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On Strassen's Version of the Law of the Iterated Logarithm for Gaussian Processes

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1. Introduction

In [8] Strassen presented the following version of the law of the iterated logarithm for the Brownian motion process $\{B(t,\omega), 0 \leqq t < \infty\}$. Define, for each ω, a sequence of functions $\{f_n(t,\omega), n \geqq 3\}$ in $C[0,1]$ with the usual sup norm $\|\cdot\|_C$ by

$$f_n(t,\omega) = B(nt,\omega)/(2n \log\log n)^{\frac{1}{2}}, \qquad 0 \leqq t \leqq 1, \qquad n = 3, 4, \ldots. \tag{1}$$

Let K be the set of all absolutely continuous functions $h \in C[0,1]$ such that

$$\int_0^1 (dh/dt)^2 \, dt \leqq 1.$$

Theorem (Strassen). *For almost every (a.e.) ω, the set of limit points of the sequence of functions $\{f_n(t,\omega), n \geqq 3\}$ coincides with the set K.*

Basing on this theorem and making use of Skorokhod representation theorem, Strassen further proved an invariance principle for the classical law of the iterated logarithm. Later Chover [2] gave a proof of Strassen's main result by using Esseen's estimate for the central limit theorem. An extension of the result to some classes of stationary random sequences satisfying mixing conditions has been given in [7].

The purpose of this paper is to generalize the above theorem of Strassen to a certain class of Gaussian processes including the Brownian motion process. Observe that the set K appearing as the set of limit points of $\{f_n(t,\omega)\}$ is the unit ball of reproducing kernel (r.k.) Hilbert space corresponding to the Brownian motion process. Thus, if we consider an analogous sequence of functions $\{f_n(t,\omega)\}$ for a Gaussian process $\{X(t,\omega), 0 \leqq t < \infty\}$, then we might expect that the set of limit points of $\{f_n(t,\omega)\}$ is characterized as a bounded set K of the r.k. Hilbert space corresponding to $\{X(t)\}$. In this paper we shall show that this is indeed the case under some conditions on $\{X(t)\}$. Precise statements of conditions and results will be given in the next section.

2. Results

Let $\{X(t,\omega), 0 \leqq t < \infty\}$ be a separable, measurable, real valued Gaussian process defined on a probability space $(\Omega, \mathscr{F}, P)$, with $X(0) \equiv 0$, $EX(t) \equiv 0$ and covariance kernel $R(s,t) = EX(s)X(t)$. Put $\sigma^2(t) = R(t,t)$.

The followint conditions will be assumed.

Condition (I). For any $T>0$, there exists a positive, nondecreasing function $g(h, T)$, $h>0$, such that

$$|R(t+h, t+h)-2R(t+h, t)+R(t, t)| \leqq g(h, T) \to 0 \quad \text{as} \quad h \to 0, \qquad \text{for all } t, t+h \in [0, T], \tag{2}$$

$$\{g(1, T)\}^{-\frac{1}{2}} \int_1^\infty g^{\frac{1}{2}}(e^{-u^2}, T)\, du \leqq C < \infty, \tag{3}$$

and

$$\sigma^2(T)/g(1, T) \uparrow \infty \quad \text{as} \quad T \to \infty. \tag{4}$$

Condition (II). There is a positive function $v(r)$, $r>0$, such that $v(r) \uparrow \infty$ and

$$R(r s, r t) = v(r) R(s, t) \qquad \text{for all } r>0, s, t \geqq 0. \tag{5}$$

Condition (II′). $R(s, t)$ has a representation of the form

$$R(s, t) = \int_0^{s \wedge t} Q(s, \lambda) Q(t, \lambda)\, d\lambda, \qquad 0 \leqq s,\ t < \infty, \tag{6}$$

where $\int_0^t Q^2(t, \lambda)\, d\lambda < \infty$ for all $t \geqq 0$ and there is a function $u(r)$ such that $Q(r t, r \lambda) = u(r) Q(t, \lambda)$ for all $r>0$, $t, \lambda \geqq 0$ and $v(r) = r u^2(r) \uparrow \infty$ as $r \to \infty$, and further

$$\sup_{0 \leqq t \leqq 1} \int_0^\delta Q^2(t, \lambda)\, d\lambda \to 0 \quad \text{as} \quad \delta \to 0. \tag{7}$$

Examples. Gaussian processes having covariance kernels

$$R(s, t) = \int_0^{s \wedge t} (s-\lambda)^\beta (t-\lambda)^\beta\, d\lambda, \qquad -1/2 < \beta < \infty, \tag{8}$$

satisfy Conditions (I) and (II′), and hence (II). This class includes the Brownian motion process $\{B(t)\}$ (with $\beta=0$) and the process $\left\{\int_0^t B(u)\, du\right\}$ (with $\beta=1$). Similarly, processes with $Q(t, \lambda) = p(t) q(\lambda)$, e.g., $p(t)=t$, $q(\lambda)=1$, satisfy Conditions (I) and (II′). Processes with stationary increments having covariance kernels

$$R(s, t) = (\tfrac{1}{2})\{s^\alpha + t^\alpha - |s-t|^\alpha\}, \qquad 0 < \alpha \leqq 2, \tag{9}$$

satisfy Conditions (I) and (II).

Remark. Under Condition (I) processes $\{X(t), 0 \leqq t \leqq T\}$ have continuous sample paths a.e. for any $T>0$ (Fernique [3]).

Define, for each $\omega \in \Omega$, a sequence of functions $\{f_n(t, \omega), n \geqq 3\}$ in $C[0, 1]$ by

$$f_n(t, \omega) = X(n t, \omega)/(2\sigma^2(n) \log\log n)^{\frac{1}{2}}, \qquad 0 \leqq t \leqq 1,\ n=3, 4, \ldots \tag{10}$$

Let $H(R_1)$ be the r.k. Hilbert space with reproducing kernel (r.k.) $R(s, t)$, $0 \leqq s, t \leqq 1$. We refer to Aronszajn [1] for the theory of reproducing kernels. Define the set K by

$$K = \{h \in H(R_1) \mid \|h\|_H \leqq 1/\sigma(1)\}, \tag{11}$$

where $\|\cdot\|_H$ denotes the norm of $H(R_1)$. Note that $H(R_1) \subset C[0, 1]$ since R is assumed to be continuous.

Our main results are the following:

Theorem 1. *If Conditions* (I) *and* (II) *are fulfilled, then, for a.e.* $\omega \in \Omega$, *the sequence of functions* $\{f_n(t, \omega), n \geqq 3\}$ *is equicontinuous.*

Theorem 2. *Under the same assumptions as in Theorem* 1 *the set of limit points of the sequence of functions* $\{f_n(t, \omega)\}$ *for a.e.* ω *is contained in the set* K.

Theorem 3. *If Conditions* (I) *and* (II′) *are satisfied, then, for a.e.* ω, *the set of limit points of* $\{f_n(t, \omega)\}$ *contains the set* K.

From Theorems 2 and 3 we have

Theorem 4. *If* $\{X(t, \omega)\}$ *satisfies Conditions* (I) *and* (II′), *then, for a.e.* ω, *the set of limit points of* $\{f_n(t, \omega)\}$ *coincides with the set* K.

3. Proof of Theorem 1

The proof is similar to that of Chover [2] except a use of Fernique's lemma [3]. We show that for any $\varepsilon > 0$ there is a $\delta = \delta(\varepsilon) > 0$ such that for a.e. ω and for some integer $N = N(\omega) \geqq 3$, we have

$$|f_n(t, \omega) - f_n(s, \omega)| < \varepsilon \tag{12}$$

if $|t-s| < \delta$ and $n \geqq N$. Let $q = q(\varepsilon)$ be an integer, which will be specified later on, and put $\delta(\varepsilon) = 2^{-q}$. From the definition of f_n, (12) may be written as

$$|X(n\,t) - X(n\,s)| < \varepsilon\big(2\sigma^2(n) \log\log n\big)^{\frac{1}{2}}, \tag{13}$$

where $|t-s| < \delta = 2^{-q}, 0 \leqq t, s \leqq 1$.

Let

$$A_n = \Big\{\omega \,\Big|\, \sup_{\substack{|t-s| < 2^{-q} \\ 0 \leqq s, t \leqq 1}} |X(n\,t) - X(n\,s)| \geqq \varepsilon\big(2\sigma^2(n) \log\log n\big)^{\frac{1}{2}}\Big\}. \tag{14}$$

It suffices to show that $P(\limsup_n A_n) = 0$. Consider the subsequence $\{n_r = 2^r, r \geqq \max(q, 3)\}$ and let

$$B_r = \Big\{\omega \,\Big|\, \max_{2^r \leqq n < 2^{r+1}} \sup_{\substack{|t-s| < 2^{-q} \\ 0 \leqq s, t \leqq 1}} |X(n\,t) - X(n\,s)| \geqq \varepsilon\big(2\sigma^2(2^r) \log\log 2^r\big)^{\frac{1}{2}}\Big\}. \tag{15}$$

Then it is enough to prove that $P(\limsup_r B_r) = 0$. Let

$$C_r = \Big\{\omega \,\Big|\, \sup_{\substack{0 \leqq h \leqq 2^{r+1-q} \\ 0 \leqq t < t+h \leqq 2^{r+1}}} |X(t+h) - X(t)| \geqq \varepsilon\big(2\sigma^2(2^r) \log\log 2^r\big)^{\frac{1}{2}}\Big\} \tag{16}$$

and

$$C_r^{(\nu)} = \Big\{\omega \,\Big|\, \sup_{t, t+h \in I(r, \nu)} |X(t+h) - X(t)| \geqq \varepsilon\big(2\sigma^2(2^r) \log\log 2^r\big)^{\frac{1}{2}}\Big\}, \tag{17}$$

where

$$I(r, \nu) = [(\nu - 1)\,2^{r-q+1}, (\nu+1)\,2^{r-q+1}], \qquad \nu = 1, 2, \ldots, 2^q - 1.$$

Since $B_r \subset C_r \subset \bigcup_{\nu=1}^{2^q-1} C_r^{(\nu)}$, it suffices to show that for each fixed ν $P\big(\limsup_r C_r^{(\nu)}\big) = 0$.

Let

$$D_r^{(\nu)} = \{\omega \mid \sup_{t \in I(r,\nu)} |X(t) - X(t_\nu)| \geqq (\varepsilon/2)(2\sigma^2(2^r)\log\log 2^r)^{\frac{1}{2}}\}, \tag{18}$$

where $t_\nu = (\nu-1)\,2^{r-q+1}$. Then we have $P(C_r^{(\nu)}) \leqq 2P(D_r^{(\nu)})$. To evaluate $P(D_r^{(\nu)})$ we need the following lemma due to Fernique [3].

Lemma (Fernique). *Let $\{Y(t), 0 \leqq t \leqq 1\}$ be a continuous, separable, real valued Gaussian process with mean zero and continuous covariance $\Gamma(s,t)$. Suppose that $E\{Y(t) - Y(s)\}^2 \leqq \Psi^2(|t-s|)$ and that $\Psi(h)$, $h \geqq 0$, is positive and increasing. Then for all positive integer p and all $x \geqq (1 + 4\log p)^{\frac{1}{2}}$, we have*

$$P\left\{\|Y\|_C \geqq x\left[\|\Gamma\|_C^{\frac{1}{2}} + 4\int_1^\infty \Psi(p^{-u^2})\,du\right]\right\} \leqq 4p^2 \int_x^\infty e^{-u^2/2}\,du, \tag{19}$$

where $\|\cdot\|_C$ is the sup *norm.*

Remark. A similar probability bound obtained by Marcus [6] may also be used.

To apply Fernique's lemma, let

$$Y(s) = X(s \cdot 2^{r-q+2} + t_\nu), \qquad 0 \leqq s \leqq 1. \tag{20}$$

Then

$$\begin{aligned} E\{Y(t) - Y(s)\}^2 &= E\{X(t \cdot 2^{r-q+2} + t_\nu) - X(s \cdot 2^{r-q+2} + t_\nu)\}^2 \\ &= v(2^{r-q+2})\left\{R\left(t + \frac{\nu-1}{2}, t + \frac{\nu-1}{2}\right)\right. \\ &\quad \left. - 2R\left(t + \frac{\nu-1}{2}, s + \frac{\nu-1}{2}\right) + R\left(s + \frac{\nu-1}{2}, s + \frac{\nu-1}{2}\right)\right\} \\ &\leqq v(2^{r-q+2})\, g(|t-s|, 2^{q-1}) \end{aligned} \tag{21}$$

and

$$\begin{aligned} |\Gamma(t,s)| &\leqq \{E[X(t \cdot 2^{r-q+2} + t_\nu) - X(t_\nu)]^2\}^{\frac{1}{2}} \{E[X(s \cdot 2^{r-q+2} + t_\nu) - X(t_\nu)]^2\}^{\frac{1}{2}} \\ &\leqq v(2^{r-q+2})\, g^{\frac{1}{2}}\left(t, \frac{\nu+1}{2}\right) g^{\frac{1}{2}}\left(s, \frac{\nu+1}{2}\right) \\ &\leqq v(2^{r-q+2})\, g(1, 2^{q-1}). \end{aligned} \tag{22}$$

Hence we have

$$\begin{aligned} P(D_r^{(\nu)}) &= \{\omega \mid \|Y\|_C \geqq (\varepsilon/2)(2\sigma^2(2^r)\log\log 2^r)^{\frac{1}{2}}\} \\ &\leqq 4p^2 \int_{y_r}^\infty e^{-u^2/2}\,du, \end{aligned} \tag{23}$$

where

$$\begin{aligned} y_r &= (\varepsilon/2)(2\log\log 2^r)^{\frac{1}{2}}\{\sigma(2^r)\, v^{-\frac{1}{2}}(2^{r-q+2})\, g^{-\frac{1}{2}}(1, 2^{q-1})\} \\ &\quad \cdot \left\{1 + 4v^{-\frac{1}{2}}(2^{r-q+2})\, g^{-\frac{1}{2}}(1, 2^{q-1}) \int_1^\infty v^{\frac{1}{2}}(2^{r-q+2})\, g^{\frac{1}{2}}(p^{-u^2}, 2^{q-1})\,du\right\}^{-1}. \end{aligned} \tag{24}$$

By the assumptions (3) and (5),

$$\begin{aligned} &1+4g^{-\frac{1}{2}}(1,2^{q-1})\int_1^\infty g^{\frac{1}{2}}(p^{-u^2},2^{q-1})\,du \\ &\quad=1+4g^{-\frac{1}{2}}(1,2^{q-1})(\log p)^{-\frac{1}{2}}\int_{(\log p)^{\frac{1}{2}}}^\infty g^{\frac{1}{2}}(e^{-u^2},2^{q-1})\,du \\ &\quad=C_1<\infty \end{aligned} \tag{25}$$

and $\sigma(2^r)\,v^{-\frac{1}{2}}(2^{r-q+2})\,g^{-\frac{1}{2}}(1,2^{q-1})=\sigma(2^{q-2})\,g^{-\frac{1}{2}}(1,2^{q-1})$, and hence

$$y_r=(\varepsilon/2\,C_1)\{\sigma(2^{q-2})\,g^{-\frac{1}{2}}(1,2^{q-1})\}(2\log\log 2^r)^{\frac{1}{2}}\to\infty \quad \text{as} \quad r\to\infty. \tag{26}$$

Choose q sufficiently large such that

$$(\varepsilon/2\,C_1)^2\,\sigma^2(2^{q-2})\,g^{-1}(1,2^{q-1})=\varepsilon'>1, \tag{27}$$

which is possible because of the assumption (4). Thus we have

$$P(C_r^{(\nu)})\leqq 8p^2\int_{y_r}^\infty e^{-u^2/2}\,du\leqq C'(\log 2^r)^{-\varepsilon'}\leqq C''\,r^{-\varepsilon'} \tag{28}$$

and $\sum_r P(C_r^{(\nu)})<\infty$. Hence, by the Borel-Cantelli lemma, $P(\limsup_r C_r^{(\nu)})=0$. This completes the proof.

The following corollary can be proved in a similar way as Corollary 2 of [2], and hence the proof is omitted.

Corollary. *For any $\varepsilon>0$ there is a $\delta=\delta(\varepsilon)$ such that for a.e. ω and for some integer $N=N(\varepsilon)$ we have $\|f_m-f_n\|_C<\varepsilon$ for all $m,n\geqq N$ with $|1-(m/n)|<\delta$.*

4. Approximation Lemma

To prove Theorems 2 and 3 we shall approximate a subsequence $\{f_{n_r}(t,\omega)\}$ by a sequence of functions in $H(R_1)$ obtained by taking partial sums of norm convergent expansion of $\{X(t)\}$ (see [4], [5]). The key lemma is the following Lemma 1.

Consider the r.k. Hilbert spaces $H(R_1)$ and $H(R_n)$ with r.k. $R_n=R(s,t)$, $0\leqq s$, $t\leqq n$. From the assumption (5) it follows that

$$\begin{aligned} \langle R(*,n\,t),R(*,n\,s)\rangle_n &=R(n\,t,n\,s)=v(n)\,R(s,t) \\ &=v(n)\,\langle R(.,t),R(.,s)\rangle_1 \\ &=\langle v^{\frac{1}{2}}(n)\,R(.,t),v^{\frac{1}{2}}(n)\,R(.,s)\rangle_1 \qquad \text{for } 0\leqq s,\ t\leqq 1, \end{aligned} \tag{29}$$

where $\langle.,.\rangle_1$ and $\langle.,.\rangle_n$ denote respectively the inner products of $H(R_1)$ and $H(R_n)$. (29) implies that there is an isometric isomorphism θ_n from $H(R_1)$ to $H(R_n)$ such that

$$\theta_n\big(v^{\frac{1}{2}}(n)\,R(.,t)\big)=R(*,n\,t), \qquad 0\leqq t\leqq 1. \tag{30}$$

Note that for any $h\in H(R_1)$

$$\begin{aligned} \theta_n\,h(n\,t) &=\langle\theta_n\,h(*),R(*,n\,t)\rangle_n \\ &=v^{\frac{1}{2}}(n)\,\langle h(.),R(.,t)\rangle_1 \\ &=v^{\frac{1}{2}}(n)\,h(t) \qquad \text{for } 0\leqq t\leqq 1, \end{aligned} \tag{31}$$

and if $\{e_j(.), j=1,2,\ldots,J\}$ is a system of orthonormal functions in $H(R_1)$, so is $\{e_{nj}(*)=\theta_n e_j(*), j=1,2,\ldots,J\}$ in $H(R_n)$.

It is well known that there is an isometric isomorphism between $H(R_n)$ and the closed linear manifold $L_n^2(X)$ spanned by $\{X(t), 0\leqq t\leqq n\}$, and if $\xi_{nj}, j=1,2,\ldots,J$, are the random variables $\in L_n^2(X)$ corresponding to orthonormal functions $e_{nj}(*)$, $j=1,2,\ldots,J$, then ξ_{nj} are independent and normally distributed with mean zero and variance one.

Lemma 1 *(Approximation Lemma). Suppose that a sequence of families of orthonormal functions $\{e_j^{(k)}(.),\ j=1,2,\ldots,J_k<\infty\}$, $k=1,2,\ldots$, in $H(R_1)$ satisfies the following condition:*

$$\sup_{0\leqq t\leqq 1}\left|R(t,t)-\sum_{j=1}^{J_k}\{e_j^{(k)}(t)\}^2\right|\to 0 \quad \text{as} \quad k\to\infty. \tag{32}$$

Let $\{\xi_{nj}^{(k)}\}$ be the Gaussian random variables corresponding to $\{e_{nj}^{(k)}(*)=\theta_n e_j^{(k)}(*)\}$. Then, for any geometric subsequence of indices $\{n_r=[c^r], c>1\}$ and any $\varepsilon>0$, there exist for a.e. ω some integers $k_0=k_0(\varepsilon)$ and $r_0=r_0(\varepsilon,\omega)$ such that

$$\sup_{0\leqq t\leqq 1}\left|f_{n_r}(t,\omega)-(2\sigma^2(1)\log\log n_r)^{-\frac{1}{2}}\sum_{j=1}^{J_k}\xi_{n_r j}^{(k)}(\omega)\,e_j^{(k)}(t)\right|<\varepsilon \tag{33}$$

for all $k\geqq k_0$ and all $r\geqq r_0$.

Remark. Let $\{e_j(.), j=1,2,\ldots\}$ be any complete orthonormal system (CONS) in $H(R_1)$. It is known [5] that the partial sums $\sum_{j=1}^{k} e_j^2(t)$ converge to $R(t,t)$ uniformly in $t\in[0,1]$. Hence the condition (32) is satisfied for the families $\{e_j(\cdot), j=1,2,\ldots,k\}$, $k=1,2,\ldots$, and we have, for all sufficiently large k and r,

$$\sup_{0\leqq t\leqq 1}\left|f_{n_r}(t,\omega)-(2\sigma^2(1)\log\log n_r)^{-\frac{1}{2}}\sum_{j=1}^{k}\xi_{n_r j}\,e_j(t)\right|<\varepsilon. \tag{34}$$

Proof. Let

$$\begin{aligned} A_r^{(k)}&=\left\{\omega\,\middle|\,\sup_{0\leqq t\leqq 1}\left|f_{n_r}(t,\omega)-(2\sigma^2(1)\log\log n_r)^{-\frac{1}{2}}\sum_{j=1}^{J_k}\xi_{n_r j}^{(k)}(\omega)\,e_j^{(k)}(t)\right|\geqq\varepsilon\right\}\\ &=\left\{\omega\,\middle|\,\sup_{0\leqq t\leqq 1}\left|v^{-\frac{1}{2}}(n_r)\,X(n_r t)-\sum_{j=1}^{J_k}\xi_{n_r j}^{(k)}\,e_j^{(k)}(t)\right|\geqq\varepsilon(2\sigma^2(1)\log\log n_r)^{\frac{1}{2}}\right\}\end{aligned} \tag{35}$$

and put

$$Y_{n_r}^{(k)}(t)=v^{-\frac{1}{2}}(n_r)\,X(n_r t)-\sum_{j=1}^{J_k}\xi_{n_r j}^{(k)}\,e_j^{(k)}(t),\qquad 0\leqq t\leqq 1. \tag{36}$$

Then $EY_{n_r}^{(k)}(t)=0$ and, noting that

$$E\{X(n_r t)\,\xi_{n_r j}^{(k)}\}=\langle R(*,n_r t), e_{n_r j}^{(k)}(*)\rangle_n=e_{n_r j}^{(k)}(n_r t)=v^{\frac{1}{2}}(n_r)\,e_j^{(k)}(t),$$

we have

$$\Gamma^{(k)}(s,t)=EY_{n_r}^{(k)}(s)\,Y_{n_r}^{(k)}(t)=R(s,t)-\sum_{j=1}^{J_k}e_j^{(k)}(s)\,e_j^{(k)}(t). \tag{37}$$

Since

$$
\begin{aligned}
E\{Y_{n_r}^{(k)}(t)-Y_{n_r}^{(k)}(s)\}^2 &= E\{v^{-\frac{1}{2}}(n_r)[X(n_r t)-X(n_r s)]\}^2-\sum_{j=1}^{J_k}\{e_j^{(k)}(t)-e_j^{(k)}(s)\}^2 \\
&\leq E\{v^{-\frac{1}{2}}(n_r)[X(n_r t)-X(n_r s)]\}^2 \\
&= R(t,t)-2R(t,s)+R(s,s) \\
&\leq g(|t-s|,1)
\end{aligned}
\tag{38}
$$

and

$$
\begin{aligned}
|\Gamma^{(k)}(s,t)| &\leq \{\Gamma^{(k)}(s,s)\}^{\frac{1}{2}}\{\Gamma^{(k)}(t,t)\}^{\frac{1}{2}} \\
&\leq \sup_{0\leq t\leq 1}\Gamma^{(k)}(t,t),
\end{aligned}
\tag{39}
$$

we can apply Fernique's lemma to obtain

$$P(A_r^{(k)})\leq 4p^2\int_{y_r^{(k)}}^{\infty} e^{-u^2/2}\,du, \tag{40}$$

where

$$y_r^{(k)}=\varepsilon(2\sigma^2(1)\log\log n_r)^{\frac{1}{2}}\{[\sup_{0\leq t\leq 1}\Gamma^{(k)}(t,t)]^{\frac{1}{2}}+4\int_1^{\infty} g^{\frac{1}{2}}(p^{-u^2},1)\,du\}^{-1}. \tag{41}$$

We may choose k and p sufficiently large such that

$$\varepsilon'=\varepsilon^2\sigma^2(1)\left\{[\sup_{0\leq t\leq 1}\Gamma^{(k)}(t,t)]^{\frac{1}{2}}+4\int_1^{\infty} g^{\frac{1}{2}}(p^{-u^2},1)\,du\right\}^{-1}>1, \tag{42}$$

because of the condition (32) and

$$\int_1^{\infty} g^{\frac{1}{2}}(p^{-u^2},1)\,du=(\log p)^{-\frac{1}{2}}\int_{(\log p)^{\frac{1}{2}}}^{\infty} g^{\frac{1}{2}}(e^{-u^2},1)\,du\to 0 \quad \text{as} \quad p\to\infty.$$

Then

$$P(A_r^{(k)})\leq C(\log c^r)^{-\varepsilon'}=C'\,r^{-\varepsilon'} \tag{43}$$

and $\sum_r P(A_r^{(k)})<\infty$ for all sufficiently large k. By the Borel-Cantelli lemma we obtain the desired conclusion.

5. Proof of Theorem 2

Let K_ε denote the ε-neighborhood of K. To prove that K contains all limit points of $\{f_n(t,\omega)\}$ it suffices to show that for arbitrary $\varepsilon>0$ the sequence $\{f_n(t,\omega)\}$ ultimately lies in $K_{3\varepsilon}$. Consider a subsequence of indices $\{n_r=[c^r], c>1\}$. Then, for any n, there are n_r and n_{r+1} such that $n_r\leq n<n_{r+1}$, and choosing $c=c(\varepsilon)$ sufficiently close to 1, we can make $|1-(n/n_r)|$ arbitrary small. Thus, by Corollary to Theorem 1, it is sufficient to show that the subsequence $\{f_{n_r}(t,\omega)\}$ ultimately lies in $K_{2\varepsilon}$. Then, by the remark following Lemma 1, it suffices to prove that

$$Z(t,\omega,k,n_r)=(2\sigma^2(1)\log\log n_r)^{-\frac{1}{2}}\sum_{j=1}^{k}\xi_{n_r j}(\omega)\,e_j(t) \tag{44}$$

with a sufficiently large k ultimately lies in K_ε. Finally it is enough to show that $\|Z\|_H \leqq (1+\varepsilon)/\sigma(1)$ ultimately, for then $(1+\varepsilon)^{-1} Z \in K$ and since

$$\|Z-(1+\varepsilon)^{-1} Z\|_C = \varepsilon(1+\varepsilon)^{-1} \|Z\|_C \leqq \varepsilon(1+\varepsilon)^{-1} \|Z\|_H \sup_{0 \leqq t \leqq 1} R^{\frac{1}{2}}(t,t) \leqq \varepsilon$$

we have $Z \in K_\varepsilon$.

Let

$$\begin{aligned} A_r &= \{\omega \,|\, \|Z(\cdot, \omega, k, n_r)\|_H^2 > (1+\varepsilon)^2 \sigma^{-2}(1)\} \\ &= \left\{\omega \,\middle|\, \left\| \sum_{j=1}^{k} \xi_{n_r j}(\omega)\, e_j(\cdot) \right\|_H^2 > (1+\varepsilon)^2 (2 \log\log n_r)\right\} \\ &= \left\{\omega \,\middle|\, \sum_{j=1}^{k} \{\xi_{n_r j}(\omega)\}^2 > (1+\varepsilon)^2 (2\log\log n_r)\right\}. \end{aligned} \tag{45}$$

If $\Psi_k(x)$ denotes the distribution function of χ^2-distribution with k degrees of freedom, we have

$$\begin{aligned} P(A_r) &= 1 - \Psi_k\big((1+\varepsilon)^2 (2\log\log n_r)\big) \\ &\leqq C\{(1+\varepsilon)^2 \log\log n_r\}^{k-1} (\log n_r)^{-(1+\varepsilon)^2} \\ &\leqq C' r^{-(1+\varepsilon)^2}, \end{aligned} \tag{46}$$

and hence, by the Borel-Cantelli lemma, $P\big(\limsup_r A_r\big) = 0$. This concludes the proof.

6. Proof of Theorem 3

First we prove the sup norm compactness of bounded sets of any r.k. Hilbert space $H(R)$ with continuous r.k. $R(s,t)$, $0 \leqq s, t \leqq 1$, and hence, in particular, that of the set K. We shall write $Q_1 \ll Q_2$ if the difference $Q_2 - Q_1$ of any two kernels Q_2 and Q_1 is nonnegative definite and denote by $\|\cdot\|_{H(Q)}$ the norm of r.k. Hilbert space $H(Q)$ with r.k. Q.

Lemma 2. *Let $f \in C[0,1]$ and $F(s,t) = f(s) f(t)$, and let a be a positive constant. If $F \ll a^2 R$, then $f \in H(R)$ and $\|f\|_{H(R)} \leqq a$. Conversely, if $\|f\|_{H(R)} \leqq a$, then $F \ll a^2 R$.*

Proof. If $F \ll a^2 R$, then $f \in H(F) \subset H(a^2 R) = H(R)$ (set theoretically) and

$$\|f\|_{H(F)} \geqq \|f\|_{H(a^2 R)} = a^{-1} \|f\|_{H(R)} \quad \text{(see [1])}.$$

Since

$$f^2(t) = F(t,t) = \|F(\cdot, t)\|_{H(F)}^2 = \|f(\cdot) f(t)\|_{H(F)}^2 = f^2(t) \|f\|_{H(F)}^2,$$

we have $a^{-1} \|f\|_{H(R)} \leqq \|f\|_{H(F)} = 1$. The latter half is obvious.

Lemma 3. *The set $K_a = \{h \in H(R) \,|\, \|h\|_{H(R)} \leqq a\}$ is compact in $C[0,1]$.*

Proof. The relative compactness of K_a is well known. That K_a is closed is easily shown by applying Lemma 2.

To prove that K is contained in the set of limit points of the sequence of functions $\{f_n(t,\omega)\}$ for a.e. ω, it suffices to show, because of compactness of K, that for any $h \in K$ and for any $\varepsilon > 0$, there are, for a.e. ω, infinitely many $f_{n_r}(t,\omega)$ in some subsequences $\{f_{n_r}(t,\omega)\}$ such that $\|f_{n_r} - h\|_C < 3\varepsilon$. To prove it we shall approximate $\{f_{n_r}(t,\omega)\}$ and h in the following way.

The assumption (6) implies that $H(R_1)$ is isometrically isomorphic to the L^2-space on $[0,1]$, $L^2_R[0,1]$, spanned by the family of functions $\{\chi(t,\lambda)\,Q(t,\lambda),\ 0\leqq t\leqq 1\}$, where $\chi(t,\lambda)=1$ for $\lambda\leqq t$ and 0 for $\lambda>t$.

For any $0<\delta<1$, define the kernel $R_\delta(s,t)$, $0\leqq s,t\leqq 1$, by

$$R_\delta(s,t)=\int_\delta^1 \chi(s,\lambda)\,Q(s,\lambda)\,\chi(t,\lambda)\,Q(t,\lambda)\,d\lambda, \tag{47}$$

and let

$$R_\delta^*(s,t)=R(s,t)-R_\delta(s,t), \qquad 0\leqq s,\ t\leqq 1. \tag{48}$$

$H(R_\delta)$ and $H(R_\delta^*)$ are isometrically isomorphic to the subspaces $L^2_R[\delta,1]$ and $L^2_R[0,\delta]$ of $L^2_R[0,1]$ spanned by

$$\{(1-\chi(\delta,\lambda))\,\chi(t,\lambda)\,Q(t,\lambda),0\leqq t\leqq 1\} \quad \textit{and} \quad \{\chi(\delta,\lambda)\,\chi(t,\lambda)\,Q(t,\lambda),0\leqq t\leqq 1\},$$

respectively, and hence $H(R_1)=H(R_\delta)\oplus H(R_\delta^*)$. Take any CONS $\{e_j(.)\}$ in $H(R_\delta)$. The convergence of $\sum_{j=1}^m e_j^2(t)$ to $R_\delta(t,t)$ is uniform in $t\in[0,1]$, and also, by the assumption (7), $\sup_{0\leqq t\leqq 1} R_\delta^*(t,t)\to 0$ as $\delta\to 0$. Therefore, first choosing δ sufficiently small and then taking a CONS $\{e_j(.)\}$ in $H(R_\delta)$ and m sufficiently large, we can make $\sup_{0\leqq t\leqq 1}\left|R(t,t)-\sum_{j=1}^m e_j^2(t)\right|$ arbitrary small. Hence, by Lemma 1, for any $\varepsilon>0$ and for any geometric subsequence $\{n_r=[c^r], c>1\}$, we have

$$\left\| f_{n_r}(t,\omega)-(2\sigma^2(1)\log\log n_r)^{-\frac12}\sum_{j=1}^m \xi_{n_r j}(\omega)\,e_j(t)\right\|_C<\varepsilon \tag{49}$$

for a.e. ω and for all r sufficiently large. Let $\{e_i^*(.)\}$ be a CONS in $H(R_\delta^*)$. Then $R_\delta^*(t,t)=\sum_{i=1}^\infty e_i^{*2}(t)$ and $h\in K$ has the expansion

$$h(t)=\sum_{j=1}^\infty h_j e_j(t)+\sum_{i=1}^\infty h_i^* e_i^*(t) \quad \text{with} \quad \sum_{j=1}^\infty h_j^2+\sum_{i=1}^\infty h_i^{*2}\leqq 1/\sigma^2(1).$$

Let $h_m(t)=\sum_{j=1}^m h_j e_j(t)$. Then

$$\begin{aligned} |h(t)-h_m(t)| &\leqq \left|\sum_{j=m+1}^\infty h_j e_j(t)\right|+\left|\sum_{i=1}^\infty h_i^* e_i^*(t)\right| \\ &\leqq \left(\sum_{j=m+1}^\infty h_j^2\right)^{\frac12}\left(\sum_{j=m+1}^\infty e_j^2(t)\right)^{\frac12}+\left(\sum_{i=1}^\infty h_i^{*2}\right)^{\frac12}\left(\sum_{i=1}^\infty e_i^{*2}(t)\right)^{\frac12}. \end{aligned} \tag{50}$$

Therefore, again choosing δ sufficiently small and then m sufficiently large, we can make, for any $\varepsilon>0$,

$$\|h-h_m\|_C<\varepsilon. \tag{51}$$

Let δ and m with a CONS $\{e_j(.)\}$ in $H(R_\delta)$ be so chosen that (49) and (51) hold. Then it is sufficient to show that for a.e. ω there are infinitely many n_r such that

$$\left\|(2\sigma^2(1)\log\log n_r)^{-\frac{1}{2}}\sum_{j=1}^{m}\xi_{n_r j}(\omega)\,e_j(t)-\sum_{j=1}^{m}h_j\,e_j(t)\right\|_C<\varepsilon. \tag{52}$$

We take the subsequence of indices $\{n_r=[(2/\delta)^r]\}$.

Let

$$A_r=\left\{\omega\,\middle|\,\left\|(2\sigma^2(1)\log\log n_r)^{-\frac{1}{2}}\sum_{j=1}^{m}\xi_{n_r j}(\omega)\,e_j(t)-\sum_{j=1}^{m}h_j\,e_j(t)\right\|_C<\varepsilon\right\} \tag{53}$$

and

$$B_r^{(j)}=\{\omega\,|\,\|\{\xi_{n_r j}(\omega)-\sigma(1)\,h_j(2\log\log n_r)^{\frac{1}{2}}\}\,e_j(t)\|_C<(\varepsilon/m)(2\sigma^2(1)\log\log n_r)^{\frac{1}{2}}\}. \tag{54}$$

Then $A_r\supset\bigcap_{j=1}^{m}B_r^{(j)}$. Noting that $\|e_j\|_C\leqq\|e_j\|_H\sup_{0\leqq t\leqq 1}R^{\frac{1}{2}}(t,t)=\sigma(1)$, let

$$C_r^{(j)}=\{\omega\,|\,|\xi_{n_r j}(\omega)-\sigma(1)\,h_j(2\log\log n_r)^{\frac{1}{2}}|<(\varepsilon/m)(2\log\log n_r)^{\frac{1}{2}}\}. \tag{55}$$

Then $B_r^{(j)}\supset C_r^{(j)}$, and hence it is enough to show that for each fixed j, $1\leqq j\leqq m$, $P(\limsup_r C_r^{(j)})=1$.

Let $\{\varphi_j,\, j=1,2,\dots,m\}$ denote the orthonormal functions in $L_R^2[\delta,1]$ corresponding to $\{e_j(.),\, j=1,2,\dots,m\}$. Then

$$\begin{aligned} e_{n_r j}(n_r t)&=v^{\frac{1}{2}}(n_r)\,e_j(t)=v^{\frac{1}{2}}(n_r)\int_0^t\varphi_j\,Q(t,\lambda)\,d\lambda\\ &=v^{\frac{1}{2}}(n_r)\int_0^{n_r t}\varphi_j(\mu/n_r)\,Q(t,\mu/n_r)\,n_r^{-1}\,d\mu\\ &=\int_0^{n_r t}n_r^{-\frac{1}{2}}\,\varphi_j(\mu/n_r)\,Q(n_r t,\mu)\,d\mu\qquad\text{for } 0\leqq t\leqq 1.\end{aligned} \tag{56}$$

Put $\varphi_{n_r j}(\mu)=n_r^{-\frac{1}{2}}\,\varphi_j(\mu/n_r)$, $0\leqq\mu\leqq n_r$. (56) shows that $\varphi_{n_r j}(\mu)$ corresponds to $e_{n_r j}(*)$ under the isometric isomorphism from $H(R_{n_r})$ to $L^2_{R[0,n_r]}$, the L^2-space spanned by functions $\{\chi(t,\mu)\,Q(t,\mu),\ 0\leqq t\leqq n_r\}$. If $p<r$, $L_R^2\,[0,n_p]$ can be regarded as a subspace of $L_R^2[0,n_r]$ in the obvious manner, and $\varphi_{n_p j}(\mu)$ corresponds to $\varphi^*_{n_p j}(\mu)=\varphi_{n_p j}(\mu)$ for $0\leqq\mu\leqq n_p$ and 0 for $n_p\leqq\mu\leqq n_r$. Accordingly, $H(R_{n_p})\subset H(R_{n_r})$ and $e_{n_p j}(.)\in H(R_{n_p})$ corresponds to

$$e^*_{n_p j}(.)=\int_0^{n_r}\varphi^*_{n_p j}(\mu)\,Q(.,\mu)\,d\mu\in H(R_{n_r}).$$

Therefore the random variable $\xi_{n_p j}\in L^2_{n_p}(X)\subset L^2_{n_r}(X)$ corresponding to $e_{n_p j}(.)$ also corresponds to $e^*_{n_p j}(.)$ under the isometric isomorphism from $H(R_{n_r})$ to $L^2_{n_r}(X)$. Now we have

$$\begin{aligned} E\,\xi_{n_p j}\,\xi_{n_r j}&=\langle e^*_{n_p j},e_{n_r j}\rangle_{n_r}=\int_0^{n_r}\varphi^*_{n_p j}(\mu)\,\varphi_{n_r j}(\mu)\,d\mu\\ &=\int_0^{n_p}\varphi_{n_p j}(\mu)\,\varphi_{n_r j}(\mu)\,d\mu=(n_p n_r)^{-\frac{1}{2}}\int_0^{n_p}\varphi_j(\mu/n_p)\,\varphi_j(\mu/n_r)\,d\mu\\ &=(n_p/n_r)^{\frac{1}{2}}\int_0^1\varphi_j(\lambda)\,\varphi_j((n_p/n_r)\,\lambda)\,d\lambda\\ &=0,\end{aligned} \tag{57}$$

since $n_p/n_r \leqq \delta$ and $\varphi_j(\lambda)=0$ for $0 \leqq \lambda \leqq \delta$. Thus, for each j, $1 \leqq j \leqq m$, $\{\xi_{n_r j}\}$ is a sequence of independent Gaussian random variables with mean 0 and variance 1.

Let Φ denote the distribution function of standard normal distribution. We have

$$\begin{aligned} P(C_r^{(j)}) &= \Phi\big(\{\sigma(1)\, h_j + (\varepsilon/m)\}(2 \log\log n_r)^{\frac{1}{2}}\big) \\ &\quad - \Phi\big(\{\sigma(1)\, h_j - (\varepsilon/m)\}(2 \log\log n_r)^{\frac{1}{2}}\big) \\ &\geqq \Phi\big(\{|\sigma(1)\, h_j| + (2\varepsilon/m)\}(2 \log\log n_r)^{\frac{1}{2}}\big) \\ &\quad - \Phi\big(|\sigma(1)\, h_j|(2 \log\log n_r)^{\frac{1}{2}}\big) \\ &\geqq C(\log\log n_r)^{-\frac{1}{2}} \exp\big(-|\sigma(1)\, h_j|^2 \log\log n_r\big), \end{aligned} \tag{58}$$

and, since $\sigma^2(1)\, h_j^2 \leqq 1$,

$$\begin{aligned} P(C_r^{(j)}) &\geqq C(\log\log n_r)^{-\frac{1}{2}} \exp(-\log\log n_r) \\ &= C(\log\log n_r)^{-\frac{1}{2}}(\log n_r)^{-1} \\ &= C'(r \log r)^{-1} \end{aligned} \tag{59}$$

and $\sum_r P(C_r^{(j)}) = \infty$. Since $\{C_r^{(j)}\}$ are independent, by the Borel-Cantelli lemma, we obtain $P(\limsup_r C_r^{(j)}) = 1$. This completes the proof.

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