\rangle[A_\omega]=0$ and $d\langle N\rangle[B_\omega]=0$. This can be done for a.e. ω by assumption. We know

$$\left[E\int_0^t |H_s K_s|\, d|\langle M,N\rangle_s|\right]^2 \leqq \left(E\int_0^t H_s^2\, d\langle M\rangle_s\right)\left(E\int_0^t K_s^2\, d\langle N\rangle_s\right)$$

and $\langle M,N\rangle_t = \int_0^t \psi_s\, d\langle N\rangle_s$ (see [3]). Now let $H_s(\omega) = 1_{A_\omega}(s)$ and $K_s \equiv 1$. Then

$$\left[E\int_0^t |1_{A_\omega}\cdot 1|\, d|\langle M,N\rangle_s|\right]^2 \leqq \left(E\int_0^t 1_{A_\omega} d\langle M\rangle_s\right)\left(E\int_0^t d\langle N\rangle_s\right) = 0.$$

Hence

$$E\int_0^t 1_{A_\omega}|\psi_s|\, d\langle N\rangle_s = E\int_0^t 1_{A_\omega}\, d|\langle M,N\rangle_s| = 0.$$

Now $1_{A_\omega} = 1$ a.e. $d\langle N\rangle$ so

$$E\int_0^t |\psi_s|\, d\langle N\rangle_s = 0.$$

Therefore $|\psi_s| = 0$ a.e. $d\langle N\rangle$ so

$$\langle M,N\rangle_t = \int_0^t \psi_s\, d\langle N\rangle_s \equiv 0.$$

Theorem. *Let M_t and N_t be continuous square integrable martingales on $(\mathscr{F}_t)$. Then M_t can be decomposed into an indefinite integral with respect to a process similar to N_t and a process strongly orthogonal to N_t.*

That is, $M_t = \int_0^t \phi_s\, dN_s^* + R_t$ where $\langle R\rangle_t \perp \langle N\rangle_t$. By symmetry N_t can also be so decomposed with respect to M_t.

Proof. By Lebesgue's decomposition theorem we know $d\langle M\rangle = d\langle M^1\rangle + d\langle M^2\rangle$ where $d\langle M^1\rangle \ll d\langle N\rangle$ and $d\langle M^2\rangle \perp d\langle N\rangle$.

Hence

$$\langle M^1\rangle_t=\int_0^t\left(\frac{d\langle M^1\rangle_s}{d\langle N\rangle_s}\right)d\langle N\rangle_s.$$

Define

$$N_t^*=\int_0^t\phi_s^{-1}\,dM_s+\int_0^t\psi_s\,d\overline{N}_s$$

where $\phi_s^{-1}=0$ whenever

$$\left(\frac{d\langle M^1\rangle_s}{d\langle N\rangle_s}\right)^{\frac{1}{2}}=0,\qquad \phi_s^{-1}=\left(\frac{d\langle M^1\rangle_s}{d\langle N\rangle_s}\right)^{-\frac{1}{2}}$$

otherwise, where

$$\psi_s=0\quad\text{when}\quad\frac{d\langle M^1\rangle_s}{d\langle N\rangle_s}\neq 0$$

$$\psi_s=1\quad\text{when}\quad\frac{d\langle M^1\rangle_s}{d\langle N\rangle_s}=0$$

and where $\overline{N}_s$ is similar to N_s and $\langle M,\overline{N}\rangle_t\equiv 0$ ([1], p. 450). Now

$$\langle N^*\rangle_t=\int_0^t\phi_s^{-2}\,d\langle M\rangle_s+\int_0^t\psi_s\,d\langle N\rangle_s$$

$$=\int_0^t\phi_s^{-2}\left(\frac{d\langle M^1\rangle_s}{d\langle N\rangle_s}\right)d\langle N\rangle_s+\int_0^t\psi_s\,d\langle N\rangle_s=\int_0^t d\langle N\rangle_s=\langle N\rangle_t.$$

Hence N_t^* is similar to N_t. Consider

$$\int_0^t\phi_s\,dN_s^*=\int_0^t\phi_s\phi_s^{-1}\,dM_s+\int_0^t\phi_s\psi_s\,d\overline{N}_s=\int_0^t(1-\psi_s)\,dM_s+0.$$

Hence

$$M_t=\int_0^t\phi_s\,dN_s^*+\int_0^t\psi_s\,dM_s.$$

Now

$$\left\langle\int_0^t\psi_s\,dM_s\right\rangle=\int_0^t\psi_s\,d\langle M\rangle_s=\int_0^t\psi_s\,d\langle M^1\rangle_s+\int_0^t\psi_s\,d\langle M^2\rangle_s$$

$$=\int_0^t\psi_s\frac{d\langle M^1\rangle_s}{d\langle N\rangle_s}\,d\langle N\rangle_s+\langle M^2\rangle_t=\langle M^2\rangle_t.$$

By choosing $R_t=\int_0^t\psi_s\,dM_s$ we have $\langle R\rangle_t\perp\langle N\rangle_t$.

Corollary. *If* $(M_t,\mathscr{F}_t)$ *and* $(N_t,\mathscr{F}_t)$ *are square integrable martingales, then* $d\langle M\rangle\perp d\langle N\rangle$ a.s. *if and only if* $\langle M,N^*\rangle_t\equiv 0$ a.s. *for all processes,* N_t^**, that are similar to* N_t*. Also* $d\langle M\rangle\ll d\langle N\rangle$ *if and only if* $M_t=\int_0^t\phi_s\,dN_s^*$ *for some process,* N_t^**, similar to* N_t*.*

Proof. (i) If $d\langle M\rangle\perp d\langle N\rangle$ then Lemma 1 shows $\langle M,N^*\rangle_t\equiv 0$.

(ii) By the theorem we know $M_t = \int_0^t \phi_s \, dN_s^* + R_t$ where $d\langle R\rangle \perp d\langle N\rangle$. Now $\langle M, N^*\rangle_t = \int_0^t \phi_s \, d\langle N\rangle_s$. Hence if $\int_0^t \phi_s \, d\langle N\rangle_s = 0$, then the above decomposition is $M_t = R_t$ so $d\langle M\rangle \perp d\langle N\rangle$.

(iii) We know $M_t = \int_0^t \phi_s \, dN_s^* + R_t$. Hence $\langle M\rangle_t = \int_0^t \phi_s^2 \, d\langle N\rangle_s + \langle R\rangle_t$ where $d\langle R\rangle \perp d\langle N\rangle$. Therefore $d\langle M\rangle = d[\int \phi_s^2 \, d\langle N\rangle_s] + d\langle R\rangle$ which is the unique Lebesgue decomposition of $d\langle M\rangle$ so if $d\langle M\rangle \ll d\langle N\rangle$, we get $d\langle R\rangle = 0$. Hence $R_t \equiv C$ which in this case is the constant zero.

(iv) If $M_t = \int_0^t \phi_s \, dN_s^*$ then $\langle M\rangle_t = \int_0^t \phi_s^2 \, d\langle N\rangle_s$ so $d\langle M\rangle \ll d\langle N\rangle$.

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