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Idempotency of the Hull-Formation $\mathbf{H}^\gamma$

By

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Suppose E is a locally convex complete Hausdorff linear space (over the real field R) and F is the space of all continuous linear functionals on E. For $S \subset E$, S_F^σ will denote the smallest σ-algebra of subsets of S which includes all sets of the form $\{s \in S : \alpha < fs < \beta\}$ (for $f \in F$, $\alpha, \beta \in R$), and γS will denote the set of all probability measures defined over S_F^σ. For $\mu \in \gamma S$, the *barycenter* of μ (if one exists) is a point b_μ of E such that $f b_\mu = \int f_S \, d\mu$ for all $f \in F$; the set of all such barycenters will be denoted by $H^\gamma S$. The hull-formation H^γ is closely related to the operations of forming the convex hull *con* S and the closed convex hull *cl con* S; in particular, *con* $S \subset H^\gamma S \subset$ *cl con* S. It is known that $H^\gamma S =$ *con* S when E is finite-dimensional (RICHTER [*9*], RUBIN-WESLER [*10*], BONNICE-KLEE [*2*]), and $H^\gamma S =$ *cl con* S when S is weakly compact (BOURBAKI [*3*], CHOQUET [*5*], HEWITT-SAVAGE [*6*], BONNICE-KLEE [*2*]), so $H^\gamma(H^\gamma S) = H^\gamma S$ in both of these cases. On the other hand, various separable Banach spaces E contain unbounded countable closed sets S for which $H^\gamma(H^\gamma S) \neq H^\gamma S$ [*2*]. In [*2*] it is conjectured that $H^\gamma(H^\gamma S) = H^\gamma S$ whenever S is a bounded Borel set in a separable Banach space E. Though still unsettled, the conjecture is verified here under the supplementary hypothesis that the space $F(= E^*)$ is also separable in its norm topology. This applies to all separable reflexive spaces, and in particular to Hilbert space.

For each topological space X, $\mathscr{B}X$ will denote the set of all *Borelian subsets* of X (i.e., the σ-algebra generated by the open sets), $\mathscr{M}^p X$ the set of all probability measures defined over $\mathscr{B}X$, and $\mathscr{M}^+ X$ the set of all measures of the form $\alpha\mu$ for $\alpha > 0$ and $\mu \in \mathscr{M}^p X$. CX will denote the Banach space of all bounded continuous real-valued functions on X. For each $\mu \in \mathscr{M}^+ X$, $\xi_\mu \in (CX)^*$ is defined as follows: $\xi_\mu \varphi = \int \varphi \, d\mu$ for all $\varphi \in CX$. There is a unique topology on $\mathscr{M}^+ X$ for which ξ is a homeomorphism into the space $(CX)^*$ in its weak* topology $\sigma((CX)^*, CX)$ (see BOURBAKI [*4*] for notation). This topology, the *weak topology* of $\mathscr{M}^+ X$, has been studied by several authors, recently by VARADARAJAN [*11, 12, 13*]. A net (μ_n) in $\mathscr{M}^+ X$ is weakly convergent to $\mu \in \mathscr{M}^+ X$ iff $(\int \varphi \, d\mu_n)$ converges to $\int \varphi \, d\mu$ for all $\varphi \in CX$. For metrizable X, other convergence criteria have been given by BILLINGSLEY [*1*].

Proposition 1. *Suppose M is a metrizable space, S is a Borelian subset of M, and J is a Borelian subset of R. Then the set $\{\mu \in \mathscr{M}^+ M : \mu S \in J\}$ is a Borelian subset of $\mathscr{M}^+ M$.*

Proof. It suffices to show that $\mathscr{A} = \mathscr{B}M$, where $\mathscr{A}$ is the set of all $A \in \mathscr{B}M$ such that for each $\alpha \in R$ the four sets

$$\{\mu : \mu A < \alpha\}, \quad \{\mu : \mu A \leqq \alpha\}, \quad \{\mu : \mu A \geqq \alpha\}, \quad \text{and} \quad \{\mu : \mu A > \alpha\}$$

are all Borelian subsets of $\mathscr{M}^+ M$. Since $\mathscr{A}$ is obviously closed under complementation, to show that $\mathscr{A} = \mathscr{B}M$ it suffices to prove that $\mathscr{A}$ includes all open sets and includes the union of each increasing sequence of members of $\mathscr{A}$.

Let G be an open subset of M. It is known (BILLINGSLEY [1], p. 252) that if a net μ_n in $\mathscr{M}^+M$ is weakly convergent to $\mu \in \mathscr{M}^+M$, then $\liminf \mu_n G \geqq \mu G$. Thus for each $\alpha \in R$ the set $\{\mu : \mu G \leqq \alpha\}$ is closed in $\mathscr{M}^+M$. Moreover,

$$\{\mu : \mu G < \alpha\} = \bigcup_{k=1}^{\infty} \{\mu : \mu G \leqq \alpha - 1/k\},$$

an F_σ set in $\mathscr{M}^+M$, and by complementation the sets $\{\mu : \mu G > \alpha\}$ and $\{\mu : \mu G \geqq \alpha\}$ are also Borelian subsets of $\mathscr{M}^+M$. Hence $\mathscr{A}$ includes all open subsets of M.

Finally, consider an increasing sequence $(A_i)_1^\infty$ of members of $\mathscr{A}$, and let $A = \bigcup_{i=1}^{\infty} A_i$. Then

$$\{\mu : \mu A \leqq \alpha\} = \bigcap_{i=1}^{\infty} \{\mu : \mu A_i \leqq \alpha\},$$

a Borelian subset of $\mathscr{M}^+M$, and the subsets

$$\{\mu : \mu A < \alpha\}, \quad \{\mu : \mu A > \alpha\}, \quad \text{and} \quad \{\mu : \mu A \geqq \alpha\}$$

are also Borelian by reasoning similar to that of the preceding paragraph. We conclude that $\mathscr{A} = \mathscr{B}M$. $\square$

We have spoken of *Borelian subsets* but not yet of Borel sets as the latter term will be used in a more absolute sense. Specifically, a *Borel set* is a homeomorph of a Borelian subset of the Hilbert cube and an *analytic* set is a metrizable space which is a continuous image of a Borel set. The necessary background material on Borel sets and analytic sets may be found in KURATOWSKI [7] and MACKEY [8]. We shall use some of this material without specific reference, especially the equivalence of the following three properties of a separable metrizable space S: S is a Borel set; S is homeomorphic with a Borelian subset of some complete metric space; S is a Borelian subset of every metric space in which it is topologically embedded.

The following result is related to one of VARADARAJAN [11; Theorem 3.5].

Proposition 2. *If S is a Borel set, so are $\mathscr{M}^+S$ and $\mathscr{M}^pS$.*

Proof. Since $\mathscr{M}^pS$ is closed in $\mathscr{M}^+S$, it suffices to consider $\mathscr{M}^+S$. We assume without loss of generality that S is a Borelian subset of the Hilbert cube Q. By a criterion of BILLINGSLEY [1; p. 252], a net (μ_n) in $\mathscr{M}^+S$ is weakly convergent to $\mu \in \mathscr{M}^+S$ iff $(\int \varphi \, d\mu_n)$ converges to $\int \varphi \, d\mu$ for all $\varphi \in US$, the space of all bounded *uniformly* continuous real-valued functions on S. But of course the members of US can be extended to members of $UQ = CQ$, and it follows that $\mathscr{M}^+S$ is homeomorphic with the set $\{\mu \in \mathscr{M}^+Q : \mu(Q \sim S) = 0\}$. Since $\mathscr{M}^+Q$ is topologically complete by [11; Theorem 3.4] and hence is a Borel set, the desired conclusion follows from Proposition 1. $\square$

Theorem. *If E is a Banach space whose conjugate space F is (norm-)separable, then $H^\gamma(H^\gamma S) = H^\gamma S$ for each bounded Borel set S in E.*

Proof. The subscript $_n$ will indicate the norm topology for subsets of E or F. Let $G = (F_n)^* = E^{**}$, the second conjugate of E_n, and let η denote the canonical embedding of E in G; i.e.,

$$\eta_x f = fx \quad \text{for all } x \in E, \quad f \in F.$$

There is a similar canonical embedding of F in the algebraic conjugate $G\#$ of G, by means of which the members of F may be regarded alternatively as linear functionals on E or on G. We shall make implicit use of this embedding though without any special notation for it. The space G will always be equipped here with its weak* topology $w = \sigma(G, F)$, and we use w also to denote the weak topology $\sigma(E, F)$ *in* E.

Let $m = \sup\{\|x\| : x \in S\}$ and $W = \{g \in G : \|g\| \leqq m\}$. Since F is assumed to be norm-separable, W_w is compact and metrizable (e.g., BOURBAKI [4; p. 66]). The restriction η_S is a continuous biunique mapping of the Borel set S_n into the Borel set W_w, and hence $(\eta S)_w$ is a Borel set [7; p. 396]. Since η_S is a homeomorphism of S_w onto $(\eta S)_w$, S_w is a Borel set and $\gamma S = \mathscr{M}^p S_w$. By Proposition 2, γS is a Borel set under the weak topology of measures.

For each measure $\mu \in \gamma S$, $\mu\eta^{-1}$ is a measure on the σ-algebra $\mathscr{B}(\eta S)_w = (\eta S)^\sigma_F$, and since ηS lies in the compact convex set W, it follows from a theorem of BOURBAKI [3; p. 81] (see also [2; Theorem 4.3]) that $\mu\eta^{-1}$ admits a barycenter $\zeta_\mu \in W$. (This is a point of W such that for all $f \in F$, $\zeta_\mu(f) = \int f_S\, d\mu$.) Note that the transformation $\mu \to \zeta_\mu$ of γS into W_w is continuous. (This is immediate from the relevant definitions in conjuction with the fact that $f_S \in C S_w$ for each $f \in F$.) This continuity implies analyticity of the set $\zeta(\gamma S)$ of all barycenters in W of measures $\mu\eta^{-1}$ for $\mu \in \gamma S$. Note further that the set $(\eta E) \cap W$, being a continuous biunique image of the Borel set $\{x \in E : \|x\| \leqq m\}_n$, must itself be a Borel set, whence of course the set $\zeta(\gamma S) \cap \eta E$ is analytic and the same is true of the set $\{\mu \in \gamma S : \zeta_\mu \in \eta E\}$ [7; p. 361]. In short, we have proved that with respect to the weak topology $w = \sigma(E, F)$ in E and the weak topology of measures in γS ($= \mathscr{M}^p S_w$), the set $T = H^\gamma S$ is analytic as is also the set δS of all measures in γS which admit barycenters in E. For each $\mu \in \delta S$, let $b(\mu) = \eta^{-1}\zeta_\mu$, the barycenter of μ; then b is a continuous mapping of δS onto T_w.

We want to prove that $H^\gamma T \subset T$. Consider an arbitrary point $p \in H^\gamma T$, p being the barycenter of a probability measure ν defined over T^σ_F ($= \mathscr{B} T_w$). Define

$$A = \{(t, \mu) : \mu \in \delta S, t = b(\mu)\} \subset T \times \delta S .$$

Since b is continuous, A is a Borelian subset of $T \times \delta S$ [7; p. 291]. Now in the terminology of MACKEY [8], ν is a finite Borel measure in the analytic Borel space T_w and hence ν is standard [8; Theorem 6.1]. By the reasoning of [8; Theorem 6.3] (which remains valid when its S_2 is an analytic Borel space rather than a standard one), there exist a set $U \subset T$, a function ψ on U to δS, and a measure $\tau \in \gamma U$ such that

U_w is a Borel set;
for each $u \in U$, $(u, \psi u) \in A$; i.e., $b(\psi u) = u$;
for each $Y \in T^\sigma_F$, $\nu Y = \tau(Y \cap U)$;
for each Borelian subset D of δS, $\psi^{-1} D$ is a Borel set in U_w.

Then of course p is the barycenter of τ, and we will show that p is also the barycenter of a measure $\varrho \in \gamma S$, thus completing the proof.

Note that the function $\psi u \,|\, u \in U$ is Borel measurable by the last statement displayed above, and for each $X \in S^\sigma_F$ the function $\mu(X) \,|\, \mu \in \delta X$ is Borel measurable by Proposition 1. Thus the composition of these two functions is also Borel measurable, and we may define

$$\varrho(X) = \int_{u\in U} (\psi u)(X)\, d\tau(u) \quad \text{for all} \quad X \in S_F^\sigma .$$

We claim $\varrho \in \gamma S$, for which it suffices to check countable additivity. Suppose $(X_i)_1^\infty$ is a sequence of pairwise disjoint members of S_F^σ. Then

$$\varrho\left(\bigcup_{i=1}^\infty X_i\right) = \int_{u\in U} (\psi u)\left(\bigcup_1^\infty X_i\right) d\tau(u) = \int_{u\in U} \left(\sum_{i=1}^\infty (\psi u)(X_i)\right) d\tau(u)$$

$$= \sum_{i=1}^\infty \left(\int_{u\in U} (\psi u)(X_i)\, d\tau(u)\right) = \sum_{i=1}^\infty \varrho(X_i),$$

where the crucial interchange is justified by Fubini's theorem.

With ϱ countably additive, it remains only to check that p is the barycenter of ϱ, i.e., that $\int f_S\, d\varrho = f_p$ for all $f \in F$. Consider an arbitrary partition of S into pairwise disjoint sets $X_1, \ldots, X_n \in S_F^\sigma$, and for each i let $\alpha_i = \inf f X$, $\beta_i = \sup f X$. Then

$$fp = \int_{u\in U} fu\, d\tau(u) = \int_{u\in U} \left(\int_{s\in S} fs\, d(\psi u)(s)\right) d\tau(u)$$

$$\leqq \int_{u\in U} \left(\sum_1^n \beta_i (\psi u)(X_i)\right) d\tau(u) = \sum_1^n \beta_i\, \varrho(X_i),$$

and similarly

$$fp \geqq \sum_1^n \alpha_i\, \varrho(X_i).$$

But then fp and $\int f_S\, d\varrho$ lie together in the interval

$$\left[\sum_1^n \alpha_i\, \varrho(X_i),\ \sum_1^n \beta_i\, \varrho(X_i)\right],$$

and the length of this interval can be made arbitrarily small for appropriate choice of the X_i's. ▯

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