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Some Contraction Semigroups in Quantum Probability

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§ 1. Introduction

In earlier papers [1–5] we have introduced and developed the idea of a quantum stochastic process, which is a generalisation to quantum mechanics of the classical Markov process of pseudo-Poisson type. In the paper [3] we showed how to construct quantum stochastic processes and characterised them in terms of their infinitesimal generators. In [4] we introduced the idea of irreducibility as a generalisation of the classical notion, and proved that irreducible processes could be divided into recurrent and transient processes, and that every irreducible finite-dimensional process is recurrent with a unique equilibrium state. The proofs were functional-analytical modifications of the classical proofs of the same theorems for classical Markov chains. In [3, 5] we showed how quantum stochastic processes allow the resolution of some conceptual difficulties in the analysis of some recent experiments on the statistics of coherent photon beams.

In this paper we study quantum stochastic process, or rather their associated semigroups, from rather a different point of view. The point of the work is to describe the Markov semigroups in more direct terms, which help to illuminate certain aspects of the theory of quantum stochastic processes, and to relate them to the time evolution of systems with an infinite number of degrees of freedom.

We start by constructing a quantum stochastic process on a boson Fock space from two self-adjoint operators, one a Hamiltonian and the other a detection or decay rate. The evolution of the pure coherent states with respect to the semigroup of the process is explicitly calculated and shown to be the same as that obtained by reduction of the wave-packet from an isometric evolution on a larger Hilbert space. This establishes that the quantum stochastic processes studied in [3] as a description of the interaction of a photon field with a measuring apparatus, are determined by isometric semigroups if one includes a quantisation of the measuring apparatus.

In the fourth section we carry out a similar procedure for the type of quantum stochastic process which we proposed in [4] as a model for the time evolution of radiating systems and again show that the evolution can be obtained by reduction of the wave-packet from an isometric semigroup. In this case the larger Hilbert space is the tensor product of the given Hilbert space and a Hilbert space which is very like a Fock space. It seems reasonable to interpret this second Hilbert space as a simplification of that part of the quantised electromagnetic field associated with the outgoing photons.

In the fifth section we describe how to construct a quantum stochastic process given a unitary representation of the real numbers and a one-parameter semigroup of probability measures on the real line. It turns out that the isometric dilation of the process can again be explicitly determined by using coherent states. The semigroup of the process is given by a formula which is known in classical probability theory as the formula for subordinating one stochastic process to another by randomising the evolution parameter.

§ 2. The Process Associated with a Quantum Field

Let $\mathscr{H}$ be a Hilbert space describing the possible states of a quantum system with Hamiltonian H. Let D be a positive, bounded, self-adjoint operator on $\mathscr{H}$, to be interpreted as a detection, absorption or decay rate. Following the methods and terminology of [3] we construct a quantum stochastic process $\mathscr{E}$ on the boson Fock space $\mathscr{F}$ over $\mathscr{H}$.

To fix notation we write $\mathscr{F}=\sum_{n=0}^{\infty}\mathscr{F}^{(n)}$ where $\mathscr{F}^{(n)}=\otimes_{\rm sym}^{n}\mathscr{H}$, and we let S_n: $\otimes^n\mathscr{H}\to\mathscr{F}^{(n)}$ be the usual symmetrisation projection. We define $\mathscr{F}_n=\sum_{r=0}^{n}\mathscr{F}^{(r)}$, $V=\mathrm{T}_s(\mathscr{F})$, $V_n=\mathrm{T}_s(\mathscr{F}_n)$ and denote the number operator by N. We denote by A_t the strongly continuous contraction semigroup on $\mathscr{H}$ whose infinitesimal generator is $-iH-\frac{1}{2}D$ and by B_t the strongly continuous contraction semigroup on $\mathscr{F}$ which is the restriction of the contraction semigroup on $\sum_{n=0}^{\infty}\otimes^n\mathscr{H}$ defined by

$$B_t(\psi_1\otimes\cdots\otimes\psi_n)=A_t\psi_1\otimes\cdots\otimes A_t\psi_n. \tag{2.1}$$

For each $\psi\in\mathscr{H}$ we define the coherent state $\tilde{\psi}\in\mathscr{F}$ by

$$\tilde{\psi}^{(n)}=(n!)^{-\frac{1}{2}}\otimes^n\psi \tag{2.2}$$

so that

$$\langle\tilde{\phi},\tilde{\psi}\rangle=\exp\langle\phi,\psi\rangle \tag{2.3}$$

and then define the state $C(\psi)\in V$ by

$$C(\psi)=\exp\{-\|\psi\|^2\}\,\tilde{\psi}\otimes\tilde{\psi}^-. \tag{2.4}$$

It is shown in [7] that the linear span of the $C(\psi)$ where $\psi\in\mathscr{H}$ is dense in V, which as usual is given the trace norm topology.

The value space X of the quantum stochastic process $\mathscr{E}$ to be constructed has only one point, so the sample space X_t defined in [3] is

$$X_t=\bigcup_{n=0}^{\infty}\{(t_1,\ldots,t_n)\colon 0<t_1<\cdots<t_n\leqq t\}. \tag{2.5}$$

As in [3] we denote by z the point in X_t corresponding to zero events and by S_t, T_t the semigroups on V defined by

$$S_t(\rho)=\mathscr{E}_t(z,\rho);\qquad T_t(\rho)=\mathscr{E}_t(X_t,\rho). \tag{2.6}$$

Since X has only one point a stochastic kernel on X can be defined as a positive linear map $\mathscr{G}$: $V\to V$.

Theorem 1. *There exists a quantum stochastic process $\mathscr{E}$ on V, X such that for all $\psi\in\mathscr{H}$ and all $\rho\in V$*

$$S_t(\rho)=B_t\,\rho\,B_t^*; \qquad T_t(C(\psi))=C(A_t\psi). \tag{2.7}$$

Proof. We construct $\mathscr{E}^{(n)}$ on V_n and then pass to a limit as $n\to\infty$.

The map $\rho\to N^{\frac{1}{2}}\rho N^{\frac{1}{2}}$ is a bounded positive linear map of $\mathrm{T}_s\left(\sum_{r=0}^{n}\otimes^r\mathscr{H}\right)$ into $\mathrm{T}_s\left(\sum_{r=1}^{n}\otimes^r\mathscr{H}\right)$. The unitary isomorphism of $\sum_{r=1}^{n}\otimes^r\mathscr{H}$ with $\mathscr{H}\otimes\left(\sum_{r=0}^{n-1}\otimes^r\mathscr{H}\right)$ induces a positive isometric embedding e of $\mathrm{T}_s\left(\sum_{r=1}^{n}\otimes^r\mathscr{H}\right)$ into $\mathrm{T}_s\left(\mathscr{H}\otimes\left(\sum_{r=0}^{n}\otimes^r\mathscr{H}\right)\right)$. Finally there is a reduction d from $\mathrm{T}_s\left(\mathscr{H}\otimes\left(\sum_{r=0}^{n}\otimes^r\mathscr{H}\right)\right)$ into $\mathrm{T}_s\left(\sum_{r=0}^{n}\otimes^r\mathscr{H}\right)$ such that for all $A\in\mathscr{L}\left(\sum_{r=0}^{n}\otimes^r\mathscr{H}\right)$ and $\rho\in\mathrm{T}_s\left(\mathscr{H}\otimes\left(\sum_{r=0}^{n}\otimes^r\mathscr{H}\right)\right)$

$$\mathrm{tr}\,[A\,d(\rho)]=\mathrm{tr}\,[(D\otimes A)\,\rho]\,.$$

The reduction d is positive and bounded. The stochastic kernel $\mathscr{G}_n$ is defined by

$$\mathscr{G}_n(\rho)=\mathrm{de}(N^{\frac{1}{2}}\rho N^{\frac{1}{2}}). \tag{2.8}$$

As it stands $\mathscr{G}_n$ is a bounded positive linear map on $\mathrm{T}_s\left(\sum_{r=0}^{n}\otimes^r\mathscr{H}\right)$, but it is easily seen that it leaves invariant the subspace V_n. If $R^{(r)}$ is the operator on $\mathscr{F}^{(r)}$

$$R^{(r)}=D\otimes 1\otimes\cdots\otimes 1+\cdots+1\otimes\cdots\otimes 1\otimes D$$

then it is easy to check directly that for all $\psi\in\mathscr{F}_n$

$$\mathrm{tr}\,[\mathscr{G}_n(\psi\otimes\psi^-)]=\sum_{r=0}^{n}\langle R^{(r)}\psi^{(r)},\psi^{(r)}\rangle\,.$$

By density arguments it follows that for all $\rho\in V_n$

$$\mathrm{tr}\,[\mathscr{G}_n(\rho)]=\mathrm{tr}\left[\sum_{r=0}^{n}R^{(r)}\rho\right] \tag{2.9}$$

so $\sum_{r=0}^{n}R^{(r)}$ is the total interaction rate of $\mathscr{G}_n$. We comment here that the unbounded positive self-adjoint operator $R=\sum_{r=0}^{\infty}R^{(r)}$ is certainly well-defined on the domain $\mathscr{D}_N$ of the number operator N.

If H_0 is the free Hamiltonian on $\mathscr{F}$ constructed from H and if Z is the infinitesimal generator of the semigroup B_t then

$$\mathscr{D}_{H_0}\cap\mathscr{F}_n=\mathscr{D}_Z\cap\mathscr{F}_n$$

for all $n=0,1,2,\ldots$ and for all ψ in this domain

$$Z\psi=-iH_0\psi-\tfrac{1}{2}R\psi\,. \tag{2.10}$$

Therefore by [3] there exists a quantum stochastic process $\mathscr{E}^{(n)}$ on V_n with infinitesimal generators $-iH_0-\frac{1}{2}R$ and $\mathscr{G}_n$. As n increases the processes $\mathscr{E}^{(n)}$ are compatible in the sense of [3], so there exists a common extension $\mathscr{E}$ to V. $\mathscr{E}$ has an unbounded stochastic kernel $\mathscr{G}$ which extends all the $\mathscr{G}_n$. To compare this with [5] we point out that if $\mathscr{H}=L^2(X,\mu)$ and D is the operator obtained by multiplying by the positive bounded function λ on X and if $a(x)$ is the annihilation operator on $\mathscr{F}$ associated with a generic point $x\in X$, then $\mathscr{G}$ is given formally by

$$\mathscr{G}(\rho)=\int_X \lambda(x)\,a(x)\,\rho\,a(x)^*\,\mu(dx). \tag{2.11}$$

To verify Eq. (2.7) we introduce the projected coherent states $\tilde{\psi}_n\in\mathscr{F}$ defined by

$$(\tilde{\psi}_n)^{(r)}=\begin{cases}(r!)^{-\frac{1}{2}}\otimes^r\psi & r\leqq n\\ 0 & r>n\end{cases}$$

and denote by W, Y the infinitesimal generators of the semigroups S_t and T_t on V respectively. For the rest of the proof we let ψ denote an arbitrary element of $\mathscr{D}_H=\mathscr{D}_{-iH-\frac{1}{2}D}$. For such ψ, $\tilde{\psi}_n\in\mathscr{D}_Z$ for all n so $e^{-\|\psi\|^2}\tilde{\psi}_n\otimes\tilde{\psi}_n^-\in\mathscr{D}_W$ and

$$W\{e^{-\|\psi\|^2}\tilde{\psi}_n\otimes\tilde{\psi}_n^-\}=e^{-\|\psi\|^2}Z\tilde{\psi}_n\otimes\tilde{\psi}_n^-+e^{-\|\psi\|^2}\tilde{\psi}_n\otimes(Z\tilde{\psi}_n)^-. \tag{2.12}$$

Moreover by [3], $\mathscr{D}_Y\cap V_n=\mathscr{D}_W\cap V_n$ and for all ρ in this domain

$$Y(\rho)=W(\rho)+\mathscr{G}_n(\rho) \tag{2.13}$$

while

$$\mathscr{G}_n(\tilde{\psi}_n\otimes\tilde{\psi}_n^-)=\langle D\psi,\psi\rangle\,\tilde{\psi}_{n-1}\otimes\tilde{\psi}_{n-1}^-. \tag{2.14}$$

Using the fact that Y and W are closed operators and going to the limit as $n\to\infty$ gives $C(\psi)\in\mathscr{D}_W\cap\mathscr{D}_Y$ and

$$W\{C(\psi)\}=e^{-\|\psi\|^2}Z\tilde{\psi}\otimes\tilde{\psi}^-+e^{-\|\psi\|^2}\tilde{\psi}\otimes(Z\tilde{\psi})^-, \tag{2.15}$$

$$Y\{C(\psi)\}=W\{C(\psi)\}+\langle D\psi,\psi\rangle\,\tilde{\psi}\otimes\tilde{\psi}^-. \tag{2.16}$$

On the other hand by direct calculation

$$\begin{aligned}\lim_{t\to0}t^{-1}\{C(A_t\psi)-C(\psi)\}&=\lim_{t\to0}t^{-1}\{e^{-\|A_t\psi\|^2}B_t\tilde{\psi}\otimes(B_t\tilde{\psi})^-\}\\&=\langle D\psi,\psi\rangle\,e^{-\|\psi\|^2}\tilde{\psi}\otimes\tilde{\psi}^-+e^{-\|\psi\|^2}Z\tilde{\psi}\otimes\tilde{\psi}^-\\&\quad+e^{-\|\psi\|^2}\tilde{\psi}\otimes(Z\tilde{\psi})^-\end{aligned} \tag{2.17}$$

which implies that

$$\lim_{t\to0}t^{-1}\{C(A_t\psi)-C(\psi)\}=Y\{C(\psi)\}. \tag{2.18}$$

Since the semigroup A_t leaves $\mathscr{D}_{-iH-\frac{1}{2}D}$ invariant it follows by the theory of one-parameter semigroups that for all $\psi\in\mathscr{D}_{-iH-\frac{1}{2}D}$ and all $t\geqq0$

$$T_t\{C(\psi)\}=C(A_t\psi).$$

This same formula is now valid for all $\psi\in\mathscr{H}$ by standard density arguments.

§ 3. An Alternative Construction of the Semigroup T_t

The alternative construction of T_t depends basically on Nagy's theorem [8], that every contraction semigroup on a Hilbert space can be obtained as the projection of a one-parameter unitary group on a larger Hilbert space. We give a constructive proof of this theorem in the case of interest since this makes the structure of the larger Hilbert space rather clear.

Proposition 2. *Let $A_t = \exp\{(-iH - \frac{1}{2}D)t\}$ be a contraction semigroup on the Hilbert space $\mathscr{H}$, where D is a bounded positive operator. Choose a representation $\mathscr{H} = L^2(X, \mu)$ such that D corresponds to multiplication by a bounded positive function d on X. Then there exists a one-parameter semigroup of isometries V_t on $\mathscr{H} \oplus L^2(X \times R^+)$ such that for all $\xi \in \mathscr{H}$, $\eta \in L^2(X \times \mathbb{R}^+)$ and $t \geqq 0$*

$$V_t(\xi \oplus \eta) = A_t \xi \oplus \eta_t. \tag{3.1}$$

Proof. We define V_t by the above formula, where

$$\eta_t(x, s) = \begin{cases} \eta(x, s-t) & \text{if } s \geqq t \\ \{A_{t-s}\xi\}(x)\, d^{\frac{1}{2}}(x) & \text{if } 0 < s < t. \end{cases} \tag{3.2}$$

Direct calculation shows that V_t is a strongly continuous semigroup. By Eq. (4.4) of [3]

$$\begin{aligned} \|V_t(\xi \oplus \eta)\|^2 &= \|A_t \xi\|^2 + \|\eta\|^2 + \int_{s=0}^{t} \langle D^{\frac{1}{2}} A_{t-s}\xi, D^{\frac{1}{2}} A_{t-s}\xi \rangle\, ds \\ &= \|A_t \xi\|^2 + \|\eta\|^2 + \int_{s=0}^{t} \operatorname{tr}[D(A_{t-s}\xi) \otimes (A_{t-s}\xi)^-]\, ds \\ &= \|\xi\|^2 + \|\eta\|^2 \end{aligned}$$

so V_t is a one-parameter semigroup of isometries.

We note that there is a standard procedure for extending a one-parameter semigroup of isometries to a one-parameter unitary group, [9], and by use of this we obtain Nagy's theorem. However, even as it stands we can interpret the summand $L^2(X \times \mathbb{R}^+)$ as being the space of single particle out-states of the measuring apparatus (see also [10]).

We denote by $\mathscr{F}$, $\mathscr{F}'$, $\mathscr{F}''$ the boson Fock spaces over $\mathscr{H}$, $\mathscr{H}'$ and $\mathscr{H}'' \equiv \mathscr{H} \otimes \mathscr{H}'$, by V, V', V'' the state spaces of those Fock spaces and by Ψ_0, Ψ_0', Ψ_0'' the corresponding vacuum states, so that $\Psi_0'' = \Psi_0 \otimes \Psi_0'$ in the isomorphism $\mathscr{F}'' = \mathscr{F} \otimes \mathscr{F}'$. We define $r\colon V'' \to V$ by the equation

$$\operatorname{tr}[A\, r(\rho)] = \operatorname{tr}[(A \otimes I)\, \rho] \tag{3.3}$$

valid for all $A \in \mathscr{L}(\mathscr{F})$ and $\rho \in V''$. It is clear that r is a bounded, positive, linear map which preserves the trace. On the other hand the injection $\phi \to \phi \otimes \Psi_0'$ of $\mathscr{F}$ into $\mathscr{F}''$ induces a positive isometric embedding $e\colon V \to V''$ and it is easy to see that re is the identity map on V.

Theorem 3. *There exists a one-parameter isometric semigroup G_t on $\mathscr{F}''$ such that for all $\rho \in V$ and $t \geqq 0$*

$$T_t(\rho) = r\{G_t(e\rho)\, G_t^*\}. \tag{3.4}$$

Proof. We define $\mathscr{H}'$ so that A_t has an isometric extension U_t to $\mathscr{H}\oplus\mathscr{H}'$, using Proposition 2. We define G_t as the isometric semigroup on $\sum_{n=0}^{\infty}\otimes^n\mathscr{H}''$ given by

$$G_t(\psi_1\otimes\cdots\otimes\psi_n)=(U_t\psi_1)\otimes\cdots\otimes(U_t\psi_n)$$

and then restrict this to the invariant subspace $\mathscr{F}''$. An easy calculation shows that for all $\xi\in\mathscr{H}$ and $\eta\in\mathscr{H}'$

$$\begin{aligned} G_t\{(\xi\oplus\eta)^{\sim}\} &= \{U_t(\xi\oplus\eta)\}^{\sim} \\ &= \{A_t\xi\oplus\eta_t\}^{\sim} \\ &= (A_t\xi)^{\sim}\otimes\eta_t^{\sim}. \end{aligned} \tag{3.5}$$

Therefore if $\rho=\tilde{\xi}\otimes\tilde{\xi}^{-}$

$$r\{G_t(e\,\rho)\,G_t^*\}=\|\tilde{\eta}_t\|^2(A_t\xi)^{\sim}\otimes(A_t\xi)^{\sim -}$$

so

$$r\{G_t(e\,C(\xi))\,G_t^*\}=\alpha_t\,C(A_t\xi).$$

Taking traces of both sides gives $\alpha_t=1$, so by Eq. (2.7), Eq. (3.4) is valid for all states of the form $\rho=C(\xi)$. Its general validity now follows from the fact that the linear span of such states is dense in V, [7].

§ 4. The Process Associated with a Radiating System

We consider here the class of quantum stochastic processes studied in detail in [4]. We suppose that $\mathscr{H}$ is a separable Hilbert space and that the value space X is a compact metric space with a finite measure μ. The process $\mathscr{E}$ is constructed as in [4] from a Hamiltonian H_0 on $\mathscr{H}$ and a strongly continuous family A_x of operators on $\mathscr{H}$ parametrised by $x\in X$. The stochastic kernel $\mathscr{G}$ of $\mathscr{E}$ is given by

$$\mathscr{G}(E,\rho)=\int_E A_x\,\rho\,A_x^*\,\mu(dx) \tag{4.1}$$

so that the total interaction rate R is

$$R=\int_X A_x^*\,A_x\,\mu(dx). \tag{4.2}$$

The semigroup B_t on $\mathscr{H}$ is the strongly continuous semigroup whose infinitesimal generator is $Z=iH_0-\frac{1}{2}R$. If z is the point of X_t corresponding to no events then as in [3]

$$S_t(\rho)=\mathscr{E}_t(z,\rho)=B_t\,\rho\,B_t^* \tag{4.3}$$

for all ρ in $V=\tau_s(\mathscr{H})$. Finally as in [3,4] we define the semigroup T_t on V by

$$T_t(\rho)=\mathscr{E}_t(X_t,\rho). \tag{4.4}$$

We comment that the processes of Section 2 are not of this class because the annihilation operators occurring in Eq. (2.11) are not bounded, even on the single particle subspace.

In order to construct the unitary group corresponding to the process $\mathscr{E}$ we have to introduce some new Hilbert spaces. If $0<t<\infty$ then the sample space X_t

defined in [4] carries a measure μ_t, the product measure constructed from μ on each component X and Lebesgue measure on each time component. We assign the measure unity to the point $z \in X_t$. A similar measure μ_∞ can be defined on the subset Y_∞ of X_∞ consisting of the sequences $\omega \in X_\infty$ of finite length. We define $\mathscr{H}'$ as the Hilbert space $L^2(Y_\infty, \mathscr{H}, \mu_\infty)$ of square-integrable functions on Y_∞ with values in $\mathscr{H}$ and let V' be the state space of $\mathscr{H}'$.

Lemma 4. *There exist positive trace-preserving linear maps e: $V \to V'$ and r: $V' \to V$ such that* $re(\rho) = \rho$ *for all $\rho \in V$.*

Proof. We define an isometric injection $\psi \to \psi^0$ of $\mathscr{H}$ into $\mathscr{H}'$ by letting $\psi_z^0 = \psi$ and $\psi_\omega^0 = 0$ for all other $\omega \in Y_\infty$. The map e: $V \to V'$ is defined as the unique positive linear isometric embedding of V into V' such that $e(\psi \otimes \psi^-) = \psi^0 \otimes \psi^{0-}$ for all $\psi \in \mathscr{H}$. For each $A \in \mathscr{L}(\mathscr{H})$ let A_0 be the operator in $\mathscr{L}(\mathscr{H}')$ defined by $(A_0 \psi)_\omega = A(\psi_\omega)$ for all $\psi \in \mathscr{H}'$ and all $\omega \in Y_\infty$. The map r: $V' \to V$ is then defined by the equation

$$\operatorname{tr}[A r(\rho)] = \operatorname{tr}[A_0 \rho] \tag{4.5}$$

valid for all $A \in \mathscr{L}(\mathscr{H})$ and $\rho \in V'$. The verification of the required formulae is immediate. We observe that if $\psi \in \mathscr{H}'$ then

$$r(\psi \otimes \psi^-) = \int_Y \psi_\omega \otimes \psi_\omega^- \, \mu_\infty(d\omega). \tag{4.6}$$

Theorem 5. *There exists a one-parameter isometric semigroup G_t on $\mathscr{H}'$ such that*

$$T_t(\rho) = r\{G_t(e\rho)\, G_t^*\} \tag{4.7}$$

for all $\rho \in V$ and $t \geqq 0$.

Definition. We call such a semigroup an *isometric dilation* of T_t.

Proof. Let $t \geqq 0$ be given and let $\omega \in Y_\infty$ be the sequence $\omega = (x_i, t_i)_{i=1}^n$. Let m be the first index with $t_{m+1} > t$ and let $\omega_1 \in Y_\infty$, $\omega_2 \in X_t$ be the sequences $\omega_1 = (x_i, t_i - t)_{i=m+1}^n$ and $\omega_2 = (x_i, t_i)_{i=1}^m$. We then define $G_t \psi \in \mathscr{H}'$ for any $\psi \in \mathscr{H}'$ by

$$(G_t \psi)_\omega = B_{t_1} A_{x_1} \dots B_{t_m - t_{m-1}} A_{x_m} B_{t - t_m} \psi_{\omega_1}. \tag{4.8}$$

We calculate the norm of $G_t \psi$ by using Eq. (4.13) of [3] together with the observation that the measure μ_∞ is the product of the measures μ_∞ and μ_t under the Borel isomorphism of Y_∞ with $Y_\infty \times X_t$ defined in [3].

$$\begin{aligned}
\int_{Y_\infty} \|(G_t \psi)_\omega\|^2 \mu_\infty(d\omega) &= \int_{Y_\infty} \int_{X_t} \operatorname{tr}[\{B_{t_1} A_{x_1} \dots B_{t-t_m} \psi_{\omega_1}\} \otimes \{B_{t_1} A_{x_1} \dots B_{t-t_m} \psi_{\omega_1}\}^-] \\
&\quad \cdot \mu_t(d\omega_2)\, \mu_\infty(d\omega_1) \\
&= \int_{Y_\infty} \operatorname{tr}[\mathscr{E}_t(X_t, \psi_{\omega_1} \otimes \psi_{\omega_1}^-)]\, \mu_\infty(d\omega_1) \\
&= \int_{Y_\infty} \|\psi_{\omega_1}\|^2 \mu_\infty(d\omega_1) = \|\psi\|^2.
\end{aligned}$$

Therefore $G_t \psi \in L^2(Y_\infty, \mathscr{H}, \mu_\infty)$ and $\|G_t \psi\| = \|\psi\|$. The verification that G_t is a strongly continuous semigroup on $\mathscr{H}'$ is now straightforward.

We now have to verify Eq. (4.7) and it is sufficient to do this for pure states ρ since both sides of the equation are continuous and linear in ρ. But if $\psi \in \mathscr{H}$ and $\rho = \psi \otimes \psi^-$ then

$$\begin{aligned} r\{G_t(e\,\rho)\,G_t^*\} &= r\{(G_t\,\psi^0) \otimes (G_t\,\psi^0)^-\} \\ &= \int_{X_t} \{B_{t_1}\,A_{x_1} \ldots B_{t-t_m}\,\psi\} \otimes \{B_{t_1}\,A_{x_1} \ldots B_{t-t_m}\,\psi\}^- \,\mu_t(d\omega_2) \\ &= T_t(\psi \otimes \psi^-) \end{aligned}$$

by Eq. (4.6), and Eq. (4.13) of [3].

We comment again that the isometric semigroup G_t can be replaced by a one-parameter unitary group, by [9]. Secondly $\mathscr{H}'$ can be regarded as the tensor product of $\mathscr{H}$ and the Fock space over $L^2(X \times (0, \infty))$, except that it is not specified whether the Fock space should be of boson or fermion type.

§ 5. Subordinated Quantum Processes

Let G denote the additive group of the real line and Γ its dual group, also isomorphic to the real line, with the coupling $\langle \gamma, g \rangle = e^{-i\gamma g}$. Let μ_t, $t \geqq 0$ be a weakly continuous one-parameter semigroup of probability measures on G and let π be a unitary representation of G on a separable Hilbert space $\mathscr{H}$ with state space V. Then the equation

$$T_t(\rho) = \int_G \pi_g\,\rho\,\pi_g^*\,\mu_t(dg) \tag{5.1}$$

defines a strongly continuous one-parameter semigroup of contractions on V. This semigroup preserves positivity and trace. The equation is an obvious modification of the formula in classical probability theory for subordinating one stochastic process to another by randomising the evolution parameter [11]. In this section we show that T_t is the semigroup of a quantum stochastic process and obtain an explicit equation for an isometric dilation of T_t. We start with the associated process, obtain its isometric dilation and only prove that T_t is the semigroup of the process at the end of the argument. This could alternatively be shown much earlier by examination of the infinitesimal generators but it would involve some rather delicate domain questions which we prefer to avoid.

The arguments we present below can be generalised to any locally compact abelian group G by using the results in [12] but we have chosen to avoid the extra complication of statement necessary to do so.

We first summarise the Lévy-Khintchine formula for the semigroup μ_t in the Hilbert space valued cocycle terminology of [12]. The semigroup μ_t is determined by a continuous character σ: $\Gamma \to \mathbb{C}$ and a measure μ on G such that $\mu(e) = 1$ and

$$\int_G |\langle \gamma, g \rangle - 1|^2\,\mu(dg) < \infty \tag{5.2}$$

for all $\gamma \in \Gamma$, in the following manner. Let $\mathscr{K}$ be the Hilbert space $L^2(G, \mu)$ and define the unitary representation U of Γ on $\mathscr{K}$ by

$$(U_\gamma\,\psi)(g) = \langle \gamma, g \rangle\,\psi(g) \tag{5.3}$$

and the element $\delta_\gamma \in \mathscr{K}$ by

$$\delta_\gamma(g) = \begin{cases} \langle \gamma, g \rangle - 1 & \text{if } g \neq e \\ \sigma(\gamma) & \text{if } g = e. \end{cases} \tag{5.4}$$

Then δ is a continuous cocycle on $\mathscr{K}$ for the representation U, that is

$$\delta_{\gamma_1 + \gamma_2} = U_{\gamma_1} \delta_{\gamma_2} + \delta_{\gamma_1} \tag{5.5}$$

for all $\gamma_1, \gamma_2 \in \Gamma$. One sees immediately that

$$\|\delta_\gamma\|^2 = \int_G |\langle \gamma, g \rangle - 1|^2 \mu(dg) + |\sigma(\gamma)|^2. \tag{5.6}$$

There exist continuous real valued functions ϕ_1 and ϕ_2 on Γ such that if $\phi(\gamma) = \phi_1(\gamma) + i\,\phi_2(\gamma)$ then

$$\int_G \langle \gamma, g \rangle \mu_t(dg) = \exp[-t\,\phi(\gamma)] \tag{5.7}$$

for all $\gamma \in \Gamma$ and $t \geqq 0$. Moreover ϕ and δ are related by the equations

$$\langle \delta(\gamma_1), \delta(-\gamma_2) \rangle = -\phi(\gamma_1 + \gamma_2) + \phi(\gamma_1) + \phi(\gamma_2) \tag{5.8}$$

and

$$\phi_1(\gamma) = \tfrac{1}{2} \langle \delta(\gamma), \delta(\gamma) \rangle. \tag{5.9}$$

We work throughout with a spectral decomposition of $\mathscr{H}$ with respect to the representation π of G. Concretely suppose $\mathscr{H} = L^2(\Lambda)$ where

$$(\pi_g \psi)(\lambda) = \langle \gamma_\lambda, g \rangle \psi(\lambda) \tag{5.10}$$

for all $\psi \in L^2(\Lambda)$, where $\gamma\colon \Lambda \to \Gamma$ is a suitable measurable function – and where we can take $\Lambda \subseteq \Gamma$ if the representation is multiplicity free. For technical reasons we suppose until nearly the end of the section that $\gamma(\Lambda)$ has compact closure in Γ, so that the vectors $\{\delta(\gamma_\lambda)\colon \lambda \in \Lambda\}$ are uniformly bounded in norm.

Lemma 6. *The equation*

$$B_t \psi = \int_G \pi_g \psi \, \mu_t(dg) \tag{5.11}$$

defines a strongly continuous one-parameter contraction semigroup on $\mathscr{H}$ with infinitesimal generator $Z = iH - \frac{1}{2}R$ where

$$(R\psi)(\lambda) = \|\delta(\gamma_\lambda)\|^2 \psi(\lambda), \tag{5.12}$$

$$(H\psi)(\lambda) = -\phi_2(\gamma_\lambda) \psi(\lambda) \tag{5.13}$$

for all $\psi \in \mathscr{H}$ and almost every $\lambda \in \Lambda$.

Proof. All the statements are immediate consequences of the equations

$$\begin{aligned} \langle B_t \xi, \eta \rangle &= \int_G \langle \pi_g \xi, \eta \rangle \mu_t(dg) \\ &= \int_G \int_\Lambda \langle \gamma_\lambda, g \rangle \xi_\lambda \eta_\lambda^- \, d\lambda \, \mu_t(dg) \\ &= \int_\Lambda e^{-t\phi(\gamma_\lambda)} \xi_\lambda \eta_\lambda^- \, d\lambda \end{aligned} \tag{5.14}$$

valid for all $\xi, \eta \in \mathscr{H}$ and all $t \geqq 0$. Note that because of our technical assumption, R is a bounded operator.

Lemma 7. *If $S: \mathscr{H} \to \mathscr{H}$ is defined by*

$$(S\psi)(\lambda) = \sigma(\gamma_\lambda)\,\psi(\lambda) \tag{5.15}$$

then the equation

$$\mathscr{G}(E,\rho) = \int_G \chi_E(g)(\pi_g - 1)\,\rho\,(\pi_g - 1)^*\,\mu(dg) + \chi_E(e)\,S\,\rho\,S^* \tag{5.16}$$

where $E \subseteq G$ and $\rho \in V$, defines a bounded stochastic kernel with total interaction rate R.

Proof. Let $\rho \in V^+$ have spectral resolution

$$\rho = \sum_{n=1}^{\infty} \alpha_n\,\xi_n \otimes \xi_n^-$$

where $\langle \xi_m, \xi_n \rangle = \delta_{mn}$ and let

$$k = \sup\{\|\delta(\gamma_\lambda)\|^2 : \lambda \in \Lambda\} < \infty.$$

The following estimates establish both that $\mathscr{G}(E,\rho)$ is well defined as a trace-class operator and that its total interaction rate is R.

$$\begin{aligned}
\operatorname{tr}[\mathscr{G}(E,\rho)] &\leqq \operatorname{tr}[\mathscr{G}(G,\rho)] \\
&= \sum_{n=1}^{\infty} \alpha_n \Big\{\int_G \|\pi_g \xi_n - \xi_n\|^2\,\mu(dg) + \|S\xi_n\|^2\Big\} \\
&= \sum_{n=1}^{\infty} \alpha_n \Big\{\int_\Lambda \int_G |\langle \gamma_\lambda, g\rangle - 1|^2\,|\xi_n(\lambda)|^2\,\mu(dg)\,d\lambda + \int_\Lambda |\sigma(\gamma_\lambda)|^2\,|\xi_n(\lambda)|^2\,d\lambda\Big\} \\
&= \sum_{n=1}^{\infty} \alpha_n \Big\{\int_\Lambda \|\delta(\gamma_\lambda)\|^2\,|\xi_n(\lambda)|^2\,d\lambda\Big\} \\
&= \sum_{n=1}^{\infty} \alpha_n \langle R\,\xi_n, \xi_n\rangle \\
&= \operatorname{tr}[R\rho] \leqq k \operatorname{tr}[\rho] < \infty .
\end{aligned}$$

Since B_t and $\mathscr{G}$ are compatible there exists a quantum stochastic process $\mathscr{E}$ on G, V which has them as infinitesimal generators by Theorem 4.7 of [3]. We can then apply the work of the last section to obtain an isometric dilation of this process.

Theorem 8. *The stochastic process $\mathscr{E}$ has an isometric dilation G_t on $L^2(\Lambda, \mathscr{F})$ where $\mathscr{F}$ is the boson Fock space over $L^2(G \times (0,\infty))$. The isometric semigroup G_t is given by*

$$(G_t\psi)(\lambda) = G_{t,\lambda}\{\psi(\lambda)\} \tag{5.17}$$

for all $\psi \in \mathscr{F}$ where $G_{t,\lambda}$ is the canonical transformation on $\mathscr{F}$ such that

$$G_{t,\lambda}(\xi^\sim) = e^{-t\phi(\gamma_\lambda)}(g_{t,\lambda} + R_t\,\xi)^\sim \tag{5.18}$$

for all coherent states $\xi^\sim \in \mathscr{F}$, *where* $g_{t,\lambda} \in L^2(G\times(0,\infty))$ *is given by*

$$g_{t,\lambda} = \delta_{\gamma_\lambda} \otimes \chi[0,t] \tag{5.19}$$

and where R_t *is the isometric semigroup on* $L^2(0,\infty)$

$$(R_t\xi)(s) = \begin{cases} \xi(s-t) & \text{if } s \geqq t \\ 0 & \text{if } s < t. \end{cases} \tag{5.20}$$

Proof. Let $Z_t = \bigcup_{n=0}^{\infty} \{G\times(0,t)\}^n$ where each subset is given the measure obtained form μ on the G coordinates and Lebesgue measure on the time coordinates. Then $Y_\infty \subseteq Z_\infty$ and by symmetrisation we can identify $L^2(Y_\infty)$ with $L^2_{\text{sym}}(Z_\infty) = \mathscr{F}$. Therefore

$$\mathscr{H}' = L^2(Y_\infty, \mathscr{H}) \simeq L^2_{\text{sym}}(Z_\infty \times \Lambda) = L^2(\Lambda, \mathscr{F}). \tag{5.21}$$

If we put $A_x = \pi_x - 1$ if $x \neq e$ and $A_e = S$ then noting that

$$(A_x\psi)(\lambda) = \delta_{\gamma_\lambda}(x)\,\psi(\lambda) \tag{5.22}$$

for all $\psi \in L^2(\Lambda)$, and using the notation of Theorem 5 we obtain

$$\begin{aligned}(G_t\psi)(\omega,\lambda) &= B_{t_1}A_{x_1}\dots A_{x_m}B_{t-t_m}\psi(\omega_1,\lambda)\\ &= e^{-t\phi(\gamma_\lambda)}\,\delta_{\gamma_\lambda}(x_1)\dots\delta_{\gamma_\lambda}(x_m)\,\psi(\omega_1,\lambda)\end{aligned} \tag{5.23}$$

for all $\psi \in L^2(Y_\infty \times \Lambda)$. Symmetrising the wave functions gives a similar equation for all $\psi \in L^2_{\text{sym}}(Z_\infty \times \Lambda)$. Now the one parameter semigroup of isometries R_t on $L^2(0,\infty)$ induces a one-parameter semigroup of isometries, which we call $R_t^\sim$, on $L^2_{\text{sym}}(Z_\infty \times \Lambda)$. The isomorphism

$$L^2(0,\infty) \simeq L^2(0,t) \oplus R_t L^2(0,\infty) \tag{5.24}$$

induces a similar isomorphism

$$L^2_{\text{sym}}(Z_\infty \times \Lambda) \simeq L^2_{\text{sym}}(Z_t \times \Lambda) \otimes R_t^\sim\, L^2_{\text{sym}}(Z_\infty \times \Lambda) \tag{5.25}$$

which allows Eq. (5.23) to be rewritten as

$$(G_t\psi)(\lambda) = e^{-t\phi(\gamma_\lambda)} \sum_{m=0}^{\infty} \frac{1}{\sqrt{m!}} \otimes^m (\delta_{\gamma_\lambda} \otimes \chi_{[0,t]}) \otimes R_t^\sim\, \psi(\lambda)$$

so that

$$\begin{aligned}(G_t\xi^\sim)(\lambda) &= e^{-t\phi(\gamma_\lambda)} \left\{ \sum_{m=0}^{\infty} \frac{1}{\sqrt{m!}} \otimes^m g_{t,\lambda} \right\} \otimes (R_t^\sim\, \xi^\sim)\\ &= e^{-t\phi(\gamma_\lambda)} (g_{t,\lambda})^\sim \otimes (R_t\xi)^\sim\\ &= e^{-t\phi(\gamma_\lambda)} (g_{t,\lambda} + R_t\xi)^\sim .\end{aligned}$$

Theorem 9. *The semigroup* T_t *of the process* $\mathscr{E}_t$ *is given by the equation*

$$T_t(\rho) = \int_G \pi_g \rho \pi_g^* \mu_t(dg). \tag{5.26}$$

Proof. Let $\xi, \eta \in \mathscr{H}$ and let A be the operator on $L^2(\Lambda, \mathscr{F})$ given by

$$\langle A\phi, \psi\rangle = \int_\Lambda \int_\Lambda \langle \phi_\lambda, \psi_{\lambda'}\rangle \bar{\eta}_\lambda \eta_{\lambda'}\, d\lambda\, d\lambda'$$

so that $A = (\eta \otimes \bar{\eta}) \otimes 1$. Then by Eq. (4.7)

$$\begin{aligned}\langle T_t(\xi \otimes \bar{\xi})\eta, \eta\rangle &= \langle A\, G_t(\xi \otimes \Psi_0),\, G_t(\xi \otimes \Psi_0)\rangle \\ &= \int_\Lambda \int_\Lambda \xi_\lambda \bar{\xi}_{\lambda'} \bar{\eta}_\lambda \eta_{\lambda'}\, e^{-t\phi(\gamma_\lambda) - t\overline{\phi(\gamma_{\lambda'})}} \langle g_{t,\lambda}^{\sim}, g_{t,\lambda'}^{\sim}\rangle\, d\lambda\, d\lambda' \\ &= \int_\Lambda \int_\Lambda \xi_\lambda \bar{\xi}_{\lambda'} \bar{\eta}_\lambda \eta_{\lambda'} \exp[-t\,\phi(\gamma_\lambda) - t\,\overline{\phi(\gamma_{\lambda'})} + t\langle \delta_{\gamma_\lambda}, \delta_{\gamma_{\lambda'}}\rangle]\, d\lambda\, d\lambda' \\ &= \int_\Lambda \int_\Lambda \xi_\lambda \bar{\xi}_{\lambda'} \bar{\eta}_\lambda \eta_{\lambda'} \exp[-t\,\phi(\gamma_\lambda - \gamma_{\lambda'})]\, d\lambda\, d\lambda' \\ &= \int_\Lambda \int_\Lambda \int_G \xi_\lambda \bar{\xi}_{\lambda'} \bar{\eta}_\lambda \eta_{\lambda'} \langle \gamma_\lambda - \gamma_{\lambda'}, g\rangle\, \mu_t(dg)\, d\lambda\, d\lambda' \\ &= \int_G \langle \pi_g \xi, \eta\rangle \langle \pi_g \xi, \eta\rangle^-\, \mu_t(dg) \\ &= \int_G \langle \pi_g(\xi \otimes \bar{\xi})\pi_g^* \eta, \eta\rangle\, \mu_t(dg).\end{aligned}$$

We have now shown that

$$\mathrm{tr}[T_t(\rho) B] = \int_G \mathrm{tr}[\pi_g \rho \pi_g^* B]\, \mu_t(dg)$$

whenever $\rho = \xi \otimes \bar{\xi}$ and $B = \eta \otimes \bar{\eta}$. The general result now follows by the usual linearity arguments.

We conclude by indicating how one would prove Theorems 8 and 9 without the assumption we made that $\gamma(\Lambda)$ has compact closure in Γ. One takes $\Lambda = \bigcup_{n=1}^{\infty} \Lambda_n$ where Λ_n form an increasing sequence of Borel sets such that $\gamma(\Lambda_n)$ have compact closures in Γ. The arguments of this section then apply to each of the increasing sequence of Hilbert spaces $\mathscr{H}_n = L^2(\Lambda_n)$. The required results then follow by taking inductive limits, making use in particular of Theorem 5.3 of [3] to obtain a quantum stochastic process with unbounded interaction rate on V as an inductive limit of processes with bounded interaction rates on $V_n = \tau_s(\mathscr{H}_n)$ for $n = 1, 2, \ldots$.

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