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Extremes and Local Dependence in Stationary Sequences

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Summary. Extensions of classical extreme value theory to apply to stationary sequences generally make use of two types of dependence restriction:

(a) a weak "mixing condition" restricting long range dependence

(b) a local condition restricting the "clustering" of high level exceedances.

The purpose of this paper is to investigate extremal properties when the local condition (b) is omitted. It is found that, under general conditions, the type of the limiting distribution for maxima is unaltered. The precise modifications and the degree of clustering of high level exceedances are found to be largely described by a parameter here called the "extremal index" of the sequence.

1. Introduction

Classical Extreme Value Theory discusses the possible limiting laws for the maximum

$$M_n = \max(\xi_1, \xi_2 \dots \xi_n) \tag{1.1}$$

of n independent identically distributed (i.i.d.) random variables (r.v.) as $n \to \infty$. Specifically it is shown that if M_n has a non-degenerate limiting distribution G i.e. if

$$P\{a_n(M_n - b_n) \leqq x\} \xrightarrow{d} G(x) \tag{1.2}$$

for some constants $a_n > 0$, b_n then G must be one of the following classical types (in the sense that $G(x) = H(ax+b)$ for some $a > 0, b$ where H is one of the listed distributions):

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$$
\begin{array}{llll}
\text{Type I} & H(x)=\exp(-e^{-x}) & -\infty<x<\infty & \\
\text{Type II} & H(x)=\exp(-x^{-\alpha}) & x>0 & (\alpha>0) \\
 & \quad =0 & x\leqq 0 & \\
\text{Type III} & H(x)=\exp(-(-x)^{\alpha}) & x\leqq 0 & (\alpha>0) \\
 & \quad =1 & x>0 &
\end{array}
$$

It may be shown that this result remains true (cf. [8, 9]) if the condition that the ξ_i be i.i.d. is replaced by the requirement that they form a stationary sequence satisfying a very weak dependence restriction. This restriction, here referred to as the *distributional mixing condition* $D(u_n)$ is defined as follows.

Write $F_{i_1\ldots i_n}(x_1\ldots x_n)=P\{\xi_{i_1}\leqq x_1\ldots \xi_{i_n}\leqq x_n\}$ for the joint distribution function of $\xi_{i_1}\ldots\xi_{i_n}$, and, for brevity, $F_{i_1\ldots i_n}(u)=F_{i_1\ldots i_n}(u,u\ldots u)$ for each $n, i_1\ldots i_n, u$. Let $\{u_n\}$ be a sequence of constants. Then the sequence $\{\xi_n\}$ is said to satisfy $D(u_n)$ if for each n, l and each choice of integers $i_1\ldots i_p, j_1\ldots j_{p'}$ such that

$$1\leqq i_1<i_2\ldots<i_p<j_1\ldots<j_{p'}\leqq n, \qquad j_1-i_p\geqq l$$

we have

$$|F_{i_1\ldots i_p j_1\ldots j_{p'}}(u_n)-F_{i_1\ldots i_p}(u_n)\,F_{j_1\ldots j_{p'}}(u_n)|<\alpha_{n,l}$$

where $\alpha_{n,l_n}\to 0$ as $n\to\infty$ for some sequence $\{l_n\}$ with $l_n=o(n)$.

In spite of the slightly complicated definition this condition is clearly much weaker than the standard forms of mixing condition (such as strong mixing) in that it requires only approximate independence of events A "from the past" and B "from the future" having the special, simple forms

$$A=\bigcap_{r=1}^{p}\{\xi_{i_r}\leqq u_n\}, \qquad B=\bigcap_{s=1}^{p'}(\xi_{j_s}\leqq u_n).$$

The specific form of the theorem referred to above (proved in [8, 9]), is as follows.

Theorem 1.1. *Let $\{\xi_n\}$ be a stationary sequence such that $M_n=\max\{\xi_1\ldots\xi_n\}$ has a non-degenerate limiting distribution G as in (1.2) for some constants $a_n>0, b_n$. Suppose that $D(u_n)$ holds for all sequences u_n given by $u_n=x/a_n+b_n$; $-\infty<x<\infty$. Then G is one of the three classical types given above.*

Thus the condition $D(u_n)$ alone is sufficient to guarantee that the central classical result concerning the possible extremal types, holds also for stationary sequences.

It is also shown in [8] that if a further condition holds – there called $D'(u_n)$, viz.

$$D'(u_n):\quad \limsup_{n\to\infty} n\sum_{j=2}^{[n/k]} P\{\xi_1>u_n, \xi_j>u_n\}\to 0 \quad \text{as } k\to\infty \tag{1.3}$$

(for each $u_n=x/a_n+b_n$), then the particular type which applies is the same as if the sequence $\{\xi_n\}$ were i.i.d. with the same marginal distribution function (d.f.) F, and the same normalizing constants may be used. In particular this means that the classical criteria for domains of attraction (cf. [9]) may be used to determine (on the basis of the behavior of the tail $1-F(x)$ for large x) which limiting law applies. These assertions result from making appropriate identifications (e.g. $u_n = x/a_n + b_n$) in the following theorem (cf. [9]) which generalizes a simple classical result.

Theorem 1.2. *Let $\{\xi_n\}$ be a stationary sequence (marginal d.f. F) and $\{u_n\}$ a sequence of constants such that $D(u_n)$, $D'(u_n)$ hold. Let $0 \leqq \tau < \infty$. Then*

$$P\{M_n \leqq u_n\} \to e^{-\tau} \tag{1.4}$$

if and only if

$$n[1-F(u_n)] \to \tau. \tag{1.5}$$

Conditions similar to $D'(u_n)$ have been used in virtually all studies of extremes of dependent sequences beginning with the early works of Watson [16] and Loynes [10] who showed in particular that (1.5) implies (1.4), using stronger dependence restrictions that $D(u_n)$. However since Theorem 1.1 does not require $D'(u_n)$ in limiting the extremal distributions to the classical types, it seems worthwhile to investigate the precise role of conditions of this kind.

It has in fact been shown by Chernick [3] (extending a result of Loynes [10]) that if for each $\tau > 0$, $u_n = u_n(\tau)$ is defined to satisfy (1.5), then under $D(u_n)$ conditions alone, any limit (function) for $P\{M_n \leqq u_n(\tau)\}$ must be of the form

$$P\{M_n \leqq u_n(\tau)\} \to e^{-\theta\tau} \tag{1.6}$$

for some θ with $0 \leqq \theta \leqq 1$.

In the present paper we extend this result in various ways. It will then follow, as a consequence, that in virtually all cases of practical interest the condition $D(u_n)$ alone is sufficient to guarantee that any asymptotic distribution for the maximum M_n is of precisely the same type as if the sequence $\{\xi_n\}$ were i.i.d. with the same marginal d.f. F. In fact the only essential difference which appears in dropping the assumption $D'(u_n)$ is that the normalizing constants in (1.2) may have to be modified from those applying to the i.i.d. case. In obtaining these results we use some ideas from O'Brien ([13, 14]).

The parameter θ in (1.6) is here (as in [9]) called the *extremal index* of the sequence $\{\xi_n\}$. The main results concerning its existence are given in Sect. 2, with particular criteria and examples cited in Sect. 3. In Sect. 4 we look briefly at the role of $D'(u_n)$ in obtaining a Poisson limit for the (time-normalized) point process of exceedances of the level u_n by the ξ_j's. When $D'(u_n)$ does not hold, the exceedances of u_n can occur in clusters, leading to multiple points in a limiting point process. As will be seen from Sect. 4 the degree of clustering is directly related to the extremal index θ.

2. Extremal Results Under $D(u_n)$

The basic technique of [8] for extending extremal theory to stationary cases is to show that

$$P\{M_n \leqq u_n\} - P^k\{M_{r_n} \leqq u_n\} \to 0 \tag{2.1}$$

for each $k=1,2\ldots$ when $D(u_n)$ holds, where $r_n=[n/k]$ (the integer part of n/k). This clearly simply reflects a form of approximate independence of the sub-maxima in the k subsets of $[n/k]=r_n$ integers $(1,2\ldots r_n)$, $(r_n+1\ldots 2r_n)\ldots$ which together comprise essentially all of $(1,2\ldots n)$. Here we obtain a somewhat more general version of this result. The notation $M(E)$ will be used (here and subsequently) to denote the maximum of ξ_j for j in the set E of integers.

Lemma 2.1. *Let $\{u_n\}$ be a sequence of constants and let $D(u_n)$ be satisfied by the stationary sequence $\{\xi_n\}$. Let $\{k_n\}$ be a sequence of constants such that $k_n=o(n)$ and, in the notation used in stating $D(u_n)$, $k_n l_n=o(n)$, $k_n\alpha_{n,l_n}\to 0$. Then*

$$P\{M_n\leqq u_n\}-P^{k_n}\{M_{r_n}\leqq u_n\}\to 0 \quad \text{as } n\to\infty \tag{2.2}$$

where $r_n=[n/k_n]$.

Proof. This will be sketched only since it is analogous to the proof of (2.1) given e.g. in [9]. We shall also assume that $n[1-F(u_n)]$ is bounded, which is not necessary (cf. [8]) but simplifying (and holds via (1.5) in the applications to be made).

Let $\{l_n\}$ be as in the definition of $D(u_n)$. Divide the integers $1\ldots n$ into intervals (i.e. sets of consecutive integers) $I_1, I_1^*, I_2, I_2^*\ldots I_{k_n}, I_{k_n}^*$ where

$$\begin{aligned}
&I_1=(1,2\ldots r_n-l_n),\quad I_1^*=(r_n-l_n+1\ldots r_n),\quad I_2=(r_n+1\ldots 2r_n-l_n),\\
&I_2^*=(2r_n-l_n+1\ldots 2r_n)\ldots I_{k_n}=((k_n-1)r_n+1\ldots k_n r_n-l_n),\\
&I_{k_n}^*=(k_n r_n-l_n+1\ldots n).
\end{aligned}$$

Thus each interval I_j contains r_n-l_n integers, with each I_j^* except I_{k_n} having l_n integers, and I_{k_n} having $n-k_n r_n+l_n\leqq k_n+l_n$ (since $r_n=[n/k_n]$). It is readily seen that

$$\begin{aligned}
0&\leqq P\left(\bigcap_{j=1}^{k_n}\{M(I_j)\leqq u_n\}\right)-P\{M_n\leqq u_n\}\\
&\leqq(k_n-1)P\{M(I_1^*)>u_n\}+P\{M(I_{k_n}^*)>u_n\}\\
&\leqq[(k_n-1)l_n+(k_n+l_n)]P\{\xi_1>u_n\}\\
&\leqq K\frac{k_n(l_n+1)}{n}\to 0 \quad \text{as } n\to\infty
\end{aligned} \tag{2.3}$$

by virtue of the stated assumptions (K being a constant).

It follows from $D(u_n)$ by a straightforward induction (cf. [8, Lemma 2.3]) that

$$\left|P\left\{\bigcap_{j=1}^{k_n}(M(I_j)\leqq u_n)\right\}-P^{k_n}\{M(I_1)\leqq u_n\}\right|\leqq k_n\alpha_{n,l_n} \tag{2.4}$$

which tends to zero by assumption. Finally it is readily checked that

$$\begin{aligned}
&|P^{k_n}\{M(I_1)\leqq u_n\}-P^{k_n}\{M_{r_n}\leqq u_n\}|\\
&\quad\leqq k_n[P\{M(I_1)\leqq u_n\}-P\{M_{r_n}\leqq u_n\}]=k_n P\{M(I_1)<u_n\leqq M(I_1^*)\}\\
&\quad\leqq k_n l_n P\{\xi_1>u_n\}\leqq K k_n l_n/n\to 0.
\end{aligned} \tag{2.5}$$

The result now follows at once by combining (2.3), (2.4) and (2.5). □

We suppose from now on that for each $\tau > 0$ a sequence $\{u_n(\tau)\}$ is defined to satisfy (1.5), viz.

$$n[1-F(u_n(\tau))] \to \tau \tag{2.6}$$

This imposes a slight restriction on the marginal d.f. F of the ξ_n, but one which will always be satisfied in the applications made. Of course if F is continuous, $u_n(\tau)$ can be defined to give equality in (2.6). In any case it is necessary and sufficient for (2.6) to hold that

$$[1-F(x-)]/[1-F(x)] \to 1 \quad \text{as } x \to \infty \tag{2.7}$$

(cf. [9]), a condition which always holds for any F in any of the three classical domains of attraction. It is also evident that if there exists $u_n(\tau)$ satisfying (2.6) for one fixed $\tau > 0$, then there exists such a $u_n(\tau)$ for all $\tau > 0$ (e.g. if $u_n(1)$ satisfies (2.6) with $\tau = 1$, define $u_n(\tau) = u_{[n/\tau]}(1)$).

The following result extends Theorem 1.2, and Theorem 1 of O'Brien [14].

Theorem 2.2. *Let $\{\xi_n\}$ be a stationary sequence and $\{u_n(\tau)\}$ constants satisfying* (2.6) *and such that $D(u_n(\tau_0))$ holds for some $\tau_0 > 0$. Then there exist constants θ, θ', $0 \leqq \theta \leqq \theta' \leqq 1$ such that*

$$\begin{aligned} \limsup_{n\to\infty} P\{M_n \leqq u_n(\tau)\} &= e^{-\theta\tau} \\ \liminf_{n\to\infty} P(M_n \leqq u_n(\tau)) &= e^{-\theta'\tau} \end{aligned} \tag{2.8}$$

for $0 < \tau \leqq \tau_0$. Hence if $P\{M_n \leqq u_n(\tau)\}$ converges for some τ, $0 < \tau \leqq \tau_0$, then $\theta = \theta'$ and $P\{M_n \leqq u_n(\tau)\} \to e^{-\theta\tau}$ for all such τ.

Proof. Note first that it is readily shown (cf. [9]) that $D(u_n(\tau))$ holds for $0 < \tau \leqq \tau_0$ since it holds for $\tau = \tau_0$. Write $\psi(\tau) = \limsup_{n\to\infty} P\{M_n \leqq u_n(\tau)\}$ and let k be a fixed integer. Then it follows from Lemma 2.1 with $k_n = k$ that

$$\limsup_{n\to\infty} P\{M_{n'} \leqq u_n(\tau)\} = \psi^{1/k}(\tau) \tag{2.9}$$

where $n' = [n/k]$. Now if $u_n(\tau) \geqq u_{n'}(\tau/k)$ it follows that

$$\begin{aligned} 0 \leqq P\{M_{n'} \leqq u_n(\tau)\} - P\{M_{n'} \leqq u_{n'}(\tau/k)\} &\leqq P\left\{\bigcup_{j=1}^{n'} (u_{n'}(\tau/k) < \xi_j \leqq u_n(\tau))\right\} \\ &\leqq n'[F(u_n(\tau)) - F(u_{n'}(\tau/k))]. \end{aligned}$$

This together with the corresponding inequality when $u_n(\tau) < u_{n'}(\tau/k)$ show that

$$\begin{aligned} |P\{M_{n'} \leqq u_n(\tau)\} - P\{M_{n'} \leqq u_{n'}(\tau/k)\}| &\leqq n' \, |F(u_n(\tau)) - F(u_{n'}(\tau/k))| \\ &= n' \left| \frac{\tau/k}{n'} (1+o(1)) - \frac{\tau}{n} (1+o(1)) \right| \end{aligned}$$

by (2.6), and this tends to zero as $n \to \infty$ since $n' \sim n/k$. But clearly

$$\limsup_{n\to\infty} P\{M_{n'} \leqq u_{n'}(\tau/k)\} = \psi(\tau/k),$$

and it thus follows that $\limsup_{n\to\infty} P\{M_{n'} \leqq u_n(\tau)\} = \psi(\tau/k)$. Combining this with (2.9) we see that

$$\psi(\tau/k) = \psi^{1/k}(\tau) \qquad 0 < \tau \leqq \tau_0, \quad k = 1, 2 \ldots \tag{2.10}$$

Now $P\{M_{n'} \leqq u_n(\tau)\} \geqq 1 - n' P\{\xi_1 > u_n(\tau)\} \to 1 - \tau/k$ as $n \to \infty$ so that by taking kth powers and using Lemma 2.1, it follows that $\liminf_{n\to\infty} P\{M_n \leqq u_n(\tau)\} \geqq (1-\tau/k)^k$, and letting $k \to \infty$ that

$$\liminf_{n\to\infty} P\{M_n \leqq u_n(\tau)\} \geqq e^{-\tau}. \tag{2.11}$$

In particular this implies that $\psi(\tau)$ is strictly positive. It is also non-increasing since if $\tau' < \tau$ it is clear that $u_n(\tau') > u_n(\tau)$ when n is sufficiently large. But the only strictly positive non-increasing solution to the functional equation (2.10) is $\psi(\tau) = e^{-\theta\tau}$ for some $\theta \geqq 0$. That is $\limsup_{n\to\infty} P\{M_n \leqq u_n(\tau)\} = e^{-\theta\tau}$ with $\theta \geqq 0$.

Similarly it may be shown that $\liminf_{n\to\infty} P\{M_n \leqq u_n(\tau)\} = e^{-\theta'\tau}$ where clearly $\theta' \geqq \theta$. By (2.11) $\theta' \leqq 1$ and hence $0 \leqq \theta \leqq \theta' \leqq 1$ as asserted. Thus the relations (2.8) follow and the final statements of the theorem are immediate from these. □

If $P\{M_n \leqq u_n(\tau)\} \to e^{-\theta\tau}$ for each $\tau > 0$ with $u_n(\tau)$ satisfying (2.6), we shall say that the sequence $\{\xi_n\}$ has *extremal index* θ (cf. Sect. 1). Use of this terminology will simplify statements of later results, and in particular gives the following obvious restatement of part of the above theorem.

Corollary 2.3. *Let $\{\xi_n\}$ be stationary and satisfy $D(u_n(\tau))$ for each $\tau > 0$ where $n[1-F(u_n(\tau))] \to \tau$. If for some $\tau_0 > 0$, $P\{M_n \leqq u_n(\tau_0)\}$ converges to a limit α then $\{\xi_n\}$ has extremal index $\theta = -\tau_0^{-1} \log \alpha$ so that $P(M_n \leqq u_n(\tau)) \to e^{-\theta\tau}$ for all $\tau > 0$.* □

In the next section we shall show that the addition of the condition $D'(u_n)$ (cf. §1) implies that $\theta = 1$, and give other criteria determining θ when $0 \leqq \theta < 1$. However here we proceed with the more general theory, showing that if $\{\xi_n\}$ has a non-zero extremal index θ, then any limiting distribution for the maximum must be of the same type as if the terms were i.i.d. with the same normalizing constants if $\theta = 1$, and simply modified constants for $0 < \theta < 1$. The basic result generalizes a theorem proved by O'Brien [14] under strong mixing assumptions. Here in addition to the previous notation we write

$$\hat{M}_n = \max(\hat{\xi}_1, \hat{\xi}_2 \ldots \hat{\xi}_n)$$

where $\hat{\xi}_1, \hat{\xi}_2 \ldots$ are i.i.d. random variables with the same d.f. F as each of the stationary sequence $\xi_1, \xi_2 \ldots$ (following Loynes [10] we call $\hat{\xi}_1, \hat{\xi}_2 \ldots$ the "associated independent sequence"). We note the well known (and easily proved) result that for any $\tau > 0$ and sequence $\{u_n\}$,

$$P\{\hat{M}_n \leqq u_n\} (= F^n(u_n)) \to e^{-\tau} \tag{2.12}$$

if and only if

$$n[1-F(u_n)]\to\tau \tag{2.13}$$

(i.e. if and only if (2.6) holds with $u_n=u_n(\tau)$).

Theorem 2.4. *Suppose that the stationary sequence $\{\xi_n\}$ has extremal index θ, $0\leqq\theta\leqq 1$. Let $\{v_n\}$ be any sequence of constants and ρ any constant with $0\leqq\rho\leqq 1$. Then*

(i) *If $\theta>0$*

$$P\{\hat{M}_n\leqq v_n\}\to\rho \quad \textit{if and only if} \quad P\{M_n\leqq v_n\}\to\rho^\theta$$

(ii) *If $\theta=0$*

(a) *if* $\liminf_{n\to\infty} P\{\hat{M}_n\leqq v_n\}>0$, *then* $P\{M_n\leqq v_n\}\to 1$

(b) *if* $\limsup_{n\to\infty} P\{M_n\leqq v_n\}<1$, *then* $P\{\hat{M}_n\leqq v_n\}\to 0$.

Proof. (i) Suppose $\theta>0$ and $P\{\hat{M}_n\leqq v_n\}\to\rho$ where $0<\rho\leqq 1$. Choose $\tau>0$ such that $e^{-\tau}<\rho$. Then

$$P\{\hat{M}_n\leqq u_n(\tau)\}\to e^{-\tau}, \qquad P\{\hat{M}_n\leqq v_n\}\to\rho>e^{-\tau}$$

so that $v_n>u_n(\tau)$ for sufficiently large n, and hence

$$\liminf_{n\to\infty} P\{M_n\leqq v_n\}\geqq \lim_{n\to\infty} P\{M_n\leqq u_n(\tau)\}=e^{-\theta\tau}.$$

Since this holds for any τ such that $e^{-\tau}<\rho$ it follows that

$$\liminf_{n\to\infty} P\{M_n\leqq v_n\}\geqq\rho^\theta.$$

It also follows in particular that if $\rho=1$ then $P\{M_n\leqq v_n\}\to 1=\rho^\theta$.

Similarly by taking $e^{-\tau}>\rho$ it may be shown that $\limsup_{n\to\infty} P\{M_n\leqq v_n\}\leqq\rho^\theta$ when $0\leqq\rho<1$. Hence $P\{M_n\leqq v_n\}\to 0$ when $\rho=0$, and for $0<\rho<1$,

$$P\{M_n\leqq v_n\}\to\rho^\theta$$

by combining the inequalities for the upper and lower limits. The proof of the converse is similar so that (i) follows.

To prove (ii) we assume $\theta=0$, so that $P\{M_n\leqq u_n(\tau)\}\to 1$ as $n\to\infty$ for each $\tau>0$. If $\liminf P\{\hat{M}_n\leqq v_n\}=\rho>0$, choose τ with $e^{-\tau}<\rho$ and hence $P\{\hat{M}_n\leqq u_n(\tau)\}\to e^{-\tau}<\rho$ so that $v_n\geqq u_n(\tau)$ for sufficiently large n. Thus

$$\liminf_{n\to\infty} P\{M_n\leqq v_n\}\geqq \lim_{n\to\infty} P\{M_n\leqq u_n(\tau)\}=1,$$

giving (a).

To show (b) note that if $\limsup P\{M_n\leqq v_n\}<1$ we must have $v_n<u_n(\tau)$ for sufficiently large n and hence

$$\limsup_{n\to\infty} P\{\hat{M}_n\leqq v_n\}\leqq\lim P\{\hat{M}_n\leqq u_n(\tau)\}=e^{-\tau}$$

for each τ. The conclusion (b) follows by letting $\tau\to\infty$. $\square$

As an immediate corollary we give conditions in terms of the extremal index under which M_n has a limiting distribution if and only if $\hat{M}_n$ does. This of course implies that in such cases, the classical domain of attraction criteria may be used in the dependent situation.

Theorem 2.5. *Let the stationary sequence $\{\xi_n\}$ have extremal index $\theta>0$. Then M_n has a non-degenerate limiting distribution if and only if $\hat{M}_n$ does, and these are then of the same type based on the same normalizing constants. In the case $\theta=1$ the limiting distributions for M_n and $\hat{M}_n$ are identical.*

Proof. If $P\{a_n(\hat{M}_n-b_n)\leqq x\}\to G(x)$, non-degenerate, then Theorem 2.4(i) shows (with $v_n=x/a_n+b_n$) that $P\{a_n(M_n-b_n)\leqq x\}\to G^\theta(x)$. But G is an extreme value distribution and it is well known (and easily checked from the possible functional forms) that G^θ is of the *same type* as G in the sense of Sect. 1 that $G^\theta(x)=G(ax+b)$ for some $a>0, b$.

The converse follows similarly, noting that if $P\{a_n(M_n-b_n)\leqq x\}\to H(x)$, non-degenerate, then $P\{a_n(\hat{M}_n-b_n)\leqq x\}\to H^{1/\theta}(x)$. As a limiting distribution for maxima from an i.i.d. sequence, $H^{1/\theta}$ must be of extreme value type and $H=(H^{1/\theta})^\theta$ must be of the same type as $H^{1/\theta}$.

The final remark for $\theta=1$ is obvious. □

For the case $0<\theta<1$ the same normalizing constants give limits e.g. $G(x)$, $G^\theta(x)=G(ax+b)$ for $\hat{M}_n$ and M_n. Of course a simple change of the set of normalizing constants for M_n will lead to the same limit $G(x)$.

It is, of course, also of interest to explore the situation when the extremal index is zero. An argument of R. Davis [5] shows (using also Theorem 2.4(ii)) that M_n and $\hat{M}_n$ cannot both have non-degenerate limiting distributions based on the same normalizing constants. This is stated precisely as follows, without proof.

Theorem 2.6. *Let the stationary sequence $\{\xi_n\}$ satisfy $D(u_n(\tau))$ where for each $\tau>0$ $u_n(\tau)$ satisfies* (2.6). *If $\{\xi_n\}$ has extremal index $\theta=0$, then M_n and $\hat{M}_n$ cannot both have non-degenerate limiting distributions based on the same normalizing constants. That is, it is not possible to have $P\{a_n(\hat{M}_n-b_n)\leqq x\}\to G(x)$, $P\{a_n(M_n-b_n)\leqq x\}\to H(x)$ for non-degenerate G, H.* □

3. Some Criteria for the Extremal Index, and Examples

In this section we first give some relationships of potential use in determining the extremal index. We are grateful to a referee for pointing out that these involve similar methods and conclusions to those of R. Davis [4], and also that calculations of Newell [12] explicitly provide the extremal index for the case of m-dependent sequences.

The first result has perhaps more theoretical then practical interest but serves as a means of extending the condition $D'(u_n)$ to apply to more dependent cases with $\theta<1$. By way of convenient notation we again write $n'=[n/k]$ for fixed k, $n=1,2\ldots$. Also as previously $F_{i_1\ldots i_r}(u)$ will denote the joint d.f. of $\xi_{i_1}\ldots\xi_{i_r}$ evaluated at $(u, u\ldots u)$.

Theorem 3.1. *Let the stationary sequence $\{\xi_n\}$ satisfy $D(u_n(\tau))$ for each $\tau>0$ where $u_n(\tau)$ satisfies* (2.6). *Then $\{\xi_n\}$ has extremal index θ $(0\leqq\theta\leqq1)$ if and only if*

$$k \limsup_{n\to\infty} |1-F_{1,\ldots,n'}(u_n)-\theta\tau_0/k| \to 0 \quad as \;\; k\to\infty \tag{3.1}$$

for some $\tau_0>0$. *Equivalently this holds if and only if*

$$1-F_{1,\ldots,n'}(u_n)\to\theta\tau_0/k+\lambda_k \quad as \;\; n\to\infty \tag{3.2}$$

where $k\lambda_k\to 0$ *as* $k\to\infty$.

Proof. For simplicity of notation we take $\tau_0=1$ and write $u_n=u_n(1)$. If $\{\xi_n\}$ has extremal index θ, and since (2.1) holds by Lemma 2.1,

$$F_{1\ldots n'}(u_n)=P\{M_{n'}\leqq u_n\}\to e^{-\theta/k}=1-\theta/k+o(1/k)$$

from which (3.2) (and hence obviously (3.1)) follow.

Conversely if (3.1) holds then

$$\begin{aligned}\limsup_{n\to\infty} P\{M_{n'}\leqq u_n\} &= \limsup_{n\to\infty} [F_{1\ldots n'}(u_n)-1+\theta/k]+1-\theta/k\\ &\leqq 1-\theta/k+\limsup_{n\to\infty}|1-F_{1\ldots n'}(u_n)-\theta/k|\\ &=1-\theta/k+o(1/k).\end{aligned}$$

Hence again by (2.1)

$$\limsup_{n\to\infty} P\{M_n\leqq u_n\}\leqq\{1-\theta/k+o(1/k)\}^k$$

for all k giving, on letting $k\to\infty$,

$$\limsup_{n\to\infty} P\{M_n\leqq u_n\}\leqq e^{-\theta}.$$

The opposite inequality for $\liminf P\{M_n\leqq u_n\}$ follows similarly so that $P\{M_n\leqq u_n\}\to e^{-\theta}$. Thus we have convergence of $P\{M_n\leqq u_n(\tau)\}$ to $e^{-\theta\tau}$ at $\tau=\tau_0=1$ and the result follows from Corollary 2.3. □

The condition $D'(u_n)$ given by (1.3) limits the probability of one exceedance of u_n being followed "closely" by another. One obvious generalization is to permit (with high probability) no more than some specified number of exceedances to occur together. One specific such restriction is to limit the quantity

$$E_{n,k}^{(r)}=k \sum_{1\leqq i_1<i_2\ldots<i_r\leqq n'} P\{\xi_{i_1}>u_n, \xi_{i_2}>u_n, \ldots \xi_{i_r}>u_n\} \tag{3.2}$$

for some r. For example the assumption $D'(u_n)$ limits $E_{n,k}^{(2)}$ so that in fact $\limsup_{n\to\infty} E_{n,k}^{(2)}\to 0$ as $k\to\infty$, from which it follows (cf. [8]) that (1.4) holds so that $\{\xi_n\}$ has extremal index 1. Generalizations of this are clearly possible to allow a non-zero limit for $\limsup_{n\to\infty} E_{n,k}^{(r)}$ as $k\to\infty$ for some values of $r\geqq 2$ as the following simplest case beyond $D'(u_n)$ shows.

Corollary 3.2. *Let the stationary sequence* $\{\xi_n\}$ *satisfy* $D(u_n(\tau))$ *for each* $\tau>0$ *where* $u_n(\tau)$ *satisfies* (2.6). *Suppose that for some* $\tau_0>0$, $u_n=u_n(\tau_0)$ *and some* θ, $0\leqq\theta\leqq 1$,

$$\limsup_{n\to\infty} |E_{n,k}^{(2)} - \tau_0(1-\theta)| \to 0 \quad \textit{as} \ k\to\infty \tag{3.3}$$

and

$$\limsup_{n\to\infty} E_{n,k}^{(3)} \to 0 \quad \textit{as} \ k\to\infty \tag{3.4}$$

Then $\{\xi_n\}$ has extremal index θ.

Proof. Since $1-F_{1\ldots n'}(u_n) = P\left\{\bigcup_{j=1}^{n'} (\xi_j > u_n)\right\}$ it follows by standard Bonferroni inequalities that

$$n'[1-F(u_n)] - k^{-1} E_{n,k}^{(2)} \leqq 1 - F_{1\ldots n'}(u_n)$$
$$\leqq n'[1-F(u_n)] - k^{-1} E_{n,k}^{(2)} + k^{-1} E_{n,k}^{(3)}$$

and hence

$$kn'[1-F(u_n)] - \tau_0(1-\theta) - |E_{n,k}^{(2)} - \tau_0(1-\theta)| \leqq k[1-F_{1\ldots n'}(u_n)]$$
$$\leqq kn'[1-F(u_n)] - \tau_0(1-\theta) + |E_{n,k}^{(2)} - \tau_0(1-\theta)| + E_{n,k}^{(3)}.$$

Since $u_n = u_n(\tau_0)$, letting $n\to\infty$ with k fixed yields

$$\theta\tau_0 - \limsup_{n\to\infty} |E_{n,k}^{(2)} - \tau_0(1-\theta)| \leqq \liminf_{n\to\infty} k[1-F_{1\ldots n'}(u_n)]$$
$$\leqq \limsup_{n\to\infty} k[1-F_{1\ldots n'}(u_n)]$$
$$\leqq \theta\tau_0 + \limsup |E_{n,k}^{(2)} - \tau_0(1-\theta)| + \limsup_{n\to\infty} E_{n,k}^{(3)}$$

from which it follows simply that

$$\limsup_{n\to\infty} |k[1-F_{1\ldots n'}(u_n)] - \theta\tau_0| \leqq \limsup_{n\to\infty} |E_{n,k}^{(2)} - \tau_0(1-\theta)| + \limsup_{n\to\infty} E_{n,k}^{(3)}$$

which tends to zero as $k\to\infty$, giving (3.1) and hence the desired conclusion by the theorem. □

The condition (3.3) may be restated in an obvious way to give the following alternative version of the corollary.

Corollary 3.3. *The result of Corollary* 3.2 *holds if* (3.3) *is replaced by*

$$\sum_{j=2}^{n'} (1-j/n') P\{\xi_j > u_n \mid \xi_1 > u_n\} \to 1 - \theta + \lambda_k \quad \textit{as} \ n\to\infty, \tag{3.5}$$

where $\lambda_k \to 0$ as $k\to\infty$.

Proof. It is simply checked that

$$E_{n,k}^{(2)} - \tau_0(1-\theta)$$
$$= n[1-F(u_n)] \sum_{j=2}^{n'} (1-j/n') P\{\xi_j > u_n \mid \xi_1 > u_n\} - \tau_0(1-\theta) \to \tau_0\lambda_k \quad \text{as} \ n\to\infty,$$

from which (3.3) follows at once. □

While "repeated limit conditions" such as (3.1) can be useful in practice, it may sometimes be more convenient to use conditions depending on a single limit only, and we shall show briefly how this may be done, giving an alternative form for Theorem 3.1.

The condition $D(u_n)$ requires that the quantity $\alpha_{n,l_n}\to 0$ as $n\to\infty$ for some $l_n=o(n)$. It is clearly possible to obtain $k_n\to\infty$ such that both

$$k_n\alpha_{n,l_n}\to 0 \tag{3.6}$$

and

$$k_n l_n=o(n) \tag{3.7}$$

hold (e.g. taking $k_n=\min(\alpha_{n,l_n}^{-\frac{1}{2}}, (n/l_n)^{\frac{1}{2}})$. Using such a sequence k_n we have the following variant of Theorem 3.1.

Theorem 3.4. *Let the stationary sequence $\{\xi_n\}$ satisfy $D(u_n(\tau))$ for each $\tau>0$ where $u_n(\tau)$ satisfies (2.6). For some $\tau_0>0$ let $k_n\to\infty$ be such that (3.6) and (3.7) hold with $u_n=u_n(\tau_0)$. If, writing $r_n=[n/k_n]$,*

$$k_n[1-F_{1\dots r_n}(u_n)]\to\theta\tau_0 \quad \textit{as } n\to\infty\ (0\leqq\theta\leqq 1) \tag{3.8}$$

then $\{\xi_n\}$ has extremal index θ. Conversely if $\{\xi_n\}$ has extremal index θ then (3.8) holds for each $\tau_0>0$ and each $k_n\to\infty$ satisfying (3.6) and (3.7) with $u_n=u_n(\tau_0)$.

Proof. If (3.8) holds then

$$P\{M_{r_n}\leqq u_n\}=F_{1\dots r_n}(u_n)=1-\frac{\theta\tau_0}{k_n}(1+o(1))$$

so that

$$P^{k_n}\{M_{r_n}\leqq u_n\}=\left[1-\frac{\theta\tau_0}{k_n}+o\left(\frac{1}{k_n}\right)\right]^{k_n}\to e^{-\theta\tau_0}$$

and hence $P\{M_n\leqq u_n\}\to e^{-\theta\tau_0}$ by Lemma 2.1 showing that $\{\xi_n\}$ has extremal index θ by Corollary 2.3.

Conversely if $\{\xi_n\}$ has extremal index θ, and $\tau_0>0$, $k_n\to\infty$ satisfying (3.6) and (3.7) then Lemma 2.1 shows that $P^{k_n}(M_{r_n}\leqq u_n)\to e^{-\theta\tau_0}$ with $u_n=u_n(\tau_0)$. It follows simply that $F_{1\dots r_n}(u_n)\to 1$ and

$$\log[1-(1-F_{1\dots r_n}(u_n))]=-\frac{\theta\tau_0}{k_n}(1+o(1))$$

so that

$$-[1-F_{1\dots r_n}(u_n)][1+o(1)]=-\frac{\theta\tau_0}{k_n}(1+o(1))$$

giving (3.8) as required. □

A simply expressed sufficient condition for (3.8) may be given as in the following corollary. In this we write $E_n^{(s)}$ for $E_{n,k_n}^{(s)}$ where this is given by (3.2) i.e.

$$E_n^{(s)}=k_n\sum_{1\leqq i_1<i\dots<i_s\leqq r_n}P\{\xi_{i_1}>u_n\dots\xi_{i_s}>u_n\} \tag{3.9}$$

(where $r_n=[n/k_n]$).

Corollary 3.5. *Let the stationary sequence $\{\xi_n\}$ satisfy $D(u_n(\tau))$ for each $\tau>0$, where $u_n(\tau)$ satisfies (2.6). For some $\tau_0>0$ let $k_n\to\infty$ be such that (3.6) and (3.7) hold with $u_n=u_n(\tau_0)$. Suppose that for each $s=1,2\ldots$ the $E_n^{(s)}$ defined by (3.9) satisfy*

$$E_n^{(s)}\to\alpha_s \quad \text{as } n\to\infty, \tag{3.10}$$

where $\alpha_s\to 0$ as $s\to\infty$. Then $\{\xi_n\}$ has extremal index

$$\theta=\tau_0^{-1}\sum_{r=1}^{\infty}(-)^{r-1}\alpha_r.$$

Proof. Write $\lambda_n=k_n[1-F_{1\ldots r_n}(u_n)]=k_n P\left\{\bigcup_{j=1}^{r_n}(\xi_j>u_n)\right\}$. Then using Bonferroni Inequalities we have for s odd, $n>s$,

$$E_n^{(1)}-E_n^{(2)}+E_n^{(3)}\ldots+E_n^{(s)}\geqq\lambda_n\geqq E_n^{(1)}-E_n^{(2)}+E_n^{(3)}\ldots-E_n^{(s+1)}$$

Writing $\underline{\lambda}=\liminf_{n\to\infty}\lambda_n$, $\bar{\lambda}=\limsup_{n\to\infty}\lambda_n$ and letting $n\to\infty$, we obtain, for each odd s,

$$\alpha_1-\alpha_2+\alpha_3\ldots+\alpha_s\geqq\bar{\lambda}\geqq\underline{\lambda}\geqq\alpha_1-\alpha_2+\alpha_3\ldots-\alpha_{s+1}. \tag{3.11}$$

Since the extreme terms differ by α_{s+1} which tends to zero as $s\to\infty$, it follows that $\bar{\lambda}=\underline{\lambda}$, say. If $T_s=\sum_{r=1}^{s}(-)^{r-1}\alpha_r$ it then follows from (3.11) that $0\leqq T_s-\lambda\leqq T_s-T_{s+1}=\alpha_{s+1}$ for s odd and similarly $0\leqq\lambda-T_s\leqq T_{s-1}-T_s=\alpha_s$ for s even, so that in both cases

$$|T_s-\lambda|\leqq\alpha_s+\alpha_{s+1}\to 0 \quad \text{as } s\to\infty.$$

Hence $\sum_{r=1}^{\infty}(-)^{r-1}\alpha_r$ converges to the value λ and (3.8) holds with $\theta\tau_0=\sum_{r=1}^{\infty}(-)^{r-1}\alpha_r$ giving the desired result. □

Finally in this section we cite some examples of sequences exhibiting all the possible types of behavior relative to the extremal index. In each of these cases $D(u_n(\tau))$ is satisfied.

The most common case is where $D'(u_n(\tau))$ holds leading to the extremal index $\theta=1$. For example this is so for a stationary normal sequence $\{\xi_n\}$ with covariance sequence $\{r_n\}$ satisfying the condition of S.M. Berman, [2], viz. $r_n\log n\to 0$ – an obviously weak condition indeed.

We have given a simple example of a case when $\theta=\frac{1}{2}$ in the discussion above. An example where a series of values of θ is possible through parameter choice in an autoregressive scheme, has been given by Chernick [3]. The stable processes considered by Rootzén [15], can have any value of θ in the range $0<\theta\leqq 1$. A simple example due to L. de Haan also exhibiting this behavior is the sequence

$$\xi_n=\max_{k\geqq 0}\rho^k\eta_{n-k}$$

where $0<\rho<1$ and $\{\eta_n\}$ is an i.i.d. sequence with common d.f. $\exp(-1/x)$. This yields an extremal index $\theta=1-\rho$.

An example of Denzel and O'Brien [6] exhibits a "chain dependent" sequence $\{\xi_n\}$ with extremal index $\theta=0$. In this case $\hat{M}_n$ has a Type II limiting distribution, but we do not know whether M_n has any sort of limiting distribution.

A further example of L. de Haan, however, provides a case where $\theta=0$ and M_n, $\hat{M}_n$ both have limiting distributions. Specifically the sequence $\{\xi_n\}$ is defined by

$$\xi_n=\max_{k\geqq 0}(\eta_{n-k}-k)$$

where η_n are i.i.d. with common d.f. $\exp(-x^{-\alpha})$ $x>0$, $(\alpha>1)$. In this case M_n has a Type II limit with parameter α and norming constants $a_n=n^{-1/\alpha}$, $b_n=0$ whereas $\hat{M}_n$ has a Type II limit with parameter $\alpha-1$ and norming constants $a_n=n^{-1/(\alpha-1)}$, $b_n=0$.

Further an example of O'Brien [13] exhibits a case in which $\{\xi_n\}$ has no extremal index at all. In this each ξ_n is uniform over the interval $[0,1]$, ξ_1, ξ_3 ... being independent and ξ_{2n} a certain function of ξ_{2n-1} for each n.

Finally while the above discussion has relied on the condition $D(u_n)$ throughout, even this condition may be weakened. For example a case where $D(u_n)$ does not hold but the extremal index exists is given by the following example of [5]. Let $\eta_1, \eta_2, \ldots$ be i.i.d. and define the sequence

$$(\xi_1, \xi_2, \xi_3, \ldots)=(\eta_1, \eta_2, \eta_2, \eta_3, \eta_3, \ldots)$$

or

$$(\eta_1, \eta_1, \eta_2, \eta_2, \ldots)$$

each with probability $\frac{1}{2}$. It follows from [5] that $\{\xi_n\}$ has extremal index $\frac{1}{2}$. Further, it is easily shown that (3.5) holds. However $D(u_n(\tau))$ does not hold, showing that even this weak (and usually satisfied) condition is not quite necessary.

4. Point Process of Clusters

As noted in Sect. 1, when $n[1-F(u_n)]\to\tau$ and $D(u_n)$ and $D'(u_n)$ both hold, the (time normalized) instants at which the sequence exceeds u_n take on a Poisson character as n becomes large. More specifically let N_n denote the point process on the unit interval $(0,1]$ consisting of those points j/n such that $\xi_j>u_n$. Then under the conditions above it may be shown ([9]) that N_n converges weakly to a Poisson Process with intensity τ on $(0,1]$.

When $D'(u_n)$ does not hold, the exceedances of u_n may tend to occur in clusters, leading to the simultaneous occurrence of multiple events i.e. a "compounding" in the limiting point process. A complete description of the limiting point process has been given by Rootzén [15] in the case where the underlying sequence $\{\xi_n\}$ belongs to a class of stable processes (cf. the above discussion in Sect. 3).

Again under a (multidimensional type of) strengthening of the condition $D(u_n)$, and assuming $D'(u_n)$, it is possible to obtain a "complete Poisson theorem" (cf. [1, 9]). This involves convergence of the point process in the plane with points at $(j/n, (\xi_j - b_n)/a_n)$, with appropriate a_n, b_n, to a certain Poisson process in the plane. Results of this type allow rather complete descriptions of (joint) asymptotic distributional results for extreme order statistics.

It is also of interest to determine the effect of eliminating the condition $D'(u_n)$ in results of this type. For example Mori [11] has shown that under strong mixing the limiting point processes are confined to a certain class (and it seems likely that this is true also under the weaker $D(u_n)$-type of condition).

We shall not investigate limiting results of these types in detail here. However it does seem interesting and useful to give the simplest of convergence results - involving the Poisson limit for the point process "positions" of the "clusters" of exceedances of high levels. This is analogous to a result of Rootzén in [15] for stable processes.

One very simple means of defining clusters of exceedances is to take a sequence r_n and consider that events occurring within a distance r_n of each other belong to the same cluster. r_n should of course be chosen so that it is at least as large as (virtually all) cluster "lengths" but small compared with cluster "separation." For many usual situations this still leaves considerable flexibility in the choice of r_n, while leading to unique results as we shall see.

More specifically we shall suppose that $D(u_n)$ holds for $u_n = u_n(\tau)$ satisfying (2.6), a sequence $k_n \to \infty$ is chosen to satisfy (3.6) and (3.7) and $r_n = [n/k_n]$. A point process N_n is defined on the unit interval $(0, 1]$ as follows. If for given $s = 1, 2 \dots k_n$ there is an exceedance of u_n by ξ_j for at least one j such that $(s-1) r_n < j \leqq s r_n$, then N_n has a single event at the point $t = s r_n/n$. That is any group of exceedances between $(s-1) r_n$ and $s r_n$ is replaced by a single event - after time-scaling - at $s r_n/n$, "representing" the original group. We refer to N_n as the "point process of cluster positions." With this construction the following result holds.

Theorem 4.1. *Let the stationary sequence* $\{\xi_n\}$ *satisfy* $D(u_n(\tau))$ *for each* $\tau > 0$ *where* $u_n(\tau)$ *satisfies* (2.6). *Let* $k_n \to \infty$ *be chosen to satisfy* (3.6) *and* (3.7) *and let* $\{\xi_n\}$ *have extremal index* θ $(0 < \theta \leqq 1)$. *Then the point process* N_n *of cluster positions for exceedances of* $u_n(\tau)$ *converges in distribution to a Poisson Process* N *on* $(0, 1]$ *with intensity parameter* $\theta\tau$.

Proof. As in previous proofs of similar results (cf. [9]) it is by a theorem of Kallenberg [7] only necessary to show that

$$\mathscr{E} N_n\{(a, b]\} \to \mathscr{E} N\{(a, b]\} \quad \text{for } 0 < a < b \leqq 1 \tag{4.1}$$

and

$$P\{N_n(E) = 0\} \to P\{N(E) = 0\} \tag{4.2}$$

for each finite disjoint union E of sets $(a_i, b_i] \subset (0, 1]$.

If v_n denotes the number of (integer) intervals $((s-1) r_n, s r_n]$ completely contained in $([na], [nb]]$ it is clear that $v_n \sim n r_n^{-1}(b-a) \sim k_n(b-a)$ and further that

$$\begin{aligned}\mathscr{E} N_n\{(a,b]\} &\sim v_n P\left\{\bigcup_{i=1}^{r_n} (\xi_i > u_n)\right\}\\ &\sim k_n(b-a)\left[1-F_{1\ldots r_n}(u_n)\right]\\ &\to (b-a)\,\theta\tau\end{aligned}$$

by (3.8). But this is just $\mathscr{E} N\{(a,b]\}$ so that (4.1) follows.

To show (4.2) we write $E=\bigcup_1^p (a_j, b_j]$ and write B_j for the integers in $([(n a_j],$ $[(n b_j]]$. Then it is readily seen that

$$\begin{aligned}P\{N_n(E)=0\} &= P\left\{\bigcap_{j=1}^{p} (M(B_j)\leqq u_n)\right\}+o(1)\\ &= \prod_{j=1}^{p} \{P(M(B_j)\leqq u_n)\} + \left[P\left\{\bigcap_{j=1}^{p} (M(B_j)\leqq u_n)\right\} - \prod_{j=1}^{p} P\{M(B_j)\leqq u_n\}\right] + o(1).\end{aligned}$$

By a straightforward induction, the difference in square brackets does not exceed $p\,\alpha_{n,n\lambda}$ in absolute value where λ is the minimum separation of the intervals $(a_j, b_j]$ (λ can be taken non-zero since abutting intervals can be combined). But $\alpha_{n,l}$ may be taken non-increasing in l (cf. [9]) and it follows from $D(u_n)$ that $\alpha_{n,n\lambda}\to 0$ as $n\to\infty$. Since $\{\xi_n\}$ has extremal index θ it follows in an obvious way that $P\{M(B_j)\leqq u_n\}\to e^{-\theta\tau(b_j-a_j)}$ and hence

$$\begin{aligned}P\{N_n(E)=0]&\to \prod_{j=1}^{p} P\{N(a_j, b_j]=0]\\ &= P\{N(E)=0\}\end{aligned}$$

proving 4.2. □

It is of interest to note an intuitively appealing interpretation of the extremal index as the inverse of mean cluster size. This may be seen even in terms of the simple approach above. For the mean cluster size can be interpreted as the (limiting) mean number of exceedances in an interval of length r_n, given at least one exceedance in that interval i.e. if Z denotes the number of exceedances of $u_n(\tau)$ in an interval of length r_n,

$$\begin{aligned}\mathscr{E}\{Z\,|\,Z\geqq 1\} &= \sum_{s=1}^{\infty} s P\{Z=s\,|\,Z\geqq 1\}\\ &= \mathscr{E} Z/P\{Z\geqq 1\}\\ &= r_n\left[1-F(u_n)\right]/\left[1-F_{1\ldots r_n}(u_n)\right]\\ &\to \theta^{-1}.\end{aligned}$$

Finally it should be noted that the limiting distributions of extreme order statistics will be affected in a more complicated way by the clustering than the maximum. These distributions would emerge from the more complete limiting result for individual exceedances. However use of the simple Poisson result given above will result in the distributions for the heights of the "kth highest clusters" rather than the kth extreme order statistics, in an obvious way. This of course is analogous to consideration of kth highest local maxima in continuous parameter situations.

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