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A Class of Limiting Distributions of High Level Excursions of Gaussian Processes*

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0. Introduction

Let $X(t)$, $-\infty<t<\infty$, be a real valued stationary Gaussian process with mean 0, variance 1, covariance function $r(t)$, and continuous sample functions. Such a process will be called a *standard* process. For $u>0$ and $T>0$, let L be the Lebesgue measure of the set $\{t\colon 0\leqq t\leqq T,\ X(t)>u\}$, the time spent above u in $[0, T]$. It is assumed that r is nonperiodic and that

$$1-r(t)\sim\gamma^\alpha |t|^\alpha, \qquad t\to 0, \tag{0.1}$$

for some $\gamma>0$ and some α, $0<\alpha\leqq 2$. Let $U(t)$, $-\infty<t<\infty$ be a Gaussian process with continuous sample functions, vanishing at $t=0$, and such that

$$E\big(U(t)-U(s)\big)\equiv 0, \qquad E|U(t)-U(s)|^2\equiv|s-t|^\alpha.$$

Our main results are:

i) The conditional distribution of $u^{2/\alpha}\gamma L$, given $X(0)=u$, converges for $u\to\infty$ to the distribution of the time spent above 0 by the process $\sqrt{2}\,U(t)-t^\alpha$, $t\geqq 0$. In the case $\alpha=2$ this provides a more general version of the "vertical window" conditional limit theorem for high level excursions, proved by Kac and Slepian [4]. We note that our earlier result [1] gives a more general version of the "horizontal window" limit theorem.

ii) In the particular case $\alpha=1$, the same limiting distribution as in i) is obtained for $u^2\gamma L$ under a different kind of conditioning, namely, conditioning by the event $L>0$. Here U is the standard Brownian motion process. The moments of the limiting distribution are explicitly computed. This conditioning is analogous to that in [1], where the case $\alpha=2$ was considered.

The limit theorem in i) may be explained in the following way. Let I_A be the indicator of the event A; then

$$L=\int_0^T I_{[X(s)>u]}\,ds$$

and so

$$u^{2/\alpha}\gamma L=\int_0^{T\gamma u^{2/\alpha}} I_{[X(s/\gamma u^{2/\alpha})>u]}\,ds$$

$$=\int_0^{T\gamma u^{2/\alpha}} I_{\{u[X(s/\gamma u^{2/\alpha})-u]>0\}}\,ds.$$

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The conditioned process $u[X(s/u^{2/\alpha})-u]$, given $X(0)=u$, converges in distribution to $\sqrt{2}\,U(s)-|s|^\alpha$; therefore, with appropriate justification, the same is true for the excursion distributions of the corresponding processes.

The result ii) implies that when $\alpha=1$ the two conditions $L>0$ and $X(0)=u$ lead to the same limiting distribution. In contrast to this, two different limiting distributions are obtained when $\alpha=2$.

1. Excursions for a Class of Gaussian Processes with Stationary Increments

Let $U(t)$ be the Gaussian process defined in § 0, and put

$$\phi(x)=(2\pi)^{-\frac{1}{2}}e^{-\frac{1}{2}x^2}, \qquad \Phi(x)=\int_{-\infty}^{x}\phi(y)\,dy.$$

Let ξ be the random variable

$$\xi=\int_0^\infty I_{[\sqrt{2}\,U(t)>t^\alpha]}\,dt.$$

This is almost surely finite because

$$E\,\xi=\int_0^\infty P\{\sqrt{2}\,U(t)>t^\alpha\}\,dt=\int_0^\infty\left[1-\Phi\left(\sqrt{t^\alpha/2}\right)\right]dt<\infty.$$

Put

$$\phi(x,y;\rho)=\frac{\exp\left\{-\dfrac{x^2-2\rho xy+y^2}{2(1-\rho^2)}\right\}}{2\pi(1-\rho^2)^{\frac{1}{2}}},$$

and note that, by the method in [3], p. 214, the variance of ξ is

$$\int_0^\infty\int_0^\infty\int_0^{r(s,t)}\phi\left(\sqrt{s^\alpha/2},\sqrt{t^\alpha/2};\rho\right)d\rho\,ds\,dt,$$

where $r(s,t)=$ correlation of $U(s)$, $U(t)$

$$=\tfrac{1}{2}\left(|s|^\alpha+|t|^\alpha-|s-t|^\alpha\right).$$

The latter is positive for all $s,t>0$ so that the variance of ξ is positive (or infinitely positive); therefore, the distribution of ξ is not degenerate.

When $\alpha=1$ $U(t)$ is the standard Brownian motion process. We shall compute the moments of ξ and show that they determine a unique distribution. For subsequent purposes the Brownian motion is conditioned by $U(0)=y_0$, not necessarily equal to 0.

Lemma 1.1. *Let* $U(t)$, $t\geqq 0$ *be the standard Brownian motion, and* ξ *defined as above* (*with* $\alpha=1$); *then*

$$E(\xi^m\,|\,U(0)=y_0)=m!\int_0^\infty\cdots\int_0^\infty\exp\left[-\sum_{i=1}^m(y_i-y_{i-1})^+\right]dy_1\ldots dy_m,\qquad m\geqq 1. \tag{1.1}$$

Proof. By the standard method of evaluating "Wiener integrals", we find that the left hand side of (1.1) is

$$m! \int_{0=s_0<s_1<\cdots<s_m<\infty} \cdots \int \int_{s_1}^{\infty} \cdots \int_{s_m}^{\infty} \prod_{i=1}^{m} \left\{ [2(s_i-s_{i-1})]^{-\frac{1}{2}} \phi\left(\frac{y_i-y_{i-1}}{\sqrt{2(s_i-s_{i-1})}} \right) \right\} dy_m \ldots dy_1\, ds_1 \ldots ds_m.$$

Change the inner variables of integration from y_i to y_i-s_i, and invert the order of integration:

$$m! \int_0^{\infty} \cdots \int_0^{\infty} \int_{0=s_0<s_1<\cdots<s_m<\infty} \cdots \int \prod_{i=1}^{m} \left\{ [2(s_i-s_{i-1})]^{-\frac{1}{2}} \phi\left(\frac{y_i-y_{i-1}}{\sqrt{2(s_i-s_{i-1})}} + \sqrt{(s_i-s_{i-1})/2} \right) \right\} \cdot ds_m \ldots ds_1\, dy_1 \ldots dy_m.$$

Integrate over $s_m, \ldots, s_1$:

$$m! \int_0^{\infty} \cdots \int_0^{\infty} \prod_{i=1}^{m} \left\{ \int_0^{\infty} \frac{1}{\sqrt{2s}} \phi\left(\frac{y_i-y_{i-1}}{\sqrt{2s}} + \sqrt{s/2} \right) ds \right\} dy_1 \ldots dy_m.$$

It follows from the form of ϕ that this is equivalent to

$$m! \int_0^{\infty} \cdots \int_0^{\infty} \exp\left[-\frac{1}{2} \sum_{i=1}^{m} (y_i-y_{i-1}) \right] \prod_{i=1}^{m} \left\{ \int_0^{\infty} \frac{e^{-s/4}}{\sqrt{2s}} \phi\left(\frac{y_i-y_{i-1}}{\sqrt{2s}} \right) ds \right\} dy_1 \ldots dy_m. \tag{1.2}$$

From the well known Laplace transform equation

$$\int_0^{\infty} e^{-st} s^{-\frac{1}{2}} \phi(x/\sqrt{s})\, ds = (2t)^{-\frac{1}{2}} e^{-\sqrt{2t}|x|},$$

it follows that

$$\int_0^{\infty} \frac{e^{-s/4}}{\sqrt{2s}} \phi\left(\frac{y_i-y_{i-1}}{\sqrt{2s}} \right) ds = e^{-\frac{1}{2}|y_i-y_{i-1}|};$$

thus; (1.2) is equal to

$$m! \int_0^{\infty} \cdots \int_0^{\infty} \exp\left\{ -\frac{1}{2} \sum_{i=1}^{m} [(y_i-y_{i-1}) + |y_i-y_{i-1}|] \right\} dy_1 \ldots dy_m.$$

This is equal to the right hand side of (1.1).

Lemma 1.2. *There is a unique distribution having the moment sequence* (1.1) *for* $y_0=0$.

Proof.

$$\begin{aligned} E(\xi^m | U(0)=0) &= \int_0^{\infty} \cdots \int_0^{\infty} P\{\sqrt{2}\, U(s_i) > s_i,\ i=1,\ldots,m | U(0)=0\}\, ds_1 \ldots ds_m \\ &= m \int_0^{\infty} \int_0^{s_m} \cdots \int_0^{s_m} P\{\sqrt{2}\, U(s_i) > s_i,\ i=1,\ldots,m | U(0)=0\}\, ds_1 \ldots ds_m \\ &\leqq m \int_0^{\infty} s^{m-1} P\{\sqrt{2}\, U(s) > s | U(0)=0\}\, ds \\ &= m \int_0^{\infty} s^{m-1} [1-\Phi(\sqrt{s/2})]\, ds. \end{aligned}$$

From this bound and the well known inequality,

$$1-\Phi(x)\leqq\phi(x)/x, \qquad x>0, \tag{1.3}$$

it follows that the series

$$\sum_{m=0}^{\infty}\frac{z^m}{m!}E(\xi^m|U(0)=0)$$

converges for $|z|<\frac{1}{4}$, so that there is a unique distribution with these moments ([2], p. 176).

This is a consequence of (1.1):

$$E(\xi^m|U(0)=0)=m\int_0^\infty E(\xi^{m-1}|U(0)=y)\,e^{-y}\,dy. \tag{1.4}$$

When $\alpha=2$, the process $U(t)$ is equivalent to ηt, where η is a random variable with a standard normal distribution. It follows that ξ is the length of the interval $(0, t)$ for which $s^2-\sqrt{2}\eta s<0$, $0<s<t$; therefore, $\xi=\sqrt{2}\eta^+$, and so its distribution has mass $\frac{1}{2}$ at the origin and a half-normal density on the positive axis.

The two-sided occupation time

$$\int_{-\infty}^{\infty} I_{[\sqrt{2}\eta t>t^2]}\,dt \tag{1.5}$$

is shown, in the same way, to be equal to $\sqrt{2}|\eta|$.

2. Conditional Limiting Distribution of the Occupation Time Given $X(0)=u$

Theorem 2.1. *Let $X(t)$ be a standard process satisfying* (0.1); *then the conditional distribution of $u^{2/\alpha}\gamma L$ given $X(0)=u$ converges to the distribution of ξ.*

Proof. There is no loss of generality in assuming that $\gamma=1$. For any $\gamma>0$ the process $X(t/\gamma)$ has the covariance $r(t/\gamma)$ which satisfies (0.1) with $\gamma=1$; furthermore:

$$u^{2/\alpha}\gamma\int_0^T I_{[X(s)>u]}\,ds=u^{2/\alpha}\int_0^{T\gamma} I_{[X(s/\gamma)>u]}\,ds.$$

If the conclusion of the theorem holds for the right hand member (for all $T>0$) then it holds for the left hand member for all $\gamma>0$.

For arbitrary $c>0$ put

$$\xi_c(u)=u^{2/\alpha}\int_0^T e^{-csu^{2/\alpha}}I_{[X(s)>u]}\,ds$$

$$\xi_c=\int_0^\infty e^{-cs}I_{[\sqrt{2}U(s)>s^\alpha]}\,ds.$$

We shall prove that the conditional distribution of $\xi_c(u)$ converges to that of ξ_c for $u\to\infty$. Since $\xi_c(u)$ and ξ_c are bounded (by $1/c$) it suffices to show that the conditional moments converge.

The process $X(t)$ is conditionally Gaussian given $X(0)=u$, and

$$E(X(t)|X(0)=u)=u\,r(t) \tag{2.1}$$

$$\text{covariance}\,(X(s), X(t)|X(0)=u)=r(s-t)-r(s)\,r(t);$$

consequently:

$$E\{u(X(t\,u^{-2/\alpha})-u)|X(0)=u\}\to -t^{\alpha},$$
$$\operatorname{Var}\{u(X(t\,u^{-2/\alpha})-X(s\,u^{-2/\alpha}))|X(0)=u\}\to 2|s-t|^{\alpha},$$

for $u\to\infty$; therefore, the conditional finite-dimensional distributions of $u(X(t\,u^{-2/\alpha})-u)$ converge to those of the process $\sqrt{2}\,U(t)-t^{\alpha}$. It follows (by dominated convergence) that

$$\begin{aligned}
&E(\xi_c^m(u)|X(0)=u)\\
&=\int_0^{Tu^{2/\alpha}}\cdots\int_0^{Tu^{2/\alpha}} e^{-c\sum_{i=1}^m s_i} P\{u(X(s_i\,u^{-2/\alpha})-u)>0,\ i=1,\dots,m|X(0)=u\}\,ds_1\dots ds_m\\
&\to\int_0^{\infty}\cdots\int_0^{\infty} e^{-c\sum_{i=1}^m s_i} P\{\sqrt{2}\,U(s_i)>s_i^{\alpha},\ i=1,\dots,m\}\,ds_1\dots ds_m\\
&=E(\xi_c^m).
\end{aligned}$$

To complete the proof it suffices to show that the distribution of ξ_c converges to that of ξ as $c\to 0$, and that the conditional distribution of $\xi_c(u)$ converges to that of $u^{2/\alpha}L$ uniformly in u. The first assertion follows from the fact that ξ_c increases as $c\downarrow 0$, and that its limit (by dominated convergence) is ξ. To verify the second assertion we observe that $u^{2/\alpha}L$ is at least equal to $\xi_c(u)$, and

$$E(|u^{2/\alpha}L-\xi_c(u)|\,|X(0)=u)=\int_0^{Tu^{2/\alpha}}(1-e^{-cs})\,P\{X(s\,u^{-2/\alpha})>u|X(0)=u\}\,ds,$$

which by (2.1), is equal to

$$\int_0^{Tu^{2/\alpha}}(1-e^{-cs})\left[1-\Phi\left(u\sqrt{\frac{1-r(s\,u^{-2/\alpha})}{1+r(s\,u^{-2/\alpha})}}\right)\right]ds. \tag{2.2}$$

Under (0.1) there exists a positive constant B such that $1-r(t)\geqq B\,t^{\alpha}$ for all $0\leqq t\leqq T$; therefore, the integral (2.2) is at most equal to

$$\int_0^{\infty}(1-e^{-cs})\,[1-\Phi(\sqrt{B\,s^{\alpha}/2})]\,ds,$$

which is independent of u and which tends to 0 with c. The proof is complete.

Let us now consider the two-sided occupation time

$$u^{2/\alpha}\int_{T_1}^{T_2} I_{[X(s)>u]}\,ds, \qquad T_1<0<T_2 \tag{2.3}$$

where the time parameter now includes a portion of the negative axis. By means of the same proof given above, with the modifications that $[T_1, T_2]$ and $e^{-c|s|}$ are used in place of $[0, T]$ and e^{-cs}, respectively, it can be shown that the conditional distribution of (2.3), given $X(0)=u$, converges to that of

$$\int_{-\infty}^{\infty} I_{[\sqrt{2}U(s)>|s|^{\alpha}]}\, ds. \tag{2.4}$$

The rest of this section is about the special case $\alpha=2$. In [1] it is shown that the conditional distribution of $u\gamma L$, given $L>0$, converges to the Rayleigh distribution. This also the limiting distribution under "horizontal window" conditioning, described in [4]. When $\alpha=2$, the random variable (2.4) has the same distribution as (1.5), which has a half-normal distribution. This is the limiting distribution under the "vertical window" conditioning in [4]. Our results apply to a larger class of processes than those in [4] because we do not assume the differentiability of the sample functions.

The limiting distribution of (2.3) may be interpreted as the limiting distribution of the length of an excursion above u—even when the sample functions are not differentiable. Under (0.1) with $\alpha=2$, $X(t)$ assumes the value u at most finitely many times in each interval, almost surely; furthermore, every t for which $X(t)$ is equal to u is a "crossing" point, not a "tangency" [8], and [3], p. 199. It follows that if $X(0)=u$ then there is an excursion above u either immediately before or after $t=0$.

3. An Asymptotic Formula for the Distribution of the Maximum of a Standard Process when $\alpha=1$

As a preliminary to the main theorem of the next section we derive an asymptotic formula for the tail of the distribution of the maximum of a standard process satisfying (0.1) with $\alpha=1$. The condition that $r(t)$ is nonperiodic is equivalent to

$$r(t)=1 \quad \text{if and only if } t=0. \tag{3.1}$$

Theorem 3.1. *Let $X(t)$, $t\geqq 0$, be a standard process satisfying* (0.1) *with $\alpha=1$, and also* (3.1); *then, for any $T>0$:*

$$P\{\max(X(t)\colon 0\leqq t\leqq T)>u\}\sim T\gamma u\phi(u), \qquad u\to\infty. \tag{3.2}$$

The plan of the proof is similar to that of Theorem 3.1 of [1]. It is also related to [5], Section 4. The proof will be completed after several preliminary lemmas.

As in the proof of Theorem 2.1 we can, by the substitution of t/γ for t, reduce the case of arbitrary γ to the special case $\gamma=1$; therefore, the hypothesis (0.1) becomes

$$1-r(t)\sim|t|, \qquad t\to 0. \tag{3.3}$$

We recall Slepian's lemma [7], which was used in [1]: if X and Y are Gaussian processes on a common index set I, and with common means 0 and common variances, and if $EX(s)X(t)\leqq EY(s)Y(t)$ for all $s, t\in I$, then

$$P\{\max(X(t)\colon t\in I)>u\}\leqq P\{\max(Y(t)\colon t\in I)>u\}. \tag{3.4}$$

Lemma 3.1. *Let $V(t)$ be a standard process with the covariance function*

$$(1-|t|)^+; \tag{3.5}$$

then, for all T, $0<T\leqq 1$, and all x,

$$\begin{aligned} P\{\max(V(t)\colon\ 0\leqq t\leqq T)\geqq x\} &= 1-\Phi(x) \\ &+x\,\phi(x)\int_0^T \Phi\big(x\sqrt{s/(2-s)}\big)\,ds \\ &+\phi(x)\int_0^T \sqrt{(2-s)/s}\,\phi\big(x\sqrt{s/(2-s)}\big)\,ds. \end{aligned} \tag{3.6}$$

Proof. The maximum of V is at least equal to x if either $V(0)\geqq x$ or $V(0)<x$ and $V(s)=x$ for some s, $0<s\leqq T$; thus, the probability in (3.6) is equal to

$$1-\Phi(x)+\int_\infty^x P\{V(s)=x \text{ for some } s,\ 0<s\leqq T\,|\,V(0)=y\}\,\phi(y)\,dy.$$

By a result of Slepian [6], the conditional probability in the integrand is equal to

$$\int_0^T \frac{|x-y|}{t}\,\frac{1}{[t(2-t)]^{\frac{1}{2}}}\,\phi\left(\frac{x-y(1-t)}{[t(2-t)]^{\frac{1}{2}}}\right)dt.$$

Multiply this by $\phi(y)$, integrate with respect to y, and invert the order of integration:

$$\int_0^T t^{-1}[t(2-t)]^{-\frac{1}{2}}\left\{\int_\infty^x (x-y)\,\phi\left(\frac{x-y(1-t)}{[t(2-t)]^{\frac{1}{2}}}\right)\phi(y)\,dy\right\}dt.$$

The inner integral can be explicitly evaluated by using the relations

$$\phi\left(\frac{x-y(1-t)}{[t(2-t)]^{\frac{1}{2}}}\right)\phi(y)=\phi\left(\frac{y-x(1-t)}{[t(2-t)]^{\frac{1}{2}}}\right)\phi(x)$$

and

$$\int_\infty^x y\,\phi(y)\,dy=-\phi(x).$$

The verification of (3.6) follows some elementary calculations.

Lemma 3.2. *Let X be a standard process satisfying* (3.3); *then for every ε, $0<\varepsilon<1$, there exists $\tau>0$ such that*

$$\begin{aligned} &P\{\max(V(t)\colon\ 0\leqq t\leqq T(1-\varepsilon))>u\} \\ &\quad\leqq P\{\max(X(t)\colon\ 0\leqq t\leqq T)>u\} \\ &\quad\leqq P\{\max(V(t)\colon\ 0\leqq t\leqq T(1+\varepsilon))>u\} \end{aligned} \tag{3.7}$$

for all u and all $0<T\leqq\tau$.

Proof. By (3.3), for every ε, $0<\varepsilon<1$, there exists $\tau>0$ such that

$$(1-|t|(1+\varepsilon))^+\leqq r(t)\leqq(1-|t|(1-\varepsilon))^+ \tag{3.8}$$

for all $|t|<\tau$. The maximum of $V(t)$ on $[0, T(1\pm\varepsilon)]$ is equivalent to the maximum of the process $V(t(1\pm\varepsilon))$ on $[0, T]$. The latter process has the covariance function $(1-|t|(1\pm\varepsilon))^+$. If $T\leqq\tau$ then, by (3.8), the covariance of X dominates that of $V(t(1+\varepsilon))$, and is dominated by that of $V(t(1-\varepsilon))$ on $[0, T]$. This and (3.4) imply (3.7).

Lemma 3.3. *Let T be fixed and let m be a positive integer. Then there exist positive numbers K and σ such that:*

If $J_1, \dots, J_m$ are closed subintervals of $[0, T]$ each of length $h<\sigma$, and J is the union of $J_1, \dots, J_m$, then

$$\limsup_{u\to\infty} \frac{P\{\max(X(t)\colon\ t\in J)>u\}}{P\{\max(X(t)\colon\ 0\leqq t\leqq T)>u\}}\leqq Kh. \tag{3.9}$$

Proof. For any arbitrary but fixed ε, $0<\varepsilon<1$, let σ be the number τ in Lemma 3.2. By Boole's inequality and stationarity:

$$P\{\max(X(t)\colon\ t\in J)>u\}\leqq m P\{\max(X(t)\colon\ 0\leqq t\leqq h)>u\}. \tag{3.10}$$

Since σ may be taken arbitrarily small we suppose that it is also smaller than T:

$$P\{\max(X(t)\colon\ 0\leqq t\leqq T)>u\}\geqq P\{\max(X(t)\colon\ 0\leqq t\leqq\sigma)>u\}. \tag{3.11}$$

It follows from Lemma 3.2 and from (3.10) and (3.11) that the ratio in (3.9) is at most equal to

$$m\frac{P\{\max(V(t)\colon\ 0\leqq t<h(1+\varepsilon))>u\}}{P\{\max(V(t)\colon\ 0\leqq t\leqq\sigma(1-\varepsilon))>u\}}. \tag{3.12}$$

Since σ may be taken arbitrarily small we suppose also that $\sigma(1+\varepsilon)<1$, and apply Lemma 3.1. (Even though there is a *strict* inequality on $\max V(t)$ in (3.12) the probability is the same as for $\max V(t)\geqq u$ because the function is (3.6) is continuous.) It follows from (3.6) and (1.3) that the ratio (3.12) converges to Kh for $u\to\infty$, where $K=m(1+\varepsilon)/\sigma(1-\varepsilon)$.

Lemma 3.4. *For every positive T and δ:*

$$\limsup_{u\to\infty} \frac{P\left\{u<\max(X(t)\colon\ 0<t<T)\leqq u+\dfrac{\delta}{u}\right\}}{Tu\phi(u)}\leqq\delta. \tag{3.13}$$

Proof. The maximum of a function on a union of disjoint intervals falls between u and $u+\delta/u$ only if the same holds for the maximum over at least one of the intervals; thus, by Boole's inequality and stationarity:

$$\begin{aligned} &P\{u<\max(X(t)\colon\ 0\leqq t\leqq T)\leqq u+\delta/u\} \\ &\quad\leqq m P\{u<\max(X(t)\colon\ 0\leqq t\leqq T/m)\leqq u+\delta/u\} \end{aligned} \tag{3.14}$$

for any positive integer m. For arbitrary ε, $0<\varepsilon<1$, let m be so large that

$$T/m\leqq\min(\tau, (1+\varepsilon)^{-1}),$$

where τ is the number in Lemma 3.2. The lemma implies:

$$P\{u<\max(X(t)\colon 0\leqq t\leqq T/m)\leqq u+\delta/u\}$$
$$\leqq P\{\max(V(t)\colon 0\leqq t\leqq T(1+\varepsilon)/m)>u\} \tag{3.15}$$
$$-P\{\max(V(t)\colon 0\leqq t\leqq T(1-\varepsilon)/m)>u+\delta/u\}.$$

Divide the right hand side by $Tu\phi(u)$ and let $u\to\infty$: by Lemma 3.1 and the estimate (1.3), the quotient converges to

$$(T/m)[1+\varepsilon-(1-\varepsilon)e^{-\delta}].$$

From (3.14), (3.15), and the arbitrariness of ε we conclude that the lim sup in (3.13) is at most equal to $1-e^{-\delta}$, which is smaller than δ.

In the following two lemmas it is shown that the maximum of X over the set $[0, T]$ is asymptotically equivalent to the maximum over a sufficiently dense finite subset.

Lemma 3.5. *For any $\delta>0$ and $T>0$:*

$$\lim_{u\to\infty}\frac{1}{u\,\phi(u)}P\{\max(X(t)\colon 0\leqq t\leqq T)>u+\frac{\delta}{u},$$
$$\max(X(j\,u^{-5})\colon 0\leqq j\leqq[T u^5])\leqq u\}=0. \tag{3.16}$$

Proof. By stationarity and Boole's inequality the probability in (3.16) is at most

$$T u^5 P\left\{X(0)\leqq u, X(u^{-5})\leqq u, \max(X(t)\colon 0\leqq t\leqq u^{-5})>u+\frac{\delta}{u}\right\}. \tag{3.17}$$

The event described in (3.17) implies that for some $n\geqq 1$ and some k, $1\leqq k\leqq 2^n$, and every c, $0<c<1$:

$$X(k\,2^{-n}u^{-5})-X((k-1)\,2^{-n}u^{-5})>\delta c^{n-1}(1-c)/u;$$

indeed, if the alternative inequality held for every n and k, and for some c, $0<c<1$, then

$$\sup_{k,n} X(k\,2^{-n}u^{-5})\leqq u+\delta/u,$$

so that, by continuity of X:

$$\max(X(t)\colon 0\leqq t\leqq u^{-5})\leqq u+\delta/u.$$

It follows that (3.17) is bounded above by

$$T u^5\sum_{n=1}^{\infty}2^n\left[1-\Phi\left(\frac{\delta c^{n-1}(1-c)}{\sqrt{2}\,u[1-r(2^{-n}u^{-5})]^{\frac{1}{2}}}\right)\right]. \tag{3.18}$$

Let u be so large that, by (3.3),

$$1-r(u^{-5})\leqq 2u^{-5};$$

then (3.18) is at most equal to

$$T u^5\sum_{n=1}^{\infty}2^n[1-\Phi(K u^{\frac{3}{2}}(c\sqrt{2})^n)],$$

where $K=\delta(1-c)/2c$. By (1.3) the expression above is at most

$$\frac{Tu^{\frac{7}{2}}}{K\sqrt{2\pi}}\sum_{n=1}^{\infty}(\sqrt{2}/c)^n \exp\left[-\frac{1}{2}K^2 u^3(2c^2)^n\right],$$

which, by the Cauchy-Schwarz inequality, is at most

$$\frac{Tu^{\frac{7}{2}}}{K\sqrt{2\pi}}\left\{\sum_{n=1}^{\infty}(2/c^2)^n e^{-K^2(2c^2)^n}\right\}^{\frac{1}{2}}\left\{\sum_{n=1}^{\infty}e^{-K^2(u^3-1)(2c^2)^n}\right\}^{\frac{1}{2}}. \tag{3.19}$$

Each of the series in (3.19) converges if $c>1/\sqrt{2}$ and $u>1$. Divide the expression (3.19) by $u\phi(u)$ and then let $u\to\infty$: the quotient is

$$\text{constant}\cdot u^{\frac{5}{2}}\left\{\sum_{n=1}^{\infty}\exp\left[-K(u^3-1)(2c^2)^n+u^2\right]\right\}^{\frac{1}{2}},$$

which converges to 0 because the term with u^3 dominates the exponent.

Lemma 3.6. *Let $I_1,\dots,I_m$ be disjoint closed subintervals of $[0,T]$, and I their union. For $u>0$, let G_k be the intersection of I_k with the set of numbers of the form ju^{-5}, where $j=0,1,\dots$; and let G be the union of $G_1,\dots,G_m$. Then:*

$$\lim_{u\to\infty}\frac{P\{\max(X(t)\colon t\in G)>u\}}{P\{\max(X(t)\colon t\in I)>u\}}=1. \tag{3.20}$$

Proof. Since $G\subset I$, it suffices to prove:

$$\lim_{u\to\infty}\frac{P\{\max(X(t)\colon t\in I)>u, \max(X(t)\colon t\in G)\leqq u\}}{P\{\max(X(t)\colon t\in I)>u\}}=0. \tag{3.21}$$

For arbitrary $\delta>0$ the numerator in (3.21) may be written as the sum of

$$P\{\max(X(t)\colon t\in I)>u+\delta u^{-1}, \max(X(t)\colon t\in G)\leqq u\} \tag{3.22}$$

and a term not more than

$$P\{u<\max(X(t)\colon t\in I)\leqq u+\delta u^{-1}\}. \tag{3.23}$$

By Lemma 3.5 the probability (3.22) is of smaller order than $u\phi(u)$. By Lemma 3.4, the ratio of (3.23) to $u\phi(u)$ has a lim sup not exceeding $T\delta$. Now by the reasoning in the proof of Lemma 3.3, the denominator in (3.21) is at least equal to a positive constant multiple of $u\phi(u)$. It follows that the lim sup of the ratio in (3.21) is at most a positive constant multiple of δ. Since δ is arbitrary, (3.21) follows.

For $u>0$ the maximum of X on the finite set G is equivalent to the maximum of the m sub-maxima over $G_1,\dots,G_m$, respectively. Now we shall prove that in estimating $P\{\max(X(t)\colon t\in G)>u\}$ for $u\to\infty$ we may suppose that the submaxima are mutually independent random variables.

Lemma 3.7.

$$\lim_{u\to\infty}\frac{1-\prod_{k=1}^{m}P\{\max(X(t)\colon t\in G_k)\leqq u\}}{P\{\max(X(t)\colon t\in G)>u\}}=1.$$

Proof. We prove this by a slight alteration of the proof of Lemma 3.4 of [1]. Put u^4 in place of $g(u)$ in that proof. By a similar argument it suffices to show that for arbitrary b, $0<b<1$, the expression $u^{10}\exp[-u^2/(2-b)]$ is of smaller order than $u\phi(u)$ for $u\to\infty$. This fact is easy to verify.

We now complete the

Proof of Theorem 3.1. The reasoning is analogous to that of Theorem 3.1 of [1]. Cut the interval $[0, T]$ into m equal sub-intervals; then clip a relatively small open segment from the right end of each so that they are mutually separated. By Lemma 3.3 the distribution of the maximum over $[0, T]$ is asymptotically nearly the same as the maximum over the union of the remaining disjoint closed subintervals. By Lemma 3.6, the maximum over these intervals is asymptotically equivalent to the maximum over a finite subset of density u^{-5}. By Lemma 3.7 the maxima over the various intervals may be considered independent random variables with a common distribution, and the upper tail of this distribution is approximately $Tu\phi(u)/m$ for large u. By the argument in [1] this is sufficient for (3.2).

4. Conditional Limiting Distribution of $u^2\gamma L$ Given $L>0$

Theorem 4.1. *Let $X(t)$ be a standard process such that r is nonperiodic and satisfies* (0.1) *for $\alpha=1$. Then the conditional distribution of $u^2\gamma L$, given $L>0$, converges to the distribution of the random variable ξ (where $U(t)$ is Brownian motion).*

Proof. By Lemma 1.2, it suffices to show that the conditional moments of $u^2\gamma L$ converge to the moments of ξ. By previous arguments, as in the proof of Theorem 2.1, it is sufficient to consider just the case $\gamma=1$; therefore, (0.1) takes the form (3.3).

For any nonnegative random variable X the conditional m-th moment, given that $X>0$, is $E(X^m)/P\{X>0\}$; therefore,

$$E[(u^2 L)^m | L>0] = \frac{E\left[u^2 \int_0^T I_{[X(s)>u]}\, ds\right]^m}{P\{\max(X(s):\ 0\leqq s\leqq T)>u\}},$$

which, by Theorem 4.1, is asymptotic to

$$E\left[u^2 \int_0^T I_{[X(s)>u]}\, ds\right]^m \Big/ Tu\phi(u) \tag{4.1}$$

for $u\to\infty$.

For $m=1$, (4.1) is equal to

$$u(1-\Phi(u))/\phi(u),$$

which, by the well known estimate associated with (1.3), converges to 1 as $u\to\infty$.

We now take $m\geqq 2$. The expression (4.1) is equal to

$$\frac{1}{Tu\phi(u)} \int_0^{Tu^2} \cdots \int_0^{Tu^2} P\{X(t_i u^{-2})>u, i=1, \ldots, m\}\, dt_1 \ldots dt_m.$$

Since the integrand above is a symmetric function of $t_1, \dots, t_m$, the integral is equal to

$$\frac{m}{Tu\phi(u)} \int_0^{Tu^2} \int_0^{t_m} \cdots \int_0^{t_m} P\{X(t_i u^{-2}) > u, i=1, \dots, m\} \, dt_1 \dots dt_m. \tag{4.2}$$

By the total probability formula, the integrand above is equal to

$$\int_u^\infty P\{X(t_i u^{-2}) > u, i=1, \dots, m-1 | X(t_m u^{-2}) = y\} \, \phi(y) \, dy,$$

which, by stationarity, is equal to

$$\int_u^\infty P\{X((t_i - t_m) u^{-2}) > u, i=1, \dots, m-1 | X(0) = y\} \, \phi(y) \, dy. \tag{4.3}$$

When (4.3) is inserted in (4.2) and integrated with respect to t_i over $[0, t_m]$, the variable of integration may be changed from t_i to $t_m - t_i$, $i=1, \dots, m-1$; therefore, (4.2) becomes

$$\frac{m}{Tu\phi(u)} \int_0^{Tu^2} \int_0^{t_m} \cdots \int_0^{t_m} \int_u^\infty P\{X(t_i u^{-2}) > u, i=1, \dots, m-1 | X(0) = y\} \, \phi(y) \, dy \, dt_1 \dots dt_m.$$

After the substitution $z = u(y-u)$ this becomes

$$\frac{m}{Tu^2} \int_0^{Tu^2} \int_0^{t_m} \cdots \int_0^{t_m} \int_0^\infty P\{X(t_i u^{-2}) > u, i=1, \dots, m-1 | X(0) = u + z u^{-1}\} \cdot e^{-z - z^2/2u^2} \, dz \, dt_1 \dots dt_m. \tag{4.4}$$

Let us now formally take limits in (4.4), both in the limits of integration and in the integrand; this will be rigorously justified later. The conditional joint distribution of $u[X(t_i u^{-2}) - u]$, $i=1, \dots, m-1$, given $X(0) = u + z u^{-1}$ is Gaussian with

$$E\{[u(X(t_i u^{-2})) - u] | X(0) = u + z u^{-1}\} = u^2 [r(t_i u^{-2}) - 1] + z r(t_i u^{-2})$$

and conditional covariances

$$u^2 \left[r\left(\frac{t_i - t_j}{u^2}\right) - r\left(\frac{t_i}{u^2}\right) r\left(\frac{t_j}{u^2}\right) \right], \quad i, j = 1, \dots, m-1.$$

As in the proof of Theorem 2.1 the joint distribution converges to that of $\sqrt{2}\, U(t_i) - t_i + z$, $i=1, \dots, m-1$, which is the finite dimensional distribution of the process, $\sqrt{2}\, U(t) - t$, conditioned by $U(0) = z$. Put this limiting probability in (4.4), and $t_m = \infty$ and $\exp(-z^2/2u^2) = 1$; then (4.4) is equal to

$$m \int_0^\infty E\left\{ \left[\int_0^\infty I_{[\sqrt{2}U(s) > s]} \, ds \right]^{m-1} \middle| U(0) = z \right\} e^{-z} \, dz,$$

which, by (1.4), is equal to $E\xi^m$.

Now we justify the foregoing limit operation on (4.4). Since the integrand is a symmetric function of $t_1, \ldots, t_{m-1}$, and since it does not depend on t_m, the expression (4.4) is equal to

$$m \int_0^{Tu^2} \left(1-\frac{t_{m-1}}{Tu^2}\right) \int_0^{t_{m-1}} \cdots \int_0^{t_{m-1}} \int_0^{\infty} P\{X(t_i u^{-2})>u, i=1, \ldots, m-1 | X(0)=u+zu^{-1}\} \cdot e^{-z-z^2/2u^2} dz\, dt_1 \ldots dt_{m-1}. \tag{4.5}$$

Put $D(t)=1$ for $t>0$, and $=0$ for $t \leqq 0$. Since the integrand in (4.5) is a symmetric function of $t_1, \ldots, t_{m-1}$, (4.5) is equivalent to

$$m! \int_0^{\infty} \cdots \int_0^{\infty} \prod_{i=1}^{m-1} D(t_i - t_{i-1}) \cdot \left(1-\frac{t_{m-1}}{Tu^2}\right)^+ \cdot P\{X(t_i u^{-2})>u, i=1, \ldots, m-1 | X(0)=u+zu^{-1}\} e^{-z-z^2/2u^2} dz\, dt_1 \ldots dt_{m-1}. \tag{4.6}$$

The integrand converges pointwise for $u \to \infty$; thus, for the convergence of the integral it is enough to show that the integrand in (4.6) is dominated by an integrable function. Such a function is

$$\prod_{i=1}^{m-1} D(t_i - t_{i-1})\, e^{-z(1-c)-c(1-c)Bt_{m-1}}, \tag{4.7}$$

where $0<c<1$ and $B>0$. To prove that this dominates the integrand, we note that the integrand is at most

$$D(Tu^2 - t_{m-1}) \cdot \prod_{i=1}^{m-1} D(t_i - t_{i-1})\, P\{X(t_{m-1} u^{-2})>u | X(0)=u+zu^{-1}\}\, e^{-z}. \tag{4.8}$$

If X is a normally distributed random variable with mean μ and σ^2, then, for any $c>0$,

$$P\{X>0\}=P\{e^{cX}>1\} \leqq E\, e^{cX} = e^{c\mu + \frac{1}{2}c^2\sigma^2}.$$

From this and the form of the conditional distribution of $X(t_{m-1} u^{-2})$ given $X(0)$ we find that

$$\begin{aligned} &P\{X(t_{m-1} u^{-2})>u | X(0)=u+zu^{-1}\} \\ &\quad = P\{u[X(t_{m-1} u^{-2})-u]>0 | X(0)=u+zu^{-1}\} \\ &\quad \leqq \exp\{czr - cu^2(1-r) + \tfrac{1}{2}c^2 u^2 (1-r^2)\} \\ &\quad \leqq \exp\{czr - c(1-c)u^2(1-r)\}, \end{aligned} \tag{4.9}$$

where $r=r(t_{m-1} u^{-2})$. As in the proof of Theorem 2.1 there exists a positive constant B such that $1-r(t) \geqq Bt$ for $0 \leqq t \leqq T$; therefore, from (4.9) we see that (4.8) is dominated by (4.7).

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