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Compactness Conditions on Markov Semi-Groups

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1. Introduction

An earlier paper, [1], was concerned with the theory of strongly continuous semi-groups of operators $\{T_t;\, t\geqq 0\}$ satisfying the condition

$$T_t - I \quad \textit{compact for some } t>0. \tag{1}$$

Let $\{P_t;\, t\geqq 0\}$ be a Markov semi-group of operators on l_1; that is, $\{P_t\}$ is the semi-group formed by the transition probabilities of a time-homogeneous, standard, discrete state Markov process: let $\{P_t\}$ have infinitesimal generator A. Then it was shown in [1] that if $\{P_t\}$ satisfies (1), A is bounded, and in fact compact.

It is natural to ask whether A must possess a corresponding compactness property when condition (1) is modified to

$$P_t - kI \qquad \textit{compact for some } k>0 \quad \textit{and} \quad t>0,$$

or, more generally,

$$P_t - \operatorname{diag}\{p_{11}(t), p_{22}(t), \ldots\} \qquad \textit{compact, and } \inf_i p_{ii}(t)>0, \qquad \textit{for some } t>0. \tag{2}$$

This paper is concerned with examining this question, in a rather more general context.

In addition to using compactness, we introduce two other types of operator on l_1, which we call the "*sc*" and "*sc**" operators, defined in terms of the uniform convergence of the row and column elements of the operator to zero. We also define a class, $\mathscr{S}$ say, of simple operators, containing, besides others, those invertible operators conformed like permutation operators. Then it is shown that, if $\{P_t\}$ satisfies the condition

$$P_t - S \qquad \textit{compact, sc, or sc* for some } t>0, \quad \textit{and} \quad S\in\mathscr{S}, \tag{3}$$

then S is, apart from a possible finite number of terms, a diagonal operator, and the infinitesimal generator A is necessarily bounded, and satisfies an appropriate condition in terms of compactness, or the *sc* or *sc** properties.

2. Preliminaries and Results

Throughout this paper we shall be concerned with a Markov semi-group $\{P_t;\, t\geqq 0\}$ of bounded linear operators on the Banach space l_1 of absolutely convergent sequences: that is, the matrix P_t, $t \geqq 0$, consists of transition probabilities $p_{ij}(t)$ satisfying

(i) $p_{ij}(t) \geqq 0$, and $\sum_j p_{ij}(t) = 1$ for all $t \geqq 0$;

(ii) $p_{ij}(s+t) = \sum_k p_{ik}(s)\, p_{kj}(t)$ for all $s,\ t \geqq 0$;

(iii) $\lim_{t \to 0+} p_{ij}(t) = \delta_{ij}$.

P_t acts on the space l_1 according to the formula

$$[P_t x]_j = \sum_i x_i\, p_{ij}(t), \qquad x \in l_1.$$

It will be assumed throughout that all operators act by right multiplication in this way.

We recall that a bounded operator $B = [b_{ij}]$ on l_1 is compact if and only if

$$\lim_{n \to \infty} \sup_i \sum_{j=n}^{\infty} |b_{ij}| = 0$$

(see e.g., [4]). The compact operators form a closed two sided ideal in the algebra of bounded linear operators on l_1.

In addition to the compact operators, we shall use two other classes of operator on l_1; these are defined now. The defining property of the first of the classes, which we denote the "*sc* property", is a weakening of compactness.

Definition. A bounded linear operator $C = [c_{ij}]$ on l_1 is said to be "*sc*" if and only if

$$\lim_{j \to \infty} \sup_i |c_{ij}| = 0.$$

That is, the *sc* property is equivalent to the uniform convergence to zero of the row elements of the matrix representation of the operator. Note that the *sc* operators may be characterised as follows: – if $\hat{l}_1$ denotes the space of elements of l_1, with the topology induced by the supremum norm, then C on l_1 is *sc* if and only if C is a compact map from l_1 to $\hat{l}_1$.

The following properties of *sc* operators will be needed for the development of the theory.

Properties of sc Operators

(I) *The set of sc operators is closed under addition, scalar multiplication, and in the uniform topology.*

This may be verified easily.

(II) *If C is sc, and B bounded, then BC is sc.*

Take $c_j \downarrow 0$ such that $\sup_i |c_{ij}| \leqq c_j$; then $\lim_j \sup_i |\sum_k b_{ik} c_{kj}| \leqq \lim_j \|B\|\, c_j = 0$.

(III) *If C is sc, and B bounded, and such that in any row or column of B there are at most K non-zero terms, then CB is sc.*

Take c_j as in (II); let the row index of the first non-zero term in the j-th column of B be denoted by $k(j)$. Then, since there are at most K non-zero terms in the j-th column of B, since $|b_{ik}| \leqq \|B\|$ for all i, k, and since c_j is decreasing in j, it follows that

$$\lim_j \sup_i \Big|\sum_k c_{ik} b_{kj}\Big| \leqq \lim_j K \|B\| \, c_{k(j)}.$$

It is now shown that, as $j \to \infty$, $k(j) \to \infty$.

If not, there exists N such that $k(j) \leqq N$ infinitely often: but this contradicts the fact that B has at most K non-zero entries in each row.

Thus $\lim_{j\to\infty} c_{k(j)} = 0$, so CB is *sc*.

The *sc* operators form a one-sided ideal in the algebra of bounded operators; property (III) plays a crucial role in giving closure on the other side in the subsequent theory. The other special class of operators we consider, defined in terms of the transpose of the *sc* property, has even weaker closure properties.

Definition. A bounded linear operator $C = [c_{ij}]$ on l_1 is said to be "*sc**" if and only if

$$\lim_i \sup_j |c_{ij}| = 0.$$

That is, the *sc** property is equivalent to the uniform convergence to zero of the column elements of the operator.

We establish some properties of *sc** operators which will be required.

Properties of sc Operators*

(I) *The set of sc* operators is closed under addition, multiplication, and in the uniform norm.*

This is easily verifiable.

(II) *If C is sc*, and D is a bounded diagonal operator, CD and DC are sc*.*

This too can be readily verified.

(III) *If B and C are sc*, so is BC.*

Take any $\varepsilon > 0$; then there exists $N > 0$ such that

$$|c_{ij}| \leqq \varepsilon \quad \text{for all } i \geqq N, \quad \text{and for all } j.$$

Then

$$\Big|\sum_i b_{ji} c_{ik}\Big| \leqq \varepsilon \|B\| + \|C\| \sum_{i=1}^{N-1} |b_{ji}|,$$

and the latter term on the right becomes arbitrarily small for large j.

Having introduced the special types of operator to be employed, it is still necessary to introduce some further notation.

Definitions. An "array" is defined to be an infinite matrix of zeros and ones; the class $\mathscr{R}$ of arrays is defined as follows:

$R \in \mathscr{R}$ if and only if R has exactly one non-zero entry in all but a finite number of rows and columns, and, in addition, an at most finite number in each column.

For R an array, the operator $P_t^{(R)}$ is defined by

$$[P_t^{(R)}]_{ij} = p_{ij}(t) \quad \text{for } (i,j) \in R,$$
$$[P_t^{(R)}]_{ij} = 0 \quad \text{for } (i,j) \notin R,$$

where "$(i,j) \in R$" denotes "$r_{ij}=1$".

With these preliminaries, we can now define precisely the sense in which the compactness condition (3) holds. For simplicity we consider the condition as holding at time $t=1$; this involves no real loss of generality, since we may change the time scale of the process by a constant multiple without altering its fundamental structure: we are then concerned with Markov semi-groups $\{P_t\}$ satisfying

$$(\alpha)\ P_1 - P_1^{(R)} \quad \textit{is compact, sc, or sc* for some } R \in \mathscr{R},$$

and (4)

$$(\beta)\ \inf_{(i,j)\in R} p_{ij}(1) > 0.$$

Condition (β) is necessary to prevent (α) holding in a trivial sense, in which case a completely different type of theory results.

In the introduction, this compactness condition was described in terms of a class $\mathscr{S}$ of simple operators. We may define $\mathscr{S}$ as follows: a real, non-negative operator $S=[s_{ij}]$ is in $\mathscr{S}$ if and only if it satisfies the following two conditions:

(α) *there exists* $R \in \mathscr{R}$ *such that* $s_{ij} > 0 \Leftrightarrow (i,j) \in R$;

(β) $\inf_{\{(i,j):\, s_{ij}>0\}} s_{ij} > 0.$

Then it is not difficult to verify that the compactness condition (4) holds if and only if

$$P_1 - S \quad \textit{is compact, sc or sc* for some } S \in \mathscr{S}.$$

Note that $\mathscr{S}$ contains the operators kI, $k>0$, and, more generally, the invertible, non-negative diagonal operators, which will prove to play a central role in the subsequent theory.

One final piece of notation is needed before we can start to develop the theory of Markov semi-groups satisfying (4):

we define the function $\rho_j(t)$, $t>0$, as

$$\rho_j(t) = \sup_i p_{ij}(t)$$

Note that

$$\rho_j(s+t) = \sup_i \sum_k p_{ik}(s)\, p_{kj}(t) \leqq \sup_k p_{kj}(t) = \rho_j(t),$$

so that $\rho_j(t)$ is non-increasing.

We are now in a position to investigate in detail the implications of the compactness conditions (4). The first result to be proved is a rather technical lemma, which is, however, of fundamental importance in the subsequent theory.

Lemma 1. *Let* $\{P_t;\ t \geqq 0\}$ *be a Markov semi-group such that condition* (4) *holds for the sc property.*

Then there exists $t_1>0$, $k_1>\frac{3}{4}$, and a positive integer J_1, such that, if $\rho_i(s)>k_1$ for some $i>J_1$, and some $s<t_1$, then

$$p_{ii}(s)=\rho_i(s), \quad \textit{and} \quad p_{ki}(s)<k_1 \qquad \textit{for } k\neq i.$$

Proof. Define $g=\inf_{(i,j)\in R} p_{ij}(1)$.

If the k-th row of R contains a unique non-zero element, denote its position by $\lambda(k)$.

Take any k_1 such that $1>k_1>\max[\frac{3}{4},(1-g)]$, and let $\varepsilon_1=g-(1-k_1)$.

Since $P_1-P_1^{(R)}$ is *sc*, there exists J_2 (taken so large that all rows and columns of R after the J_2-th contain a unique non-zero element) such that

$$p_{ij}(1)<\varepsilon_1 \quad \text{for all } j>J_2, \quad \text{and} \quad (i,j)\notin R. \tag{5}$$

Since there are only a finite number of non-zero elements in each column of R, we may choose $J_1\geqq J_2$ such that, for all $k>J_1$, $\lambda(k)>J_2$.

Then take $t_1<1$ such that

$$p_{ii}(t)>k_1 \quad \text{for all } t\leqq t_1, \quad \text{and} \quad i\leqq J_1. \tag{6}$$

Now suppose that there exists $i>J_1$, $s<t_1$, and states l, j, with $l\neq j$, such that $p_{li}(s)\geqq k_1$, $p_{ji}(s)\geqq k_1$: suppose, without loss of generality, that $p_{li}(s)\geqq p_{ji}(s)$. Note that both l and j are $>J_1$: for, suppose l is $\leqq J_1$; then, from (6), $p_{ll}(s)>k_1$, and so $1\geqq p_{ll}(s)+p_{li}(s)>2k_1>\frac{3}{2}$, which is impossible; an exactly similar argument holds for j.

Then,

$$\begin{aligned} g\leqq p_{j\lambda(j)}(1)&=\sum_k p_{jk}(s)\,p_{k\lambda(j)}(1-s)\\ &\leqq p_{ji}(s)\,p_{i\lambda(j)}(1-s)+\sum_{k\neq i} p_{jk}(s)\\ &=p_{ji}(s)\,p_{i\lambda(j)}(1-s)+\big(1-p_{ji}(s)\big)\\ &\leqq p_{ji}(s)\,p_{i\lambda(j)}(1-s)+(1-k_1). \end{aligned}$$

Thus

$$p_{l\lambda(j)}(1)\geqq p_{li}(s)\,p_{i\lambda(j)}(1-s)\geqq p_{ji}(s)\,p_{i\lambda(j)}(1-s)\geqq g-(1-k_1)=\varepsilon_1.$$

But $j>J_1$, so $\lambda(j)>J_2$: thus, by (5), $(l,\lambda(j))\in R$; but this is impossible, since $(j,\lambda(j))\in R$, $\lambda(j)>J_2$, and there is exactly one non-zero element in each column of R after the J_2-th.

Thus, for $i>J_1$, $p_{ki}(s)\geqq k_1$ for at most one k for each $s<t_1$.

Now suppose that $i>J_1$, and $\rho_i(s)>k_1$, for some $s<t_1$: then, since $\rho_i(\bullet)$ is a non-increasing function, it follows that $\rho_i(t)\geqq\rho_i(s)$, $0<t\leqq s$. By the preceding remark, there exists a unique k for each such t such that

$$\rho_i(t)=p_{ki}(t), \quad \text{and} \quad p_{fi}(t)<k_1 \quad \text{for all } f\neq k.$$

Thus, by the continuity of the functions $p_{fi}(t)$, $\rho_i(t)=p_{ki}(t)$ for a fixed k, $0<t\leqq s$. Thus $k=i$, by standardness, and the assertion is proved. [Note that this lemma, and its proof, is reminiscent of the type of argument used by Speakman, [3], in her Lemma 2.]

We recall that the infinitesimal generator A of the semi-group $\{P_t\}$ is defined by

$$A x = \lim_{t \to 0+} t^{-1}(P_t - I)\, x,$$

for all x in l_1 for which the limit exists. Boundedness of A is equivalent to the uniform continuity of the semi-group at the origin; in this case,

$$P_t = \exp(t A),$$

and A is the uniform limit of $t^{-1}(P_t - I)$ as $t \to 0+$.

The essential difficulty in establishing the results of this paper is to prove that A is necessarily bounded when $\{P_t\}$ satisfies condition (4). This step is carried out in the following theorem, with the help of Lemma 1: we are also able to deduce in this theorem that, if (4) holds, R must be a diagonal array, apart from a possible finite number of terms.

As is a standard notation, the function $g(t)$ is defined by

$$g(t) = \inf_i p_{ii}(t), \qquad t > 0.$$

Theorem 1. *Let $\{P_t; t \geqq 0\}$ be a Markov semi-group with infinitesimal generator A: let $R \in \mathscr{R}$ be such that*

(α) $P_1 - P_1^{(R)}$ *is sc,*

(β) $\inf_{(i,j) \in R} p_{ij}(1) > 0$.

Then A is bounded, and R is, apart from a possible finite number of terms, a diagonal array.

Proof. As in the proof of Lemma 1, let $g = \inf_{(i,j) \in R} p_{ij}(1)$, and, if R has a unique non-zero element in its j-th row, denote its position by $\lambda(j)$. Since the postulates of the theorem are precisely those of Lemma 1, the lemma holds, and we take t_1, k_1 and J_1 to be as they appear in the statement of the lemma.

Take a fixed integer $n > 1$:

let j be $> J_1$; then, since each column of R after the J_1-th contains a non-zero element, it follows that $\rho_j(1) \geqq g$.

Thus, since $\rho_j(\cdot)$ is non-increasing, $\rho_j(1 - n^{-1}) \geqq g$.

Take a fixed ε in the range $0 < \varepsilon < g$, and define the function $\phi(j)$ as follows:

$$\phi(j) \text{ is some integer such that } p_{\phi(j)\, j}(1 - n^{-1}) \geqq (g - \varepsilon):$$

note that this is well defined for $j > J_1$ since $\rho_j(1 - n^{-1}) \geqq g$. Notice that, for any i,

$$\phi(j) = i \text{ for at most } (g - \varepsilon)^{-1} \text{ values of } j. \tag{7}$$

Now, from the properties of $\mathscr{R}$, the domain of $\lambda(\cdot)$ consists of all but a finite number of states, and there exists a constant K', say, such that, for any j, $\lambda(\cdot) = j$ at most K' times. By this remark, and (7), there exists $J_3 > J_1$ such that,

$$\text{for } j > J_3,\ \phi(j) \text{ is in the domain of } \lambda, \text{ and } \lambda[\phi(j)] > J_1. \tag{8}$$

The operator $Q_{1/n}$ is now defined as follows:

$$[Q_{1/n}]_{ji}=0, \qquad j\leqq J_3;$$
$$[Q_{1/n}]_{ji}=\delta_{i\lambda[\phi(j)]}\, p_{ji}(n^{-1}), \qquad j>J_3.$$

Then the operator $Q_{1/n}$ has at most one non-zero entry in each row, and, by (7) and (8), at most $(g-\varepsilon)^{-1}K'$ non-zero entries in each column: further, the first J_1 columns of $Q_{1/n}$ are zero.

Now, since $P_1-P_1^{(R)}$ is *sc*,

$P_{\phi(j)\,i}(1)\to 0$ as $i\to\infty$ through values other than $\lambda[\phi(j)]$, uniformly in $j>J_3$.

But,
$$p_{\phi(j)\,i}(1)\geqq p_{\phi(j)\,j}(1-n^{-1})\, p_{ji}(n^{-1})\geqq (g-\varepsilon)\, p_{ji}(n^{-1}),$$
by definition of ϕ.

Thus $p_{ji}(n^{-1})\to 0$ as $i\to\infty$ through values other than $\lambda[\phi(j)]$, uniformly in $j>J_3$. That is, since the *sc* property is not affected by the behaviour of any finite number of rows,
$$P_{1/n}-Q_{1/n} \quad \text{is } sc. \tag{9}$$

Suppose now that, for some $r\geqq 1$, $P_{r/n}-Q_{1/n}^r$ is *sc*: then consider the identity
$$P_{(r+1)/n}-Q_{1/n}^{r+1}=P_{1/n}(P_{r/n}-Q_{1/n}^r)+(P_{1/n}-Q_{1/n})\,Q_{1/n}^r.$$

The first term on the right is *sc* by Property II of *sc* operators; the second term on the right is *sc* by Property III of *sc* operators, and the remarks above on the form of $Q_{1/n}$: thus, by Property I,
$$P_{(r+1)/n}-Q_{1/n}^{r+1} \quad \text{is } sc.$$

Thus, using induction on (9), it follows in particular that
$$P_1-Q_{1/n}^n \quad \text{is } sc,$$
and hence that
$$Q_{1/n}^n-P_1^{(R)} \quad \text{is } sc. \tag{10}$$

$Q_{1/n}^n$ has exactly one non-zero element in each row with sufficiently large row index: it follows from (10) that, for large enough row index j, this element is in the $\lambda(j)$-th position, and, further, becomes arbitrarily close to $p_{j\lambda(j)}(1)$.

Recall the definitions of t_1 and k_1 in Lemma 1, and take ε as before such that $0<\varepsilon<g$: take n so large that
$$k_1^n\leqq g-\varepsilon, \quad \text{and} \quad n^{-1}<t_1. \tag{11}$$

By the preceding remarks, there exists N such that, for $r\geqq N$, there is a non-zero element in the r-th row of $Q_{1/n}^n$, and this element is greater than $g-\varepsilon$.

Take $r\geqq N$: then the non-zero element in the r-th row of $Q_{1/n}^n$ is of the form
$$p_{ri_1}(n^{-1})\, p_{i_1 i_2}(n^{-1})\dots p_{i_{n-1} i_n}(n^{-1}),$$
where, since the first J_1 columns of $Q_{1/n}$ are zero, $i_k>J_1$ for $k=1,\dots,n$. By (11), at least one of the $p_{i_{j-1}i_j}(n^{-1})$ is greater than k_1. Hence, since $n^{-1}<t_1$, and $i_j>J_1$, it follows from Lemma 1 that $i_{j-1}=i_j$: thus $i_1=\cdots=i_n=r$.

That is, the non-zero entry in the r-th row of $Q_{1/n}$ is the r-th entry, and $p_{rr}(n^{-1}) > k_1$: further, from the proof of Lemma 1,

$$p_{rr}(t) > k_1 \quad \text{for all } t \leqq n^{-1}.$$

Since this holds for all $r \geqq N$, it follows that

$$\liminf_{t \to 0+} [g(t)] \geqq k_1,$$

and so, by a well known result of Reuter, (see, e.g., [2]), A is bounded.

The assertion about the form of R follows immediately, either by taking powers of $Q_{1/n}$, which we now know to be diagonal, apart from a finite number of terms, or on noting, by an elementary argument, that $g(t) \geqq e^{-\frac{1}{2}\|A\| t}$, and hence, in particular, $g(1) > 0$, and considering this in conjunction with the postulates of the theorem.

This concludes the proof.

As an immediate corollary, the following holds.

Corollary. *The conclusions of Theorem* 1 *are still valid if compactness or the sc* property replaces the sc property in the statement of the theorem.*

Proof. It is trivial to verify that, if $P_1 - P_1^{(R)}$ is compact or *sc**, then $P_1 - P_1^{(R)}$ is *sc*.

The important part of Theorem 1 is the implication that A is bounded when the compactness conditions (4) hold: we have seen that the other assertion, on the form of the array R, can be made to follow as an easy consequence of this. Using the boundedness of A, and restricting attention to the diagonal case, as we are now justified in doing, it is not difficult to prove the following theorem.

Theorem 2. *Let* $\{P_t; t \geqq 0\}$ *be a Markov semi-group with infinitesimal generator* A. *If*

(α) $P_1 - \operatorname{diag}\{p_{11}(1), p_{22}(1), \ldots\}$ *is compact, sc or sc*, and*

(β) $g(1) > 0$,

then A *is bounded, and*

$$A - \operatorname{diag}\{a_{11}, a_{22} \ldots\} \text{ is compact, sc, or sc* respectively.}$$

Further,

$$P_t - \operatorname{diag}\{p_{11}^t(1), p_{22}^t(1), \ldots\}$$

and

$$P_t - \operatorname{diag}\{p_{11}(t), p_{22}(t), \ldots\}$$

are correspondingly compact, sc or sc for all* $t > 0$.

Proof. (a) The result is proved first for compactness.

By Theorem 1, A is bounded, and so $g(t) > 0$ for all t.

Let $D_t = \operatorname{diag}\{p_{11}(t), p_{22}(t) \ldots\}$, and $D_A = \operatorname{diag}\{a_{11}, a_{22}, \ldots\}$. Then, since

$$p_{ij}(1) \geqq p_{ii}(1 - n^{-1})\, p_{ij}(n^{-1}) \geqq g(1 - n^{-1})\, p_{ij}(n^{-1}),$$

$P_{1/n} - D_{1/n}$ is majorised, element wise, by a constant multiple of $P_1 - D_1$, and so

$$P_{1/n} - D_{1/n} \text{ is compact for all } n \geqq 1. \tag{12}$$

Since A is the uniform limit of $n(P_{1/n}-I)$, and D_A of $n(D_{1/n}-I)$, as $n\to\infty$, it follows from (12) that $A-D_A$ is compact, thus establishing the first assertion of the theorem.

From the identity

$$(P-Q)(R-S)=(PR-QS)-Q(R-S)-(P-Q)S, \tag{13}$$

it is seen that if $P-Q$ and $R-S$ are compact, so is $PR-QS$. It follows readily from this, and (12), that

$$P_1-D_{1/n}^n \text{ is compact for all } n\geqq 1.$$

Thus, since P_1-D_1 is compact by hypothesis,

$$D_{1/n}^n-D_1 \text{ is compact}.$$

But it is readily verified that

$$\begin{aligned} &D_{1/n}^n-D_1 \text{ is compact} \\ &\quad\Leftrightarrow [p_{jj}^n(1/n)-p_{jj}(1)]\to 0 \qquad \text{as } j\to\infty \\ &\quad\Leftrightarrow [p_{jj}(1/n)-p_{jj}^{1/n}(1)]\to 0 \qquad \text{as } j\to\infty \\ &\quad\Leftrightarrow D_{1/n}-(D_1)^{1/n} \text{ is compact}. \end{aligned}$$

Thus $P_{1/n}-(D_1)^{1/n}$ is compact for all n, and, by the identity (13) again,

$$P_{m/n}-(D_1)^{m/n} \text{ is compact for all } m, n.$$

It follows, on taking uniform limits, that

$$P_t-(D_1)^t \text{ is compact for all } t>0.$$

It is a straightforward deduction from this that P_t-D_t is compact for all $t>0$, on noting that the moduli of the elements of $P_t-(D_1)^t$ majorise the elements of P_t-D_t.

(b) The proof of the theorem for the *sc* and *sc** cases is on exactly similar lines: while care has to be taken with use of the identity (13), it is not difficult to verify that the manipulations used in (a) can always be justified by means of the closure properties for *sc* and *sc** operators outlined above. It is interesting to note that the above proof still holds in the compact and *sc* cases if (13) is replaced by the simple identity

$$PR-QS=P(R-S)+(P-Q)S,$$

but that the weaker closure properties of the *sc** operators necessitate the use of the more cumbersome identity.

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