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On the Influence of Moments on Approximations by Portion of a Chebyshev Series in Central Limit Convergence

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1. Introduction

Let X_i, $i=1,2,3,\ldots$ be a sequence of independent and identically distributed random variables with $EX_i=0$ and $\operatorname{var} X_i=1$. Write $F(x)$ for the distribution function and $f(t)$ for the characteristic function of X_i and put $S_n=\sum_{i=1}^{n} X_i$. Then,

$$F_n(x)=P(S_n\leqq x\sqrt{n})\to\Phi(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{1}{2}u^2}\,du$$

as $n\to\infty$. We shall herein be concerned with the influence of moments of X_i on the rate of convergence to zero of

$$A_{kn}=\sup_x |F_n(x)-G_{kn}(x)|$$

where

$$G_{kn}(x)=\Phi(x)+\frac{1}{\sqrt{2\pi}}\,e^{-\frac{1}{2}x^2}\sum_{s=1}^{k} Q_s(x)\,n^{-\frac{1}{2}s}$$

is a given portion of the Chebyshev series corresponding to the X_i (see for example Gnedenko and Kolmogorov [2], Section 38), the $Q_j(x)$ being polynomials of degree $3j-1$ whose coefficients depend on the first $(j+2)$ moments of X_i. Now, Cramér (see [2], Section 45) has shown that for distributions satisfying the condition (C) (that is, $\limsup_{t\to\infty}|f(t)|<1$) and if $E|X_i|^{k+2}<\infty$ $(k\geqq 1)$, then $A_{kn}=o(n^{-k/2})$) as $n\to\infty$. Furthermore, Ibragimov [4] has produced necessary and sufficient conditions, under (C), for (i) $A_{kn}=o(n^{-k/2})$ $(k\geqq 1)$ and (ii) $A_{kn}=O(n^{-(k+\delta)/2})$, $0<\delta\leqq 1$, $k\geqq 1$, but these conditions are not in general moment conditions. We shall provide, also under (C), some necessary and sufficient conditions in terms of moments on the rate of convergence of A_{kn} to zero.

In order to avoid presupposing the existence of moments of higher order than the second, we shall follow the formulation of [4] and prescribe an arbitrary numerical sequence $\beta_1=0$, $\beta_2=1$, $\beta_3,\beta_4,\ldots$. On the basis of this sequence we form polynomials $Q_k(x)$ in such a way that their coefficients are expressed in terms of $\beta_1,\ldots,\beta_{k+2}$ in the same way as the coefficients of the classical polynomials $Q_k(x)$ are expressed in terms of the cumulants $\kappa_1,\ldots,\kappa_{k+2}$ of X_i. That is,

$$Q_k(x)=-\sum\frac{1}{j_1!\cdots j_k!}\left(\frac{\beta_3}{3!}\right)^{j_1}\cdots\left(\frac{\beta_{k+2}}{(k+2)!}\right)^{j_k} H_{3j_1+\cdots+(k+2)j_k-1},$$

where the summation is over all non-negative solutions of $j_1+2j_2+\cdots+kj_k=k$ and $H_m(x)$ is the Hermite-Chebyshev polynomial

$$H_m(x)=(-1)^m e^{\frac{1}{2}x^2}\frac{d^m}{dx^m}e^{-\frac{1}{2}x^2}$$

(Petrov [8]). $Q_k(x)$ will henceforth be interpreted in this way.

Let $\alpha_1=0$, $\alpha_2=1$, $\alpha_3, \alpha_4, \ldots$ be the "moment" sequence corresponding to the "cumulant" sequence $\beta_1=0$, $\beta_2=1$, $\beta_3, \beta_4, \ldots$. We shall establish the following results.

Theorem 1. *In order that*

$$\sum_{n=1}^{\infty} n^{-1+(k+\delta)/2} \sup_x |F_n(x)-G_{kn}(x)|<\infty \tag{1}$$

where k is a non-negative integer and $0<\delta<1$, it is necessary and for $k=0$ or for distributions satisfying (C) also sufficient that

$$E|X_i|^{k+2+\delta}<\infty \quad \text{and} \quad \alpha_j=EX_i^j, \quad j=1,2,\ldots,k+2. \tag{2}$$

Theorem 2. *In order that the relation (1) hold, where $0<\delta<1$, it is necessary and for distributions satisfying (C) or for $k=0$ also sufficient that*

$$f(t)=\exp\left\{\sum_{s=2}^{k+2}\frac{(it)^s}{s!}\beta_s+|t|^{k+2}\gamma(t)\right\}, \tag{3}$$

where for $A>0$,

$$\int_0^A \frac{|\gamma(t)|}{t^{1+\delta}}\,dt<\infty. \tag{4}$$

Unfortunately it has not been possible to treat the case $\delta=0$ in general and then not without certain presuppositions on the existence of moments. In this case we find the following result.

Theorem 3. *Suppose $E|X_i|^{k+2}<\infty$ where k is a non-negative even integer and $\alpha_j=EX_i^j$, $j=1,2,\ldots,k+2$. Then, for (1) to hold with $\delta=0$ it is necessary and for $k=0$ or for distributions satisfying (C) also sufficient that $E|X_i|^{k+2}\log(1+|X_i|)<\infty$.*

These theorems extend the work of Heyde [3] where the results for the case $k=0$ were obtained.

2. Preliminary Lemmas

Lemma 1. *Suppose $E|X_i|^r<\infty$ for some integer $r\geqq 2$. Then $f(t)$ is representable in the form*

$$f(t)=\exp\left\{\sum_{s=2}^{r}\frac{(it)^s}{s!}\kappa_s+|t|^r\gamma(t)\right\} \tag{5}$$

where $\gamma(t)=o(1)$ as $t\to 0$ and κ_s denotes the s-th cumulant of X_i. Furthermore, there exists an $\varepsilon>0$ such that for $0<t<\varepsilon$, $|\gamma(t)|>0$ or $|\gamma(t)|\equiv 0$. If $|\gamma(t)|\equiv 0$ for $0<t<\varepsilon$, X_i has a normal distribution.

Proof. The representation in the form (5) with $\gamma(t)=o(1)$ as $t\to 0$ follows simply from a Taylor expansion of $\log f(t)$ (e.g. [2], p. 64).

Next, suppose $\gamma(t)=0$ for all $t\in\{t_k\}$ where $\{t_k\}$ is a sequence of non-zero real numbers converging to zero. Then,

$$\exp\left\{\sum_{s=2}^{r}\frac{(i\,t)^s}{s!}\kappa_s+|t|^r\gamma(t)\right\}=\exp\left\{\sum_{s=2}^{r}\frac{(i\,t)^s}{s!}\kappa_s\right\} \tag{6}$$

for all $t\in\{t_k\}$ and applying Theorem 4.2.1 of Linnik [5], we have that (6) holds for all real t. However, this is impossible unless $r=2$ since the left hand side of (6) represents a characteristic function and the right hand side does not, in view of Marcinkiewicz's Theorem (e.g. Lukacs [7], p. 147), unless $r=2$. Thus, if $r>2$ we must be able to choose $\varepsilon>0$ so that $\gamma(t)$ has no zeros in $(0,\varepsilon)$. If $r=2$, on the other hand, either (6) holds for all t in which case $\gamma(t)\equiv 0$ and X_i has a normal distribution or zero is not a limit point of a sequence of zeros of $\gamma(t)$ and hence we can choose an interval $(0,\varepsilon)$ containing no zeros of $\gamma(t)$.

Lemma 2. *Suppose $E|X_i|^r<\infty$ for some integer $r\geqq 2$. Then, $f(t)$ is representable in the form*

$$f(t)=\sum_{s=0}^{r}\frac{(i\,t)^s}{s!}\mu_s+|t|^r\beta(t) \tag{7}$$

where $\beta(t)=o(1)$ as $t\to 0$ and μ_s denotes the s-th moment of X_i. Furthermore, for any $A>0$ and $0\leqq\delta<1$, the conditions $\int_0^A|\beta(t)|\,t^{-(1+\delta)}\,dt<\infty$ and $\int_0^A|\gamma(t)|\,t^{-(1+\delta)}\,dt<\infty$ are equivalent, $\gamma(t)$ being given by (5) and these conditions are in turn equivalent to $E|X_i|^{r+\delta}<\infty$ if $\delta>0$, $E|X_i|^r\log(1+|X_i|)<\infty$ if r is even and $\delta=0$.

Proof. Firstly, the representation of $f(t)$ in the form (7) with $\beta(t)=o(1)$ as $t\to 0$ follows simply from a Taylor expansion of $f(t)$. Also, from Lemma 3 of [4] we find that as $t\to 0$,

$$\log f(t)-\sum_{s=2}^{r}\frac{(i\,t)^s}{s!}\kappa_s=f(t)-\sum_{s=0}^{r}\frac{(i\,t)^s}{s!}\mu_s+\frac{\Lambda_{r+1}}{(r+1)!}(i\,t)^{r+1}+o(t^{r+1}),$$

where Λ_{k+1} is a constant, so that

$$|t|^r\gamma(t)=|t|^r\beta(t)+\frac{\Lambda_{r+1}}{(r+1)!}(i\,t)^{r+1}+o(t^{r+1}) \tag{8}$$

and we readily deduce the equivalence of

$$\int_0^A|\beta(t)|\,t^{-(1+\delta)}\,dt<\infty \quad\text{and}\quad \int_0^A|\gamma(t)|\,t^{-(1+\delta)}\,dt<\infty,\qquad 0\leqq\delta<1.$$

Now suppose that

$$\int_0^A|\beta(t)|\,t^{-(1+\delta)}\,dt<\infty.$$

This implies

$$\int_0^A|\mathrm{Re}\,\beta(t)|\,t^{-(1+\delta)}\,dt<\infty$$

where Re denotes the real part and hence that

$$\left|\int_0^A \operatorname{Re}\beta(t)\, t^{-(1+\delta)}\, dt\right| < \infty .$$

It is with this last condition that we shall work. We have

$$\begin{aligned}
\left|\int_0^A \frac{\operatorname{Re}\beta(t)}{t^{1+\delta}}\, dt\right| &= \left|\int_0^A \frac{1}{t^{r+\delta+1}} \left(\int_{-\infty}^{\infty} \operatorname{Re}\left(e^{itx} - \sum_{s=0}^{r} \frac{(i t x)^s}{s!}\right) dF(x)\right) dt\right| \\
&= \left|\int_{-\infty}^{\infty} dF(x) \int_0^A \frac{\operatorname{Re}\left(e^{itx} - \sum_{s=0}^{r} \frac{(i t x)^s}{s!}\right)}{t^{r+\delta+1}}\, dt\right| \\
&= \left|\int_{-\infty}^{\infty} dF(x) \int_0^A \frac{\left(\cos t x - \sum_{s=0}^{R} (-1)^s \frac{(t x)^{2s}}{(2s)!}\right)}{t^{r+\delta+1}}\, dt\right|
\end{aligned} \tag{9}$$

where $R=[r/2]$, the integer part of $r/2$. But, after two integrations by parts,

$$\begin{aligned}
&\int_0^A \frac{\left(\cos t x - \sum_{s=0}^{R} (-1)^s \frac{(t x)^{2s}}{(2s)!}\right)}{t^{r+\delta+1}}\, dt \\
&= \frac{-\left(\cos A x - \sum_{s=0}^{R} (-1)^s \frac{(A x)^{2s}}{(2s)!}\right)}{(r+\delta)\, A^{r+\delta}} - \frac{x\left(\sum_{s=0}^{R-1} (-1)^s \frac{(A x)^{2s+1}}{(2s+1)!} - \sin A x\right)}{(r+\delta)(r+\delta-1)\, A^{r+\delta-1}} \\
&\quad + \frac{x^2}{(r+\delta)(r+\delta-1)} \int_0^A \frac{\left(\sum_{s=0}^{R-1} (-1)^s \frac{(t x)^{2s}}{(2s)!} - \cos t x\right)}{t^{r+\delta-1}}\, dt,
\end{aligned}$$

so that, recalling that $E|X_i|^r < \infty$, we must have using (9),

$$\left|\int_{-\infty}^{\infty} x^2\, dF(x) \int_0^A \frac{\left(\cos t x - \sum_{s=0}^{R-1} (-1)^s \frac{(t x)^{2s}}{(2s)!}\right)}{t^{r+\delta-1}}\, dt\right| < \infty .$$

Continuing this reduction, we find ultimately that

$$\left|\int_{-\infty}^{\infty} x^{2R}\, dF(x) \int_0^A \frac{(\cos t x - 1)}{t^{r+\delta+1-2R}}\, dt\right| < \infty$$

which transforms to give

$$\int_{-\infty}^{\infty} |x|^{r+\delta} \left\{\int_0^{A|x|} \frac{1-\cos u}{u^{r+\delta+1-2R}}\, du\right\} dF(x) < \infty .$$

Thus, if r is even, $R=r/2$ and

$$\int_{-\infty}^{\infty} |x|^{r+\delta} \left\{\int_0^{A|x|} \frac{1-\cos u}{u^{1+\delta}}\, du\right\} dF(x) < \infty , \tag{10}$$

while if r is odd, $R=(r-1)/2$ and

$$\int_{-\infty}^{\infty} |x|^{r+\delta}\left\{\int_0^{A|x|} \frac{1-\cos u}{u^{2+\delta}}\,du\right\} dF(x)<\infty. \tag{11}$$

(10) and (11) are clearly equivalent to $E|X_i|^{r+\delta}<\infty$ if $\delta>0$. On the other hand, when $\delta=0$, we have for $|x|>1$

$$\int_0^{A|x|} \frac{1-\cos u}{u}\,du=\int_0^{A} \frac{1-\cos u}{u}\,du+\int_A^{A|x|}\frac{du}{u}-\int_A^{A|x|}\frac{\cos u}{u}\,du\sim\log|x|$$

as $|x|\to\infty$ so that (10) is equivalent to the condition $E|X_i|^r\log(1+|X_i|)<\infty$ when $\delta=0$. We have thus shown that $\int_0^A |\beta(t)|\,t^{-(1+\delta)}\,dt<\infty$ implies $E|X_i|^{r+\delta}<\infty$ if $0<\delta<1$, $E|X_i|^r\log(1+|X_i|)<\infty$ if r is even and $\delta=0$.

Finally, suppose that $E|X_i|^{r+\delta}<\infty$, $\delta>0$, or that $E|X_i|^r\log(1+|X_i|)<\infty$ if r is even and $\delta=0$. Then,

$$\begin{aligned}
\int_0^A \frac{|\beta(t)|}{t^{1+\delta}}\,dt &= \int_0^A \frac{\left|f(t)-\sum_{s=0}^{r}\frac{(i\,t)^s}{s!}\mu_s\right|}{t^{r+1+\delta}}\,dt\\
&=\int_0^A \frac{1}{t^{r+1+\delta}}\left|\int_{-\infty}^{\infty}\left(e^{itx}-\sum_{s=0}^{r}\frac{(i\,t\,x)^s}{s!}\right)dF(x)\right|dt\\
&\le \int_{-\infty}^{\infty}\left\{\int_0^A\left|\frac{\left(e^{itx}-\sum_{s=0}^{r}\frac{(i\,t\,x)^s}{s!}\right)}{t^{r+1+\delta}}\right|dt\right\}dF(x)\\
&=\int_{-\infty}^{\infty}|x|^{r+\delta}\left\{\int_0^{A|x|}\frac{\left|e^{iu}-\sum_{s=0}^{r}\frac{(i\,u)^s}{s!}\right|}{u^{r+1+\delta}}\,du\right\}dF(x).
\end{aligned} \tag{12}$$

But, using the inequalities

$$\left|e^{ix}-\sum_{s=0}^{r}\frac{(i\,x)^s}{s!}\right|\le\frac{|x|^{r+1}}{(r+1)!}\qquad\text{for } |x|\le 1,$$

$$\left|e^{ix}-\sum_{s=0}^{r}\frac{(i\,x)^s}{s!}\right|\le(1+e)|x|^r\qquad\text{for } |x|\ge 1,$$

(for the first of these see e.g. Lemma 1 of [4] while the second is obtained by taking the modulus of each of the terms and bounding this) we have

$$\begin{aligned}
I(|x|)&=\int_0^{A|x|}\frac{\left|e^{iu}-\sum_{s=0}^{r}\frac{(i\,u)^s}{s!}\right|}{u^{r+1+\delta}}\,du\\
&\le\begin{cases}\dfrac{1}{(r+1)!}\displaystyle\int_0^1\frac{du}{u^\delta} & \text{for } |x|\le A^{-1}\\[2ex]
\dfrac{1}{(r+1)!}\displaystyle\int_0^1\frac{du}{u^\delta}+(1+e)\int_1^{A|x|}\frac{du}{u^{1+\delta}} & \text{for } |x|>A^{-1}.\end{cases}
\end{aligned}$$

Thus, $I(|x|) \leqq c_1$ if $\delta > 0$, $I(|x|) \leqq c_2 \log(|x|+1)$ if $\delta = 0$, where c_1 and c_2 are positive constants and using these results in (12) we have

$$\int_0^A |\beta(t)|\, t^{-(1+\delta)}\, dt < \infty$$

if $\delta > 0$ or if r is even and $\delta = 0$. This completes the proof of the lemma.

Lemma 3. *Suppose* (1) *holds and that* $E|X_i|^r < \infty$ *for some integer* $2 \leqq r \leqq k+2$. *Then,* $\alpha_j = EX_i^j$, $j = 1, 2, \ldots, r$.

Proof. The result of the lemma is true by specification for $r=2$ and we develop a proof by induction.

Suppose that $E|X_i|^s < \infty$, some $s \geqq 2$ and $\alpha_j = EX_i^j$, $j = 1, 2, \ldots, s$. Then, if $E|X_i|^{s+1} < \infty$, let $Q_j^*(x)$, $1 \leqq j \leqq s-1$ be the classical Chebyshev polynomials expressed in terms of the cumulants κ_j, $j = 1, 2, \ldots, s+1$ of X_i and write

$$G_{s-1,n}^*(x) = \Phi(x) + \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \sum_{j=1}^{s-1} Q_j^*(x) \frac{1}{n^{j/2}}.$$

We have

$$\sup_x |G_{s-1,n}(x) - G_{s-1,n}^*(x)| \leqq \sup_x |F_n(x) - G_{s-1,n}(x)| + \sup_x |F_n(x) - G_{s-1,n}^*(x)|, \tag{13}$$

and from Theorem 1 of [4],

$$n^{(s-1)/2} \sup_x |F_n(x) - G_{s-1,n}^*(x)| = o(1) \tag{14}$$

as $n \to \infty$. Also, from (1),

$$\liminf_{n\to\infty} n^{(k+\delta)/2} \sup_x |F_n(x) - G_{kn}(x)| = 0,$$

so that

$$\begin{aligned} &\liminf_{n\to\infty} n^{(s-1)/2} \sup_x |F_n(x) - G_{s-1,n}(x)| \\ &\quad \leqq \liminf_{n\to\infty} n^{(s-1)/2} \sup_x |F_n(x) - G_{kn}(x)| + \liminf_{n\to\infty} n^{(s-1)/2} \sup_x |G_{kn}(x) - G_{s-1,n}(x)| \quad (15) \\ &\quad = 0 \end{aligned}$$

since

$$G_{kn}(x) - G_{s-1,n}(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \sum_{j=s}^{k} Q_j(x)\, n^{-j/2}.$$

Consequently, using (14) and (15) in (13),

$$\liminf_{n\to\infty} n^{(s-1)/2} \sup_x |G_{s-1,n}(x) - G_{s-1,n}^*(x)| = 0. \tag{16}$$

But,

$$G_{s-1,n}(x) - G_{s-1,n}^*(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} [Q_{s-1}(x) - Q_{s-1}^*(x)] \frac{1}{n^{(s-1)/2}}$$

since $\alpha_j = EX_i^j$, $j = 1, 2, \ldots, s$ implies $Q_j(x) = Q_j^*(x)$, $1 \leqq j \leqq s-2$, and hence (16) can only hold if $Q_{s-1}(x) = Q_{s-1}^*(x)$. This gives $\kappa_{s+1} = \beta_{s+1}$ upon identifying coefficients and hence $\alpha_{s+1} = EX_i^{s+1}$ as required.

3. Proof of Theorems

We start by proving Theorems 1 and 2 simultaneously in the following three steps. Firstly we note that the equivalence of (2) and (3), (4) follows immediately from Lemmas 1 and 2. Next we shall show that (3), (4) ensures (1) under (C) or if $k=0$ and lastly that (1) ensures (2).

(3), (4) $\Rightarrow$ (1) *under* (C) *or if* $k=0$.

Firstly we note that $G_{kn}(x)$ has a bounded first derivative; let $|G'_{kn}(x)|\leqq B$. Then, using a bound due to Esseen ([2], Section 39), we have for any $T>0$,

$$|F_n(x)-G_{kn}(x)|\leqq\frac{1}{\pi}\int_{-T}^{T}\left|\frac{f_n(t)-g_{kn}(t)}{t}\right|dt+c\frac{B}{T} \tag{17}$$

where $f_n(t)$ and $g_{kn}(t)$ are the characteristic functions corresponding to $F_n(x)$ and $G_{kn}(x)$ respectively and c is a positive constant.

Now, using (3) we have

$$\begin{aligned} f_n(t)&=\left[f\left(\frac{t}{\sqrt{n}}\right)\right]^n\\ &=\exp\left\{\sum_{s=2}^{k+2}\frac{(i\,t)^s}{s!}\,\frac{\beta_s}{n^{(s-2)/2}}+n\left|\frac{t}{\sqrt{n}}\right|^{k+2}\gamma\left(\frac{t}{\sqrt{n}}\right)\right\}, \end{aligned}$$

while from Lemma 5 of [4],

$$g_{kn}(t)=\exp\left\{\sum_{s=2}^{k+2}\frac{(i\,t)^s}{s!}\,\frac{\beta_s}{n^{(s-2)/2}}+D(t)\right\},$$

where $D(t)=O(|n^{-\frac{1}{2}}\,t|^{k+1})$ as $t\to 0$ so that

$$\begin{aligned} |f_n(t)-g_{kn}(t)|\leqq&\left|\exp\left\{\sum_{s=2}^{k+2}\frac{(i\,t)^s}{s!}\,\frac{\beta_s}{n^{(s-2)/2}}\right\}\right|\left|\exp\left(n\left|\frac{t}{\sqrt{n}}\right|^{k+2}\gamma\left(\frac{t}{\sqrt{n}}\right)\right)-1\right|\\ &+\left|\exp\left\{\sum_{s=2}^{k+2}\frac{(i\,t)^s}{s!}\,\frac{\beta_s}{n^{(s-2)/2}}\right\}-g_{kn}(t)\right|. \end{aligned} \tag{18}$$

Now in view of Lemma 1, we may choose a, $0<a<1$, so small that $\max_{|t|<a}|\gamma(t)|\leqq\frac{1}{4}$. Then for $|t|<a\sqrt{n}$, using the inequality $|e^x-1|\leqq|x|\,e^{|x|}$, the first term on the right hand side of (18) is bounded by

$$\begin{aligned} &\exp\left\{\sum_{s=1}^{[\frac{1}{2}(k+2)]}(-1)^s\frac{t^{2s}}{(2s)!}\,\frac{\beta_{2s}}{n^{s-1}}\right\}\frac{|t|^{k+2}}{n^{k/2}}\left|\gamma\left(\frac{t}{\sqrt{n}}\right)\right|\exp\left\{\frac{|t|^{k+2}}{n^{k/2}}\left|\gamma\left(\frac{t}{\sqrt{n}}\right)\right|\right\}\\ &\leqq\exp\left\{n\left(\sum_{s=1}^{[\frac{1}{2}(k+2)]}(-1)^s\frac{t^{2s}}{(2s)!}\,\frac{\beta_{2s}}{n^s}+\frac{1}{4}\left|\frac{t}{\sqrt{n}}\right|^{k+2}\right)\right\}n\left|\frac{t}{\sqrt{n}}\right|^{k+2}\left|\gamma\left(\frac{t}{\sqrt{n}}\right)\right|\\ &\leqq\exp(-t^2/8)\,n\left|\frac{t}{\sqrt{n}}\right|^{k+2}\left|\gamma\left(\frac{t}{\sqrt{n}}\right)\right| \end{aligned} \tag{19}$$

for n sufficiently large since $\beta_2=1$. Also, from Lemma 5 of [4], the second term on the right hand side of (18) is bounded by

$$\frac{c}{n^{(k+1)/2}}\left(|t|^{3(k+1)}+|t|^{k+1}\right)e^{-t^2/4} \tag{20}$$

for $|t|<b\sqrt{n}$ when b is sufficiently small, c being a positive constant. Then, choosing $\alpha=\min(a,b)$ and using (18), (19) and (20), we have

$$\begin{aligned}\sum_{n=1}^{\infty} n^{-1+(k+\delta)/2} &\int_{-\alpha\sqrt{n}}^{\alpha\sqrt{n}}\left|\frac{f_n(t)-g_{kn}(t)}{t}\right| dt\\ &\leqq c\sum_{n=1}^{\infty} n^{-1-(1-\delta)/2}\int_{-\alpha\sqrt{n}}^{\alpha\sqrt{n}}\left(|t|^{3k+2}+|t|^{k}\right)e^{-t^2/4}\,dt\\ &\quad+\sum_{n=1}^{\infty} n^{-1+\delta/2}\int_{-\alpha\sqrt{n}}^{\alpha\sqrt{n}} e^{-t^2/8}\,|t|^{k+1}\left|\gamma\left(\frac{t}{\sqrt{n}}\right)\right| dt\\ &\leqq A+\sum_{n=1}^{\infty} n^{(k+\delta)/2}\int_{-\alpha}^{\alpha}|u|^{k+1}|\gamma(u)|\,e^{-nu^2/8}\,du,\end{aligned} \tag{21}$$

A being a finite constant. Furthermore, using a standard Abelian theorem (e.g. Feller [1], Vol. II, p. 423), we have

$$\lim_{t\uparrow 1}(1-t)^{(k+2+\delta)/2}\sum_{n=1}^{\infty} n^{(k+\delta)/2}\,t^n=\Gamma\big((k+2+\delta)/2\big)$$

so that for $u\neq 0$ it is possible to choose a constant $K_1>0$ such that

$$\sum_{n=1}^{\infty} n^{(k+\delta)/2}\,e^{-nu^2/8}\leqq K_1\left(1-e^{-u^2/8}\right)^{-(k+2+\delta)/2}.$$

Also, $|u|\leqq\alpha<1$, so that

$$1-e^{-u^2/8}>\tfrac{1}{8}u^2\left(1-\tfrac{1}{16}\alpha^2\right)$$

and hence

$$\sum_{n=1}^{\infty} n^{(k+\delta)/2}\,e^{-nu^2/8}\leqq K_2\,|u|^{-(k+2+\delta)}$$

for some $K_2>0$. Consequently, noting that $|\gamma(t)|$ is symmetric in t, we have

$$\begin{aligned}\sum_{n=1}^{\infty} n^{(k+\delta)/2}&\int_{-\alpha}^{\alpha}|u|^{k+1}|\gamma(u)|\,e^{-nu^2/8}\,du\\ &=2\int_0^{\alpha}u^{k+1}|\gamma(u)|\left\{\sum_{n=1}^{\infty} n^{(k+\delta)/2}\,e^{-nu^2/8}\right\}du\\ &\leqq 2K_2\int_0^{\alpha}\frac{|\gamma(u)|}{u^{1+\delta}}\,du<\infty\end{aligned} \tag{22}$$

in view of (4).

Next write $T_\alpha = \{t: \alpha\sqrt{n} \leqq |t| \leqq \alpha n^{(k+1)/2}\}$, noting that T_α is empty if $k=0$. Then, using condition (C), $\max_{|t| \geqq \alpha} |f(t)| = \theta < 1$, so that for t in T_α,

$$|f_n(t)| = \left| f\left(\frac{t}{\sqrt{n}}\right)\right|^n \leqq \theta^n$$

and hence

$$\int_{T_\alpha} \left|\frac{f_n(t)}{t}\right| dt \leqq 2\theta^n \int_{\alpha\sqrt{n}}^{\alpha n^{(k+1)/2}} \frac{dt}{t} \sim \theta^n (k+1) \log n$$

as $n \to \infty$. Consequently,

$$\sum_{n=1}^{\infty} n^{-1+(k+\delta)/2} \int_{T_\alpha} \left|\frac{f_n(t)}{t}\right| dt < \infty. \tag{23}$$

Furthermore, by the rules for forming the polynomials $Q_j(x)$,

$$g_{kn}(t) = e^{-\frac{1}{2}t^2}\left[1 + \sum_{j=1}^{k} P_j(i\,t)\, n^{-\frac{1}{2}j}\right]$$

where P_j is a polynomial of degree $3j$ determined from the formal identity

$$\exp\left\{\sum_{j=3}^{\infty} \frac{\beta_j}{j!} \frac{(i\,t)^j}{n^{(j-2)/2}}\right\} = 1 + \sum_{j=1}^{\infty} P_j(i\,t)\, n^{-j/2}.$$

Thus,

$$|g_{kn}(t)| \leqq e^{-\frac{1}{2}t^2}\left[1 + \sum_{j=1}^{3k} a_{nj} |t|^j\right]$$

where the a_{nj} are polynomials in $n^{-\frac{1}{2}}$ which tend to zero as $n \to \infty$. Consequently, we certainly have

$$\int_{T_\alpha} \left|\frac{g_{kn}(t)}{t}\right| dt \leqq K\, e^{-\alpha^2 n/2}\, n^{3k(k+1)/2}$$

for K a positive constant which gives

$$\sum_{n=1}^{\infty} n^{-1+(k+\delta)/2} \int_{T_\alpha} \left|\frac{g_{kn}(t)}{t}\right| dt < \infty, \tag{24}$$

and from (23) and (24) we obtain

$$\int_{T_\alpha} \left|\frac{f_n(t) - g_{kn}(t)}{t}\right| dt \leqq \int_{T_\alpha} \left|\frac{f_n(t)}{t}\right| dt + \int_{T_\alpha} \left|\frac{g_{kn}(t)}{t}\right| dt < \infty. \tag{25}$$

The required result (1) then follows, using (17) with $T = \alpha n^{\frac{1}{2}(k+1)}$, in view of (21), (22) and (25).

(1) $\Rightarrow$ (2)

Firstly we symmetrize the X_i's. Consider the sequence Y_i, $i = 1, 2, 3, \ldots$ of independent symmetrized random variables; each Y_i having the distribution of the difference between two independent X_i's. Clearly the Y_i have characteristic function $|f(t)|^2$ and the distribution function of $Z_n = n^{-\frac{1}{2}} \sum_{i=1}^{n} Y_i$ is $F_n(x) * (1 - F_n(-x-0)) = F_n^*(x)$. Write $G_{kn}^*(x)$ for the convolution $G_{kn}(x) * (1 - G_{kn}(-x))$. Then,

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{-1+(k+\delta)/2} \sup_x |F_n^*(x) - G_{kn}^*(x)| \\
&\quad = \sum_{n=1}^{\infty} n^{-1+(k+\delta)/2} \sup_x |F_n(x) * (1 - F_n(-x-0)) - G_{kn}(x) * (1 - G_{kn}(-x))| \\
&\quad \leqq \sum_{n=1}^{\infty} n^{-1+(k+\delta)/2} \sup_x |F_n(x) * (1 - F_n(-x-0)) - G_{kn}(x) * (1 - F_n(-x-0))| \\
&\quad + \sum_{n=1}^{\infty} n^{-1+(k+\delta)/2} \sup_x |G_{kn}(x) * (1 - F_n(-x-0)) - G_{kn}(x) * (1 - G_{kn}(-x))| \\
&\quad < \infty
\end{aligned} \tag{26}$$

in view of (1).

Next we shall show that (26) implies $E|Y_i|^{k+2} < \infty$ and hence $E|X_i|^{k+2} < \infty$. We note that

$$\sum_{n=1}^{\infty} n^{-1+(k+\delta)/2} \{1 - G_{kn}^*(x_n)\} < \infty$$

where $x_n = \{(k+\delta+1) \log n\}^{\frac{1}{2}}$, so that from (26),

$$\sum_{n=1}^{\infty} n^{-1+(k+\delta)/2} P\left(\left|\sum_{i=1}^{n} Y_i\right| > n^{\frac{1}{2}} x_n\right) < \infty. \tag{27}$$

But, for symmetric random variables,

$$P\left(\left|\sum_{i=1}^{n} Y_i\right| > n^{\frac{1}{2}} x_n\right) \geqq \tfrac{1}{2} P\left(\max_{1 \leqq k \leqq n} |Y_k| > n^{\frac{1}{2}} x_n\right) \tag{28}$$

(e.g. [1], Vol. II, p. 147) and from Bonferroni's inequalities (e.g. [1], Vol. I, p. 100) we have

$$\begin{aligned}
&n P(|Y_i| > n^{\frac{1}{2}} x_n) \{1 - \tfrac{1}{2}(n-1) P(|Y_i| > n^{\frac{1}{2}} x_n)\} \\
&\quad \leqq P\left(\max_{1 \leqq k \leqq n} |Y_k| > n^{\frac{1}{2}} x_n\right) \leqq n P(|Y_i| > n^{\frac{1}{2}} x_n),
\end{aligned}$$

while $n P(|Y_i| > n^{\frac{1}{2}} x_n) \to 0$ as $n \to \infty$ since $E Y_i^2 = 2 E X_i^2 < \infty$. Consequently,

$$P\left(\max_{1 \leqq k \leqq n} |Y_k| > n^{\frac{1}{2}} x_n\right) \sim n P(|Y_i| > n^{\frac{1}{2}} x_n) \tag{29}$$

as $n \to \infty$ and hence, from (27), (28) and (29),

$$\sum_{n=1}^{\infty} n^{(k+\delta)/2} P(|Y_i| > n^{\frac{1}{2}} x_n) < \infty. \tag{30}$$

But,

$$E|Y_i|^{k+2} = -\int_0^\infty x^{k+2}\, dP(|Y_i|>x)$$

$$\leqq (k+2)\int_0^\infty x^{k+1} P(|Y_i|>x)\, dx$$

$$=(k+2)\sum_{n=0}^{\infty} \int_{x_n n^{\frac{1}{2}}}^{x_{n+1}(n+1)^{\frac{1}{2}}} x^{k+1} P(|Y_i|>x)\, dx$$

$$\leqq \sum_{n=0}^{\infty} P(|Y_i|>n^{\frac{1}{2}} x_n)\left\{\left((n+1)\, x_{n+1}^2\right)^{(k+2)/2} - (n\, x_n^2)^{(k+2)/2}\right\}$$

$$\leqq c\sum_{n=0}^{\infty} n^{k/2} (\log n)^{k/2} P(|Y_i|>n^{\frac{1}{2}} x_n) < \infty$$

in view of (30), c being a positive constant. It then follows that $E|X_i|^{k+2}<\infty$ and an appeal to Lemma 3 gives $\alpha_j = EX_i^j$, $j=1,2,\ldots,k+2$.

Now we note that the characteristic functions of $F_n^*(x)$ and $G_{kn}^*(x)$ are $|f_n(t)|^2$ and $|g_{kn}(t)|^2$ respectively. Then, integrating by parts in the equation

$$|f_n(t)|^2 - |g_{kn}(t)|^2 = \int_{-\infty}^{\infty} e^{itx}\, d\{F_n^*(x) - G_{kn}^*(x)\},$$

we obtain

$$-\frac{|f_n(t)|^2 - |g_{kn}(t)|^2}{it} = \int_{-\infty}^{\infty} e^{itx} \{F_n^*(x) - G_{kn}^*(x)\}\, dx.$$

Also,

$$it\, e^{-t^2/2} = \int_{-\infty}^{\infty} e^{itx} \frac{x}{\sqrt{2\pi}} e^{-x^2/2}\, dx,$$

and we obtain from Parseval's identity

$$\int_{-\infty}^{\infty} \{|f_n(t)|^2 - |g_{kn}(t)|^2\}\, e^{-t^2/2}\, dt = \sqrt{2\pi} \int_{-\infty}^{\infty} \{F_n^*(x) - G_{kn}^*(x)\}\, x\, e^{-x^2/2}\, dx.$$

Thus from (26)

$$\sum_{n=1}^{\infty} n^{-1+(k+\delta)/2} \left| \int_{-\infty}^{\infty} \{|f_n(t)|^2 - |g_{kn}(t)|^2\}\, e^{-t^2/2}\, dt \right|$$

$$= \sqrt{2\pi} \sum_{n=1}^{\infty} n^{-1+(k+\delta)/2} \left| \int_{-\infty}^{\infty} \{F_n^*(x) - G_{kn}^*(x)\}\, x\, e^{-x^2/2}\, dx \right| \tag{31}$$

$$\leqq 2\sqrt{2\pi} \sum_{n=1}^{\infty} n^{-1+(k+\delta)/2} \sup_x |F_n^*(x) - G_{kn}^*(x)| < \infty.$$

Furthermore, we note that for $0<\alpha<\frac{1}{4}$,

$$\sum_{n=1}^{\infty} n^{-1+(k+\delta)/2} \left| \int_{n^\alpha}^{\infty} e^{-t^2/2} \{|f_n(t)|^2 - |g_{kn}(t)|^2\}\, dt \right|$$

$$\leqq 2\sum_{n=1}^{\infty} n^{-1+(k+\delta)/2} \int_{n^\alpha}^{\infty} e^{-t^2/2}\, dt < \infty,$$

so that in view of (31)

$$\sum_{n=1}^{\infty} n^{-1+(k+\delta)/2} \left| \int_0^{n^\alpha} e^{-t^2/2} \{|f_n(t)|^2 - |g_{kn}(t)|^2\} \, dt \right| < \infty. \tag{32}$$

Now from Lemma 5 of [4], letting $R = [\frac{1}{2}(k+2)]$, we have for $t < c'\sqrt{n}$, c' some suitably small positive constant,

$$\left| |g_{kn}(t)|^2 - \exp\left(\sum_{s=1}^{R} (-1)^s \frac{t^{2s} 2\kappa_{2s}}{(2s)!\, n^{s-1}}\right) \right| \leqq \frac{c}{n^{(k+1)/2}} (|t|^{3(k+1)} + |t|^{k+1}) \, e^{-t^2/4}$$

for some $c > 0$ since $\alpha_j = EX_i^j$, $j = 1, 2, \ldots, k+2$ ensures $\beta_j = \kappa_j$, $j = 1, 2, \ldots, k+2$. Thus,

$$\begin{aligned} &\sum_{n=1}^{\infty} n^{-1+(k+\delta)/2} \left| \int_0^{n^\alpha} \left\{ \exp\left\{ \sum_{s=1}^{R} (-1)^s \frac{t^{2s}}{(2s)!} \frac{2\kappa_{2s}}{n^{s-1}} \right\} - |g_{kn}(t)|^2 \right\} e^{-t^2/2} \, dt \right| \\ &\qquad \leqq c \sum_{n=1}^{\infty} n^{-1-(1-\delta)/2} \int_0^{n^\alpha} e^{-3t^2/4} (t^{3(k+1)} + t^{k+1}) \, dt < \infty. \end{aligned} \tag{33}$$

Also, from Lemma 1,

$$|f_n(t)|^2 = \left| f\left(\frac{t}{\sqrt{n}}\right) \right|^{2n} = \exp\left\{ \sum_{s=1}^{R} (-1)^s \frac{t^{2s}}{(2s)!} \frac{2\kappa_{2s}}{n^{s-1}} + 2 \frac{|t|^{k+2}}{n^{k/2}} \operatorname{Re} \gamma\left(\frac{t}{\sqrt{n}}\right) \right\},$$

and using this result in conjunction with (32) and (33),

$$\begin{aligned} &\sum_{n=1}^{\infty} n^{-1+(k+\delta)/2} \\ &\quad \cdot \left| \int_0^{n^\alpha} e^{-t^2/2} \exp\left\{ \sum_{s=1}^{R} (-1)^s \frac{t^{2s}}{(2s)!} \frac{2\kappa_{2s}}{n^{s-1}} \right\} \left[1 - \exp\left(2 \frac{t^{k+2}}{n^{k/2}} \operatorname{Re} \gamma\left(\frac{t}{\sqrt{n}}\right) \right) \right] dt \right| < \infty. \end{aligned}$$

Now Lemma 1 tells us that for n large enough, $\operatorname{Re} \gamma\left(\frac{t}{\sqrt{n}}\right)$ will be of constant sign for $0 < t \leqq n^\alpha$ and hence

$$\begin{aligned} &\sum_{n=1}^{\infty} n^{-1+(k+\delta)/2} \\ &\quad \cdot \int_0^{n^\alpha} e^{-t^2/2} \exp\left\{ \sum_{s=1}^{R} (-1)^s \frac{t^{2s}}{(2s)!} \frac{2\kappa_{2s}}{n^{s-1}} \right\} \left| 1 - \exp\left(2 \frac{t^{k+2}}{n^{k/2}} \operatorname{Re} \gamma\left(\frac{t}{\sqrt{n}}\right) \right) \right| dt < \infty \end{aligned}$$

so that

$$\begin{aligned} &\sum_{n=1}^{\infty} n^{-1+(k+\delta)/2} \\ &\quad \cdot \int_0^{1} \exp\left\{ -3t^2/2 + \sum_{s=2}^{R} (-1)^s \frac{t^{2s}}{(2s)!} \frac{2\kappa_{2s}}{n^{s-1}} \right\} \left| 1 - \exp\left(2 \frac{t^{k+2}}{n^{k/2}} \operatorname{Re} \gamma\left(\frac{t}{\sqrt{n}}\right) \right) \right| dt < \infty \end{aligned}$$

which implies

$$\sum_{n=1}^{\infty} n^{-1+(k+\delta)/2} \int_0^1 \left| 1 - \exp\left(2 \frac{t^{k+2}}{n^{k/2}} \operatorname{Re} \gamma\left(\frac{t}{\sqrt{n}}\right) \right) \right| dt < \infty.$$

However, as $n \to \infty$,

$$\left|1-\exp\left(2\frac{t^{k+2}}{n^{k/2}}\operatorname{Re}\gamma\left(\frac{t}{\sqrt{n}}\right)\right)\right| = \frac{2|t|^{k+2}}{n^{k/2}}\left|\operatorname{Re}\gamma\left(\frac{t}{\sqrt{n}}\right)\right|(1+o(1)),$$

so that

$$\sum_{n=1}^{\infty} n^{-1+\delta/2}\int_0^1 t^{k+2}\left|\operatorname{Re}\gamma\left(\frac{t}{\sqrt{n}}\right)\right| dt<\infty,$$

and, upon making the transformation $u=t/\sqrt{n}$, this yields

$$\sum_{n=1}^{\infty} n^{(k+1+\delta)/2}\int_0^{1/\sqrt{n}} u^{k+2}|\operatorname{Re}\gamma(u)|\,du<\infty. \tag{34}$$

Now $\int_0^{1/\sqrt{x}} u^{k+2}|\operatorname{Re}\gamma(u)|\,du$ is monotone decreasing as x increases so for $X>1$,

$$\begin{aligned}
&\int_1^X x^{(k+1+\delta)/2}\left\{\int_1^{1/\sqrt{x}} u^{k+2}|\operatorname{Re}\gamma(u)|\,du\right\}dx \\
&\quad\leqq \sum_{n=1}^{[X]}\int_n^{n+1} x^{(k+1+\delta)/2}\left\{\int_1^{1/\sqrt{x}} u^{k+2}|\operatorname{Re}\gamma(u)|\,du\right\}dx \\
&\quad\leqq \sum_{n=1}^{[X]}(n+1)^{(k+1+\delta)/2}\int_1^{1/\sqrt{n}} u^{k+2}|\operatorname{Re}\gamma(u)|\,du \\
&\quad\leqq c\sum_{n=1}^{[X]} n^{(k+1+\delta)/2}\int_1^{1/\sqrt{n}} u^{k+2}|\operatorname{Re}\gamma(u)|\,du,
\end{aligned}$$

c being a positive constant such that $(n+1)^{(k+1+\delta)/2}<c\,n^{(k+1+\delta)/2}$ for all positive integral n. Consequently, using (34), we have

$$\int_1^{\infty} x^{(k+1+\delta)/2}\left\{\int_0^{1/\sqrt{x}} u^{k+2}|\operatorname{Re}\gamma(u)|\,du\right\}dx<\infty. \tag{35}$$

Now in view of (35) we must have

$$\int_{\omega}^{2\omega} x^{(k+1+\delta)/2}\left\{\int_0^{1/\sqrt{x}} u^{k+2}|\operatorname{Re}\gamma(u)|\,du\right\}dx \to 0$$

as $\omega\to\infty$ and

$$\begin{aligned}
&\int_{\omega}^{2\omega} x^{(k+1+\delta)/2}\left\{\int_0^{1/\sqrt{x}} u^{k+2}|\operatorname{Re}\gamma(u)|\,du\right\}dx \\
&\quad\geqq \omega^{(k+3+\delta)/2}\int_0^{1/\sqrt{2\omega}} u^{k+2}|\operatorname{Re}\gamma(u)|\,du \geqq 0,
\end{aligned}$$

so that, putting $v=1/\sqrt{2\omega}$, we conclude that

$$v^{-(k+3+\delta)}\int_0^{v} u^{k+2}|\operatorname{Re}\gamma(u)|\,du\to 0 \tag{36}$$

as $v \to 0$. Then, upon making the transformation $v = 1/\sqrt{x}$, (35) becomes

$$\int_0^1 \left\{ \int_0^v u^{k+2} |\mathrm{Re}\, \gamma(u)|\, du \right\} v^{-(k+4+\delta)}\, dv < \infty$$

and, in view of (36), the fact that $\int_0^1 |\mathrm{Re}\, \gamma(t)|\, t^{-(1+\delta)}\, dt < \infty$ holds follows immediately from an integration by parts. Lemma 2 then enables us to conclude that $E|Y_i|^{k+2+\delta} < \infty$ from which we deduce that $E|X_i|^{k+2+\delta} < \infty$. This completes the proof that (1) $\Rightarrow$ (2) and hence the proof of Theorems 1 and 2.

For the proof of Theorem 3, we note firstly that the above proof that (3), (4) $\Rightarrow$ (1) under (C) or if $k=0$ together with Lemmas 1 and 2 show that $E|X_i|^{k+2} \log(1+|X_i|) < \infty$ implies (1) with $\delta = 0$ under (C) or if $k=0$. On the other hand, the above proof that (1) $\Rightarrow$ (2) shows that (1) with $\delta = 0$ implies $\int_0^1 |\mathrm{Re}\, \gamma(t)|\, t^{-1}\, dt < \infty$. Thus, from Lemmas 1 and 2 we have $E|Y_i|^{k+2} \log(1+|Y_i|) < \infty$ from which we deduce the required result that $E|X_i|^{k+2} \log(1+|X_i|) < \infty$.

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